

BREAKING THE INFINITE BARRIER IN THE $\frac{1}{3}-\frac{2}{3}$ CONJECTURE

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ABSTRACT. The balance constant is a poset parameter measuring how well random linear extensions can be split according to their relative order on two elements. We give an ε -improvement for the balance constant over the bound by Brightwell, Felsner and Trotter [BFT95], for some small $\varepsilon > 0$. This is the first general result towards the $\frac{1}{3}-\frac{2}{3}$ conjecture in over 30 years.

1. INTRODUCTION

When a major open problem remains unresolved for decades, one can certainly ask why that is. In most cases, there are too many plausible answers. But for the $\frac{1}{3}-\frac{2}{3}$ *conjecture*, there is a standard answer that has also been known for decades: while the conjecture is stated for finite posets, it is *false* for *infinite* posets, and there are no known tools which make a distinction. This *infinite barrier* has plagued the area for over 30 years, repelling numerous attempts at solving the conjecture. In this paper we break the infinite barrier by a microscopic margin.

1.1. Background. Let $P := (X, \prec)$ be a finite poset, where X is a finite set of size $n := |X|$, and $\prec$ is a partial order on X . We write $[n] := \{1, \dots, n\}$. A *linear extension* of P is a bijection $f : X \rightarrow [n]$ that is order preserving: $x \prec y$ in P implies $f(x) < f(y)$ for all $x, y \in X$. Let $\mathcal{E}(P)$ denote the set of linear extensions of P .

Conjecture 1.1 ($\frac{1}{3}-\frac{2}{3}$ *conjecture*). *Let $P := (X, \prec)$ be a finite poset that is not a chain. Then there exist elements $x, y \in X$, such that*

$$(1.1) \quad \frac{1}{3} \leq \mathbf{P}[f(x) < f(y)] \leq \frac{2}{3},$$

where the probability is over uniform random $f \in \mathcal{E}(P)$.

The conjecture is a central open problem in order theory, independently proposed by Kislitsyn [Kis68] and Fredman [Fre75], see also [Lin84], in connection to sorting with partial information. The simplicity and elegance of the $\frac{1}{3}-\frac{2}{3}$ conjecture have attracted much attention, and numerous special cases of the conjecture have been established in the literature.

These known special cases include: width two posets [Lin84], posets with symmetries [GHP87], semiorders [Bri89], posets of height $O(\log \log n)$ [Fri93], 6-thin posets [Pec08], N -free posets [Zag12], Young diagrams [OS18], posets whose cover graph is a forest [Zag19], and posets with at most 14 elements [Gup26]. We refer to [CP25, §11] for an introduction to the problem, additional special cases and further references.

In a major breakthrough, Kahn and Saks [KS84] used geometric inequalities to show that the inequality (1.1) holds with slightly weaker constants:

$$(1.2) \quad \frac{3}{11} \leq \mathbf{P}[f(x) < f(y)] \leq \frac{8}{11}.$$

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Note that $\frac{1}{3}-\frac{2}{3}$ is tight for the 3-element poset consisting of a 2-element chain and an isolated element, so the bound $\frac{3}{11} \approx 0.2727 < \frac{1}{3}$ leaves relatively little room for improvement. Despite various attempts and simplifications, see e.g. [KL91] and [Mat02, §12.3], the *Kahn–Saks bound* stood unimproved for a decade.

In another major breakthrough, Brightwell, Felsner and Trotter [BFT95] used both the combinatorial *Ahlsvede–Daykin inequality* [AD78] and the geometric *Alexandrov–Fenchel inequalities* [BZ88] to improve on the Kahn–Saks bound:

$$(1.3) \quad \frac{5 - \sqrt{5}}{10} \leq \mathbf{P}[f(x) < f(y)] \leq \frac{5 + \sqrt{5}}{10}.$$

Note that $\frac{5 - \sqrt{5}}{10} \approx 0.2764$, so this might seem an incremental improvement which could perhaps be improved as new poset inequalities are added to the mix. Yet this *BFT bound* remained the best known general bound prior to the present work.

