Discrete Probability Worksheet 3 Solutions

1. Suppose you flip two fair coins. Let A be the event that the first coin is heads, B the event that the second coin is heads and C the event that both coins show the same face. Are A and B independent? A and C? B and C? How about A, B, and C?

Solution: At first, it might seem that only A and B are independent. But often our intuition about probability can be misleading. So let's try directly checking which are independent using the definition of independence.

Let's define our sample space to be all pairs of tails and heads. So there are 4 outcomes and since the coins are fair, each outcome is equally likely. Now observe that each of the events A, B, and C consists of two outcomes (the two outcomes for C are HH and TT). So each of these events has probability 1/2. In other words,

$$P(A) = P(B) = P(C) = \frac{1}{2}.$$

The event $A \cap B$ just means that both coins show heads. The event $A \cap C$ means that the first coin is heads and both coins are the same—in other words, $A \cap C$ just means that both coins are heads. And so does the event $B \cap C$. So $A \cap B$, $A \cap C$ and $B \cap C$ are actually all the same event, and all have the same probability, 1/4. In other words,

$$P(A \cap B) = P(A \cap B) = P(B \cap C) = \frac{1}{4}.$$

The event $A \cap B \cap C$ means that the first coin is heads, the second coin is heads, and both coins are the same. But this, too, is just equivalent to saying that both coins are heads. So we also have

$$P(A \cap B \cap C) = \frac{1}{4}.$$

Using the definition of independence, all of this means that the following are independent:

A and B
A and C
B and C

but A, B, and C are not independent.

2. Show that if A, B, and C are independent events then

$$P(A \mid B \cap C) = P(A)$$

Solution: Using the definitions of conditional probability and independence,

$$P(A \mid B \cap C) = \frac{P(A \cap (B \cap C))}{P(B \cap C)} = \frac{P(A \cap B \cap C)}{P(B \cap C)} = \frac{P(A)P(B)P(C)}{P(B)P(C)} = P(A).$$

3. Suppose you draw five cards from a standard deck of 52.

   (a) What is the probability that you get exactly three red cards?

   Solution: Let Ω be all unordered sets of five distinct cards. Let A be the event that there are exactly three red cards. Since each outcome in Ω is equally likely, $P(A) = \frac{|A|}{|\Omega|}$. To pick a set of five cards with exactly three red cards, we need to choose 3 cards out of the 26 red cards and 2 black cards out of the 26 black cards. So

   $$|A| = \binom{26}{3}\binom{26}{2}.$$

   And since Ω is all ways to choose 5 cards from 52 where order doesn't matter,

   $$|\Omega| = \binom{52}{5}.$$

   Therefore

   $$P(A) = \frac{\binom{26}{3}\binom{26}{2}}{\binom{52}{5}}.$$

   (b) Now suppose that if you don't get exactly three red cards, you replace all five cards, reshuffle the deck, and try again. You keep doing this until you do get exactly three red cards. What is the probability that you have to repeat this exactly seven times?

   Solution: Let p be the answer to part (a)—i.e. the probability of success. In order for it to take exactly 7 attempts, we need to fail exactly six times and then succeed once. Since we reshuffle the cards, each attempt is independent. Thus we can multiply the probabilities to get $(1-p)^6 p$.

   (c) The two parts above are examples of what two types of distributions?

   Solution: The first one is an example of a hypergeometric distribution and the second is an example of a geometric distribution (where the parameter is the answer to part (a)).

4. **Challenge Question:** Suppose your friend offers you the following deal: she will start flipping a coin until she gets heads. If n is the number of times she has to flip the coin before getting heads, she will give you 2^n dollars. What is the expected amount of money you will win in this game? Is there anything surprising or troubling about your answer? How much money would you be willing to pay your friend to play this game with her?

   Comment: This is known as the St. Petersburg paradox.