Math 10B Combinatorics Worksheet

1. Suppose you own a shop that sells t-shirts. There are

   - 6 sizes
   - 5 colors
   - 20 designs
   - 3 types of fabric
   - 6 languages
   - 2 styles (long or short sleeved).

   Is it reasonable for you to stock 10 of each possible type of t-shirt in your store?

2. Suppose that it takes one second to try a password on a computer (I realize that this assumption is not realistic). You want to make sure that nobody can brute-force the password—i.e. determine what the password is simply by sequentially trying all possibilities. In each of the following cases, how long must the password be to make this infeasible? Assume that anything that takes more than a year is not feasible.

   (a) The password may contain only digits.

   (b) The password may contain only uppercase letters.

   (c) The password may contain only digits and uppercase letters.

   (d) The password may contain either only digits or only uppercase letters, but may not contain both digits and uppercase letters.

3. How many functions from $\{1, 2, 3, 4, 5\}$ to $\{A, B, C\}$ are there?

4. How many possible answer keys are there for a 30 question true/false test?

5. How many numbers below 100 are divisible by 2,3, or 5?

6. How many functions from $\{1, 2, 3, 4\}$ to $\{1, 2, 3\}$ are onto?

7. Find a simple formula for $\frac{d^n}{dx^n} xe^x$ and use induction to prove it is correct. [*Hint:* Try calculating the first few derivatives and see if you can notice a pattern. Once you think you have found a pattern, try to use induction to prove that your guess is correct.]

8. Find a simple formula for $\frac{d^n}{dx^n} x^2e^x$ and use induction to prove it is correct.

9. (a) Out of a class of 20 students, how many ways are there to form a study group? Assume that a study group must have at least 2 students.

   (b) How many ways are there to form a study group that contains at least one of Bob, Sue, and Alicia?

10. Consider the following game with 2 players. The game is played with a pile of stones. On your turn you can remove either one or two stones from the pile. You lose if the pile is empty at the start of your turn. Prove that when the initial number of stones is divisible by 3, the second player can always win.

11. Show that there are more than 100 people alive right now who were born in the same minute.

12. Show that at least 19 subsets of $\{1, 2, \ldots, 10\}$ have the same sum. For a challenge, try to improve this result as much as possible.

13. How many anagrams does "obfuscated" have?

14. (a) How many anagrams does "banana" have?
    (b) How many anagrams does "banana" have in which the three 'a's are next to each other?
    (c) How many anagrams does "banana" have in which the three 'a's are next to each other and the two 'n's are *not* next to each other?

15. Suppose that 4 people are standing in line. How many ways are there to rearrange the line so that nobody is standing in their original place?

16. (a) Starting from a pool of n people, how many ways are there to select a committee of k people, one of whom is the president of the committee?
    (b) Use your answer to part (a) to prove that

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

17. Using any method you like (i.e. any chain of sound reasoning), prove that for any m, n, and $k \leq m + n$

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}.$$

18. How many ways are there to pay your employees if you have \$1000 and 5 employees? Assume that you are allowed to pay employees nothing and that you don't have to spend all \$1000. Also assume that you must pay employees in dollar amounts (e.g. you cannot pay someone \$4.53).

19. How many ways are there to arrange 20 books on a bookcase with 3 shelves? Assume, as in real life, that books are distinguishable and that the order of the books on each shelf matters.

20. Suppose you have 8 boxes labelled 1 through 8 and 16 indistinguishable red balls. How many ways are there to put the balls into the boxes if:
    (a) No odd box can be empty.
    (b) Odd boxes must have an odd number of balls and even boxes, an even number of balls.
    (c) You also have 16 indistinguishable green balls and want to distribute both the red and green balls into the boxes.

21. How many rhyme schemes are there for a poem with n lines?

22. Show that when using the version of the stable marriage algorithm in which the men propose to the women, it is possible that every woman gets her last choice.

23. Show that when using the version of the stable marriage algorithm in which the men propose to the women, it is possible that some women have an incentive to lie—i.e. if they lie about their preferences they can end up with a higher ranked man (according to their true preferences) than the one they would end up with if they did not lie.