The reason the BFT bound remained unimproved for over 30 years was the *infinite barrier* that was also introduced in the original paper [BFT95, §7], where (1.3) was extended to infinite posets. The authors then considered an example of an *infinite Fibonacci poset* (see below), first introduced in [Bri88], and for which (1.3) is sharp. Despite many recent developments in both geometric and combinatorial inequalities, the infinite barrier gave little hope for improvement, see e.g. [Bri99].

1.2. Main result. In this paper, we overcome this infinite poset barrier and obtain the first improvement on the bound in (1.3), as stated in the following theorem.

Theorem 1.2 (*Breaking the infinite barrier*). *There exists a fixed constant $\varepsilon > 0$, such that for every finite poset $P := (X, \prec)$ that is not a chain, there exist elements $x, y \in X$, s.t.*

$$(1.4) \quad \frac{5 - \sqrt{5}}{10} + \varepsilon \leq \mathbf{P}[f(x) < f(y)] \leq \frac{5 + \sqrt{5}}{10} - \varepsilon,$$

where the probability is over uniform random $f \in \mathcal{E}(P)$.

The constant ε can be made explicit, but is microscopically small. Our proof allows us to take

$$(1.5) \quad \varepsilon := 3^{-3^{16 \times 10^{18}}},$$

and we note that already $3^{3^{16}}$ has over 20 million decimal digits. As our primary goal is to establish a separation between finite and infinite posets in this context, we make no attempt to optimize this constant. For simplicity, we omit the explicit derivation of ε .

The proof of Theorem 1.2 relies on three main ingredients, each of independent interest. Indeed, we divide the posets into three classes:

- posets of small range,
- posets of large width, and
- posets of large range and small width.

We treat these cases completely differently: we break the infinite barrier for the first case, we prove the $\frac{1}{3}-\frac{2}{3}$ Conjecture for the second case, and use the asymptotic bound by Kahn and the first author [AK25b], to obtain the result in the third case. We now proceed to discuss each case separately, give formal statements and present the background (cf. §7.1).

1.3. Posets of small range. To state them, we first define the *balance constant* of elements $x, y \in X$ of a finite poset $P = (X, \prec)$ as

$$\delta(P; x, y) := \min \{ \mathbf{P}[f(x) < f(y)], \mathbf{P}[f(y) < f(x)] \},$$

and of the poset itself:

$$\delta(P) := \max_{x, y \in X, x \neq y} \delta(P; x, y).$$

Here and throughout the paper, the probability is over uniform random $f \in \mathcal{E}(P)$.

Denote $C_{\text{BFT}} := \frac{5-\sqrt{5}}{10}$. In this notation, the $\frac{1}{3}-\frac{2}{3}$ Conjecture asserts that $\delta(P) \geq 1/3$ for every finite poset P that is not a chain. Similarly, Theorem 1.2 says that there exists an absolute constant $\varepsilon > 0$ such that $\delta(P) \geq C_{\text{BFT}} + \varepsilon$ for every such P .

For $x \in X$, let $\pi(x)$ denote the number of elements of P incomparable to x . Define the *range*¹ of P as

$$\pi(P) := \max_{x \in X} \pi(x).$$

The first ingredient in the proof of Theorem 1.2 is an explicit bound on the balance constant for posets of small range.

Theorem 1.3 (*The case of small range*). *For an integer $D \geq 2$, let $P = (X, \prec)$ be a finite poset that is not a chain, such that $\pi(P) \leq D$. Then:*

$$\delta(P) > \frac{5 - \sqrt{5}}{10} + \eta_D \quad \text{where} \quad \eta_D := \frac{(D+1)^{-4(D+1)}}{4096}$$

We will prove Theorem 1.3 in Section 4. Note that for $D \leq 5$, Brightwell and Wright [BW92] proved a stronger bound $\delta(P) \geq \frac{1}{3}$, establishing the $\frac{1}{3}-\frac{2}{3}$ Conjecture for such posets. In [Pec08], Peczarski extended this result to $D \leq 6$. Unfortunately, their methods break down for larger D .

