Math 10B Probability Worksheet 3

1. Suppose that there are two slot machines, one of which pays out 10% of the time and the other pays out 20% of the time. Unfortunately, you have no idea which is which. Suppose you randomly choose a machine and put in a quarter. If you don't get a jackpot, what is the chance that you chose the machine that pays out 20% of the time? If you had instead gotten a jackpot, what would be the chance that you chose the one that pays out 20% of the time?

First let's formalize the information we have been given. Let A be the event that you chose the 20% machine and let B be the event that you win. Then we are trying to find $P(A \mid B^c)$ and $P(A \mid B)$. Rephrasing the information given in the question, we know that

$$\begin{aligned} P(A) &= 0.5 \\ P(B \mid A) &= 0.2 \\ P(B \mid A^c) &= 0.1. \end{aligned}$$

So we have enough information to use Bayes' theorem:

$$\begin{aligned} P(A \mid B^c) &= \frac{P(B^c \mid A)P(A)}{P(B^c)} \\ &= \frac{P(B^c \mid A)P(A)}{P(B^c \mid A)P(A) + P(B^c \mid A^c)P(A^c)} \\ &= \frac{0.8 \cdot 0.5}{0.8 \cdot 0.5 + 0.9 \cdot 0.5} \\ &= \frac{0.4}{0.85} \\ &\approx 0.471. \end{aligned}$$

So if you lose, the probability that you picked the "good" machine is 47%—in other words, losing should not affect your beliefs very much because both machines are likely to lose. By the way, humans have a tendency to change their beliefs too much in situations like this, where the only evidence available is very weak.
Now let's calculate the probability that you chose the "good" machine given that you won. We will use Bayes' theorem again:

$$\begin{aligned} P(A \mid B) &= \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A^c)P(A^c)} \\ &= \frac{0.2 \cdot 0.5}{0.2 \cdot 0.5 + 0.1 \cdot 0.5} \\ &= \frac{0.1}{0.15} \\ &\approx 0.667. \end{aligned}$$

2. Suppose you flip two fair coins. Let A be the event that the first coin is heads, B the event that the second coin is heads and C the event that both coins show the same face. Are A and B independent? A and C? B and C? How about A, B, and C?

At first, it might seem that only A and B are independent. But often our intuition about probability can be misleading. So let's try directly checking which are independent using the definition of independence.

Let's define our sample space to be all pairs of tails and heads. So there are 4 outcomes, and since the coins are fair, each outcome is equally likely. Now observe that each of the events A, B, and C consists of two outcomes (the two outcomes for C are HH and TT). So each of these events has probability 1/2. In other words,

$$P(A) = P(B) = P(C) = \frac{1}{2}.$$

The event $A \cap B$ just means that both coins show heads. The event $A \cap C$ means that the first coin is heads and both coins are the same—in other words, $A \cap C$ just means that both coins are heads. And so does the event $B \cap C$. So $A \cap B$, $A \cap C$ and $B \cap C$ are actually all the same event, and all have the same probability, 1/4. In other words,

$$P(A \cap B) = P(A \cap B) = P(B \cap C) = \frac{1}{4}.$$

The event $A \cap B \cap C$ means that the first coin is heads, the second coin is heads, and both coins are the same. But this, too, is just equivalent to saying that both coins are heads. So we also have

$$P(A \cap B \cap C) = \frac{1}{4}.$$

Using the definition of independence, all of this means that the following are independent:

A and B

A and C

B and C

but A, B, and C are not independent.

3. Show that if A, B, and C are independent events then

$$P(A \mid B \cap C) = P(A)$$

Using the definitions of conditional probability and independence,

$$P(A \mid B \cap C) = \frac{P(A \cap (B \cap C))}{P(B \cap C)} = \frac{P(A \cap B \cap C)}{P(B \cap C)} = \frac{P(A)P(B)P(C)}{P(B)P(C)} = P(A).$$

4. Suppose we roll a fair 6-sided die three times. Let the random variable X be the sum of the rolls. What is $P(X = 4)$? What about $P(X = 8)$? For a challenge, consider the following problem. Imagine that your friend offers to let you roll the three dice and promises to give you as many dollars as the sum of the three dice. How much money should you be willing to pay for this opportunity?

Let's define the sample space to be all ordered lists of three numbers between 1 and 6. Since there are 6 options for each number and 3 numbers, the size of the sample space is 6^3. Since the die is fair, each outcome in the sample space is equally likely.
We can calculate both probabilities in the same way. We need to count the number of ways to find three positive numbers that add up to 4 for the first question and 8 for the second. We can use the stars and bars trick to count these. And fortunately, if three positive numbers add up to 4 or 8 then none of them can be greater than 6, so we don't need to worry about excluding possibilities where some number is larger than 6. Using the stars and bars trick, we get

$$\binom{4-3+3-1}{4-3} = 3$$

ways to find three positive numbers that add up to 4 (recall that since no number can be 0, we first put one "ball" into each "box") and

$$\binom{8-3+3-1}{8-3} = \binom{7}{5}$$

ways to find three positive numbers that add up to 8. Thus the probabilities are

$$P(X = 4) = \frac{3}{6^3}$$
$$P(X = 8) = \frac{\binom{7}{5}}{6^3}$$