Dynamics Worksheet 6 Solutions: Review

1. (a) Solve the following recurrence relation.

$$a_n = 2a_{n-1} - 2a_{n-2}$$
$$a_0 = 1 \quad a_1 = 1$$

Solution: We can find the solution using the characteristic equation, which here is $\lambda^2 - 2\lambda + 2 = 0$. The roots of the characteristic equation are $1+i$ and $1-i$. Therefore the general solution is

$$a_n = C_1(1+i)^n + C_2(1-i)^n.$$

Now we need to use the initial conditions to solve for C_1 and C_2. Plugging in $n = 0$ and $n = 1$ we have

$$\begin{aligned} 1 = a_0 &= C_1 + C_2 \\ 1 = a_1 &= C_1(1+i) + C_2(1-i). \end{aligned}$$

Solving this system of linear equations gives us $C_1 = 1/2$ and $C_2 = 1/2$. So the final solution is

$$a_n = \frac{1}{2}(1+i)^n + \frac{1}{2}(1-i)^n.$$

(b) What is a_{100}?

Solution: Of course, by part (a), $a_{100} = \frac{1}{2}(1+i)^{100} + \frac{1}{2}(1-i)^{100}$. But by using Euler's formula we can write something a little nicer looking. By Euler's formula, $1 + i = \sqrt{2}e^{\pi i/4}$ and $1 - i = \sqrt{2}e^{-\pi i/4}$. Therefore

$$\begin{aligned} a_{100} &= \frac{1}{2}(\sqrt{2}e^{\pi i/4})^{100} + \frac{1}{2}(\sqrt{2}e^{-\pi i/4})^{100} \\ &= \frac{1}{2}\left(2^{50}e^{25\pi i}\right) + \frac{1}{2}\left(2^{50}e^{-25\pi i}\right) \\ &= \frac{1}{2}(-2^{50}) + \frac{1}{2}(-2^{50}) \\ &= -2^{50}. \end{aligned}$$

2. Compute the following indefinite integral.

$$\int \frac{x}{x^2 - 4}\, dx$$

Solution: We can either use partial fraction decomposition or just use substitution. Note that $x^2 - 4 = (x+2)(x-2)$. So to find the partial fraction decomposition we need to find constants A and B such that

$$\frac{x}{x^2-4} = \frac{A}{x+2} + \frac{B}{x-2} = \frac{A(x-2) + B(x+2)}{x^2-4}.$$

Therefore we need to find A and B such that $Ax - 2A + Bx + 2B = x$, which means that $A + B = 1$ and $-2A + 2B = 0$. Solving this system of linear equations, we find that $A = 1/2$ and $B = 1/2$. Therefore we have

$$\int \frac{x}{x^2-4}\,dx = \frac{1}{2}\int \frac{1}{x-2} + \frac{1}{x+2}\,dx = \frac{1}{2}(\ln|x-2| + \ln|x+2|) + C.$$

3. Solve the following differential equation.

$$y' = \frac{y^2-4}{t^2y+y}$$

Solution: This equation is separable. We can rewrite it as

$$\frac{yy'}{y^2-4} = \frac{1}{t^2+1}.$$

So we have

$$\int \frac{y}{y^2-4}\,dy = \int \frac{1}{t^2+1}\,dt.$$

We calculated the first integral in problem 2 and the second integral evaluates to $\arctan t + C$. Therefore the solution y satisfies

$$\frac{1}{2}(\ln|y-2| + \ln|y+2|) = \arctan t + C.$$

In this case, it is difficult to find an explicit solution for y (on an exam though, you should find an explicit solution for y unless the question says that you don't need to).

4. Solve the following differential equation

$$(t^2-4)y' + t(y+8) = 2t^3.$$

Solution: We can use an integrating factor to solve this differential equation. First we rewrite it as

$$y' + \frac{t}{t^2-4}y = \frac{2t^3-8t}{t^2-4} = 2t.$$

Then the integrating factor is

$$I(t) = e^{\int \frac{t}{t^2-4}\,dt} = e^{1/2(\ln|t+2|+\ln|t-2|)} = \left(e^{\ln|t+2|}e^{\ln|t-2|}\right)^{1/2} = \sqrt{|t+2||t-2|}.$$

Let's suppose for a moment that $t \geq 2$ so that this last expression is equal to

$$\sqrt{(t+2)(t-2)} = \sqrt{t^2-4},$$

in which case

$$I(t)y(t) = \int 2t\sqrt{t^2-4}\,dt$$

Substituting $u = t^2 - 4$ we get

$$I(t)y(t) = \frac{2}{3}(t^2-4)^{3/2} + C.$$

Solving for $y(t)$ we get

$$y(t) = \frac{2}{3}(t^2-4) + \frac{C}{\sqrt{t^2-4}}$$

Note that when C is nonzero, this is not defined at $t = 2$ and $t = -2$. But when $C = 0$ it gives us a valid solution to the differential equation that is defined everywhere.