We emphasize that the finiteness assumption in Theorem 1.3 is essential. Define the *infinite Fibonacci poset* $\mathcal{F} = (X, \prec)$ on $X = \{x_i : i \in \mathbb{Z}\}$, where $x_i \prec x_j$ if and only if $j - i \geq 2$. Let $\mathcal{F}_m$, where $m \geq 1$, be the (finite) *Fibonacci poset* obtained by restriction of $\mathcal{F}$ to a finite subset $X_m := \{x_{-m}, \dots, x_m\}$, cf. [MPP18]. Define

$$\delta(\mathcal{F}; x_i, x_j) := \lim_{m \rightarrow \infty} \delta(\mathcal{F}_m; x_i, x_j)$$

and let

$$\delta(\mathcal{F}) := \sup_{i, j \in \mathbb{Z}, i < j} \delta(\mathcal{F}; x_i, x_j).$$

This poset satisfies $\pi(\mathcal{F}) = 2$, since each element is incomparable with exactly two others, and yet it can be shown that $\delta(\mathcal{F}) = C_{\text{BFT}}$, see [BFT95, §7].²

Our proof builds on the approach of Brightwell–Felsner–Trotter [BFT95], which in turn builds on the approach of Kahn–Saks [KS84]. The original strategy is to find a special triple of elements called a *BFT triple*, and the BFT analysis shows that it contains a pair with balance at least C_{BFT} . Our strategy is to improve this bound by choosing the BFT triple more carefully.

The finite restrictions of the Fibonacci poset illustrate why this choice matters. Consider the restriction to X_m and note that as $m \rightarrow \infty$, the adjacent pairs at any fixed i, j , the balance constant $\delta(\mathcal{F}_m; x_i, x_j) \rightarrow C_{\text{BFT}} \approx 0.2764$, while for the pairs i, j at the endpoints we have

¹For $D \geq 2$, a poset P is called *D-thin* if $\pi(P) \leq D$. We prefer *range* over *thinness* which is more standard in the literature.

²As one would expect, we have $\max \delta(\mathcal{F}_m; x_i, x_j) \geq \frac{1}{3}$. This is a typical case when the order of operations matters: $\lim_{m \rightarrow \infty} \max_{i < j} \delta(\mathcal{F}_m; x_i, x_j) \neq \sup_{i < j} \lim_{m \rightarrow \infty} \delta(\mathcal{F}_m; x_i, x_j)$.

$\delta(\mathcal{F}; x_i, x_j) \rightarrow \frac{3-\sqrt{5}}{2} \approx 0.3820$. Thus these finite posets contain substantially better-balanced pairs near their boundaries. In the two-sided infinite limit, however, the endpoints recede to infinity, and these favorable pairs are lost. This explains the obstruction posed by the infinite example: choosing triples near the center *cannot* yield a uniform improvement over (1.3). In other words, we need a way to distinguish the better-balanced triples from those that reproduce the behavior of the infinite poset.

Our main tool for making this distinction is the displacement of an element from its position in the height order. Formally, let

$$h(x) := \mathbf{E}[f(x)]$$

denote the *height* of a uniform random linear extension $f \in \mathcal{E}(P)$. Now let $X = \{v_1, \dots, v_n\}$, where elements v_i are written in nondecreasing order of h . Finally, define

$$d_i := h(v_i) - i \quad \text{and} \quad M := \max_{1 \leq i \leq n} |d_i|,$$

the *displacement* and *maximal absolute displacement*, respectively. In $\mathcal{F}_m$, the displacement $\rightarrow 0$ on every element x_i , as $m \rightarrow \infty$, while the maximal absolute displacement remains bounded away from zero. In other words, the displacement detects the boundary behavior that disappears in the limit.

Our strategy is therefore to choose a BFT triple with an endpoint attaining the maximum absolute displacement M . We then derive a lower bound on the best pair balance of the selected triple in terms of M . In turn, M admits a positive lower bound depending only on the range $\pi(P)$, which explains the role of the range assumption in the theorem. Together, these bounds yield Theorem 1.3. Establishing them accounts for most of the technical work in the proof.

