MATH 10B, SPRING 2017, QUIZ 12

(1) Find the eigenvalues of the following matrix and for each eigenvalue, find a corresponding eigenvector.

$$\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$$

Let A be the matrix in the question. The eigenvalues are the roots of

$$\det(A - \lambda I) = (1 - \lambda)(-4 - \lambda) - 6 = \lambda^2 + 3\lambda - 10 = (\lambda + 5)(\lambda - 2).$$

So the eigenvalues are -5 and 2.
Eigenvector for 2: Finding an eigenvector with eigenvalue 2 means finding x and y such that

$$\begin{bmatrix} 1-2 & 2 \\ 3 & -4-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This means that $-x + 2y = 0$. Setting $y = 1$, this implies that $x = 2$. So an eigenvector for A with eigenvalue 2 is

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Note that this is not the only valid answer: any scalar multiple of this is also an eigenvector.
Eigenvector for -5: By the same process as above, one eigenvector for A with eigenvalue -5 is

$$\begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

(2) Solve the following initial value problem.

$$y' = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix} y \qquad y(0) = \begin{bmatrix} 0 \\ -16 \end{bmatrix}$$

By problem (1), the general solution is

$$y(t) = C_1 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 e^{-5t} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

Using the given initial values,

$$\begin{bmatrix} 0 \\ -16 \end{bmatrix} = y(0) = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 2C_1 + C_2 \\ C_1 - 3C_2 \end{bmatrix}.$$

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Solving this system of linear equations, we have $C_1 = \frac{-16}{7}$ and $C_2 = \frac{32}{7}$. So the final solution is

$$y(t) = \frac{-16}{7} e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{32}{7} e^{-5t} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$