5. For what values of a does the following differential equation have a solution?

$$y'' + ay' + \frac{5a^2}{4}y = 0$$
$$y(0) = 1 \quad y(\pi) = 2$$

Solution: First we will use the characteristic polynomial method to solve the differential equation. The characteristic equation is

$$\lambda^2 + a\lambda + \frac{5a^2}{4} = 0.$$

Using the quadratic formula, we can find that its roots are

$$\lambda = \frac{-a \pm \sqrt{a^2 - 5a^2}}{2} = \frac{-a \pm 2ai}{2} = -\frac{a}{2} \pm ai.$$

Which are complex unless $a = 0$. First let's assume that $a \neq 0$. So the general solution is

$$y(t) = C_1 e^{-at/2}\cos at + C_2 e^{-at/2}\sin at.$$

Using the initial conditions to solve for C_1 and C_2, we find that

$$\begin{aligned} 1 &= y(0) = C_1 \\ 2 &= y(\pi) = C_1 e^{-a\pi/2}\cos a\pi + C_2 e^{-a\pi/2}\sin a\pi. \end{aligned}$$

As long as $\sin a\pi \neq 0$, this has a solution. And $\sin a\pi = 0$ nif and only if a is an integer in which case $\cos a\pi$ is either 1 or -1 and we get

$$2 = C_1 e^{-a\pi/2}\cos a\pi.$$

Since C_1 must be equal to 1 and since $e^{-a\pi/2}$ is always positive, this means we must have $\cos a\pi = 1$ and $e^{-a\pi/2} = 2$. But there is no integer a that satisfies both of these.

Now suppose that $a = 0$. Then the characteristic equation is $\lambda^2 = 0$ and 0 is a double root. So the general solution in that case is

$$y(t) = C_1 e^{0t} + C_2 t e^{0t} = C_1 + C_2 t.$$

Using the initial conditions to solve for C_1 and C_2 we find that $y(t) = 1 + \frac{1}{\pi}t$.

So the original differential equation has a solution if and only if a is either 0 or not an integer.

6. You poll 140 randomly selected people and find the following results: 30 people like ketchup and support NAFTA, 50 people dislike ketchup and support NAFTA, 50 people like ketchup and oppose NAFTA, and 10 people dislike ketchup and oppose NAFTA. You believe that liking ketchup is not independent from supporting NAFTA. Test this hypothesis.

Solution: We will perform a χ^2 test for independence. Our null hypothesis is that supporting NAFTA is independent from liking ketchup. The alternative hypothesis is that they are not independent. The table of observed frequencies is

	supports NAFTA	doesn't support NAFTA
likes ketchup	30	50
doesn't like ketchup	50	10

The table of expected frequencies (i.e. expected given that the two random variables are independent) is

	supports NAFTA	doesn't support NAFTA
likes ketchup	$\frac{80\cdot 80}{140}$	$\frac{80\cdot 60}{140}$
doesn't like ketchup	$\frac{80\cdot 60}{140}$	$\frac{60\cdot 60}{140}$

The χ^2 statistic is

$$\left(\frac{\left(\frac{80\cdot 80}{140} - 30\right)^2}{\frac{80\cdot 80}{140}}\right) + \left(\frac{\left(\frac{80\cdot 60}{140} - 50\right)^2}{\frac{80\cdot 60}{140}}\right) + \left(\frac{\left(\frac{80\cdot 60}{140} - 50\right)^2}{\frac{80\cdot 60}{140}}\right) + \left(\frac{\left(\frac{60\cdot 60}{140} - 10\right)^2}{\frac{60\cdot 60}{140}}\right) \approx 29.4.$$

The number of degrees of freedom is $(2-1)(2-1) = 1$.

If you look at a table you will find that the χ^2 value is large enough to reject the null hypothesis at the 0.05 (or even 0.01) significance level.

7. There is an urn with an unknown number of balls, some of which are white and some of which are black. You draw 20 balls from the urn with replacement and 7 of them are black. Find a 95% confidence interval for the fraction of balls in the urn that are black.

Solution: We are trying to estimate the mean of a Bernoulli random variable using 20 samples. Our estimate for the mean is

$$\widehat{p} = \frac{7}{20}.$$

Since it is Bernoulli, our estimate for the variance is

$$\widehat{\mathrm{Var}} = \frac{7}{20}\left(1 - \frac{7}{20}\right) = \frac{91}{400}.$$

Therefore the 95% confidence interval is

$$\left(\frac{7}{20} - 2\sqrt{\frac{91/400}{20}}, \frac{7}{20} + 2\sqrt{\frac{91/400}{20}}\right)$$