1.4. Poset of large width. Note that the improvement over C_{BFT} in Theorem 1.3 tends to zero as the range $\pi(P)$ grows: $\eta_D \rightarrow 0$ as $D \rightarrow \infty$. Thus, we need a different strategy in this case. Recall that the *width* $w(P)$ of a poset P is the maximum size of a set of pairwise incomparable elements of P .

Theorem 1.4 (The case of large width). *There exists a universal constant $K > 0$, such that for every finite poset $P = (X, <)$ of width $w(P) > K$, we have:*

$$\delta(P) > e^{-1} - 10^{-100}.$$

We prove Theorem 1.4 in Section 6. Note that $e^{-1} \approx 0.3679 > \frac{1}{3}$, so the result implies the $\frac{1}{3}$ – $\frac{2}{3}$ Conjecture in this case. The constant e^{-1} in Theorem 1.4 is an artifact of our use of Grünbaum-type inequalities [Gru60] as the log-concavity input in the proof. The celebrated *Kahn–Saks conjecture* [KS84, p. 114] claims a stronger conclusion, that $\delta(P) \rightarrow \frac{1}{2}$ as $w(P) \rightarrow \infty$. This conjecture was recently resolved by the first author in a companion paper [Air26] (in fact, that paper gave an inspiration for this work), by incorporating the more refined *Komlós selection theorem* [AK25a, Thm. 8.1], together with substantially more involved technical estimates. We present the e^{-1} bound here because it admits a much simpler proof and already suffices for our purposes.

The proof of Theorem 1.4 centers on *canopy antichains*, namely antichains of the form

$$\max(\{y \in X : h(y) \leq h(x)\}) \quad \text{or} \quad \min(\{y \in X : h(y) > h(x)\}),$$

where $x \in X$, and $\min(S)$ and $\max(S)$ denote the sets of minimal and maximal elements of the poset induced by S , respectively. Assuming that $\delta(P)$ lies below e^{-1} by a fixed positive amount and that $w(P)$ is sufficiently large, we use these sets to construct a large antichain whose elements have closely clustered average heights in P . We then show that these two properties are incompatible with the assumed bound on $\delta(P)$, proving the theorem. Constructing

such an antichain accounts for the bulk of the technical work and draws on the recent work of Aires and Kahn [AK25a] on *window estimates* of posets, Shepp's *XYZ correlation inequality* [She82] (see also [AS16, CP25]), and Haqi's recent remarkable proof of the *Haqi–Kahn ideal inequality* [Haq26].

1.5. Poset of small width and large range. It remains to show that Theorem 1.2 holds for posets with *small width* and *large range*. This is achieved by the following theorem of Aires and Kahn [AK25b] which gives a much stronger asymptotic bound than we need:

Theorem 1.5 (*The case of small width and large range* [AK25b, Thm 1.6]). *For all $K, \epsilon > 0$, there exist $L = L(K, \epsilon) > 0$, such that for every finite poset $P = (X, \prec)$ of width $w(P) < K$ and range $\pi(P) > L$, we have:*

$$\delta(P) > \frac{1}{2} - \epsilon.$$

In other words, for posets of bounded width $w(P) < K$, we have:

$$\pi(P) \rightarrow \infty \quad \implies \quad \delta(P) \rightarrow \frac{1}{2}.$$

Clearly, by taking $\epsilon = \frac{1}{6}$ we obtain the $\frac{1}{3}-\frac{2}{3}$ Conjecture for all posets of bounded width and large enough range. Theorem 1.2 now follows from combining Theorem 1.3, Theorem 1.4, and Theorem 1.5.

We note that all constants in Theorems 1.4 and 1.5 can be made explicit. We omit their values for readability, as determining them is not a primary aim of this paper, see also §7.4. In fact, the proof in [AK25b] uses only elementary probabilistic estimates.

1.6. Paper structure. The paper is organized as follows. In Section 2, we review results in the literature, and in Section 3 we establish new technical lemmas we need towards the proof of Theorem 1.3. The proof of Theorem 1.3 is given in Section 4. Then, in Section 5, we review the results in the literature towards the proof of Theorem 1.4. The proof itself is given in Section 6. We conclude with final remarks in Section 7.

