MATH 10B, SPRING 2017, QUIZ 9

(1) Solve the following recursion relation with the given initial conditions.

$$a_n = -4a_{n-1} + 5a_{n-2}$$
$$a_0 = 3,\ a_1 = 9.$$

The characteristic equation for this recursion relation is

$$\lambda^2 = -4\lambda + 5.$$

Which can be rewritten as $\lambda^2 + 4\lambda - 5 = 0$. This polynomial factors into $(\lambda + 5)(\lambda - 1)$. Thus the roots are -5 and 1. So the general solution of the recursion relation is

$$a_n = C_1(-5)^n + C_2(1)^n$$

where C_1 and C_2 are constants. We now need to use the initial conditions to solve for C_1 and C_2. We have

$$3 = a_0 = C_1(-5)^0 + C_2(1)^0 = C_1 + C_2$$
$$9 = a_1 = C_1(-5)^1 + C_2(1)^1 = -5C_1 + C_2.$$

Solving this system of equations gives us $C_1 = -1$ and $C_2 = 4$. Therefore the final solution is

$$a_n = -(-5)^n + 4(1)^n.$$

(2) Compute the following indefinite integral.

$$\int \frac{3x - 9}{x^2 + 4x - 5}\, dx.$$

We will use the method of partial fraction decomposition to simplify the integrand. The denominator factors into $(x + 5)(x - 1)$. So we want to find constants A and B such that

$$\frac{3x - 9}{x^2 + 4x - 5} = \frac{A}{x + 5} + \frac{B}{x - 1} = \frac{A(x - 1) + B(x + 5)}{(x + 5)(x - 1)}.$$

In other words, A and B must satisfy

$$Ax - A + Bx + 5B = 3x - 9$$

for all x. Therefore A and B must satisfy

$$A + B = 3$$
$$-A + 5B = -9.$$

Date: April 11, 2017.

This is (almost) the same system of equations we solved in question 1. So $A = 4$ and $B = -1$.
Now we need to actually compute the integral. By what we have just done,

$$\begin{aligned}\int \frac{3x-9}{x^2+4x-5}\,dx &= \int \frac{4}{x+5} - \frac{1}{x-1}\,dx \\ &= 4\ln|x+5| - \ln|x-1| + C.\end{aligned}$$

(3) Find a solution to the following differential equation.

$$\frac{dy}{dt} = \frac{3t-9}{y^2(t^2+4t-5)}.$$

This equation is separable. We have

$$\int y^2\,dy = \int \frac{3t-9}{t^2+4t-5}\,dt.$$

Using the answer to question 2, we find that

$$\frac{y^3}{3} = 4\ln|t+5| - \ln|t-1| + C$$

and solving for y we get a final answer of

$$y = (12\ln|t+5| - 3\ln|t-1| + C)^{1/3}$$

(where we have absorbed a factor of 3 into the constant C).