2. PRELIMINARIES FOR THEOREM 1.3

2.1. Preliminaries. Throughout this paper, let $P = (X, \prec)$ be a finite poset, where X is the ground set and $\prec$ is a partial order on X . Write $n := |X|$ and $[n] := \{1, \dots, n\}$. We use f to denote a uniformly random linear extension of P , and $\mathbf{P}$ and $\mathbf{E}$ to denote the corresponding probability measure and expectation, respectively.

Let

$$C_{\text{BFT}} := \frac{5 - \sqrt{5}}{10} \approx 0.276393 \dots$$

Let $D \geq 2$ be a fixed integer throughout Sections 2–4, and let η_D be the constant in Theorem 1.3. We will assume throughout Sections 2–4 that $\delta(P) \leq C_{\text{BFT}} + \eta_D$, as otherwise the theorem follows immediately. Note that 0.2764 is strictly greater than $C_{\text{BFT}} + \eta_D$, the lower bound for $\delta(P)$ in Theorem 1.3. We begin with a lemma that will be used repeatedly throughout the next three sections.

Lemma 2.1 ([KL91, p. 365], [AK25a, Cor. 5.1(b)]). *Let x, y be distinct elements satisfying $h(x) \leq h(y)$. Then*

$$\mathbf{P}[f(y) < f(x)] \leq 1 - e^{-1}.$$

This result was first proved by Kahn and Linial [KL91, p. 365] via Grünbaum-type inequalities [Gru60]. Note that they stated the result for the case $h(x) = h(y)$, although their proof also covers the general case. A proof of the general statement appears in [AK25a, Cor. 5.1(b)]. The following consequence of Lemma 2.1 will be especially useful.

Corollary 2.2. *Suppose that $\delta(P) \leq 0.2764$. Then, for all distinct $x, y \in X$ satisfying $h(x) \leq h(y)$,*

$$\mathbf{P}[f(y) < f(x)] \leq 0.2764.$$

Proof. Suppose to the contrary that the claim is false, then

$$\delta(P; x, y) = \min\{\mathbf{P}[f(x) < f(y)], \mathbf{P}[f(y) < f(x)]\} > \min\{e^{-1}, 0.2764\} = 0.2764,$$

contradicting the assumption that $\delta(P) \leq 0.2764$. $\square$

We now review the results of Brightwell, Felsner, and Trotter [BFT95] that will be used in the proof of Theorem 1.3. For an element x of a poset P , the *height* $h(x) := \mathbf{E}[f(x)]$ is the expected position of x in a uniformly random linear extension f of P . We call a triple (x, y, z) of distinct elements of P a *BFT triple* if these elements do not form a chain and

$$h(x) \leq h(y) \leq h(z) \leq h(x) + 2.$$

The use of height functions in this setting goes back to the breakthrough work of Kahn and Saks [KS84], who used pairs of incomparable elements x, y satisfying $h(x) \leq h(y) \leq h(x) + 1$ to establish the bound

$$\delta(P; x, y) \geq 3/11 \approx 0.2727.$$

Building on this approach, Brightwell, Felsner, and Trotter [BFT95] showed that a BFT triple (x, y, z) yields

$$\max\{\delta(P; x, y), \delta(P; x, z), \delta(P; y, z)\} > C_{\text{BFT}} \approx 0.276393 \dots$$

In this paper, we refine this approach by showing that every finite nonchain poset P with at least three elements and $\pi(P) \leq D$ contains a BFT triple (x, y, z) satisfying the stronger bound

$$\max\{\delta(P; x, y), \delta(P; x, z), \delta(P; y, z)\} > C_{\text{BFT}} + \eta_D,$$

thereby proving Theorem 1.3.

Write $u \parallel v$ if u and v are incomparable in P . Following [BFT95, Section 4], and passing to the dual poset and reversing the order of the triple if necessary, we split the analysis into the following four cases:

- Case A. $x \prec z$, $y \prec z$, and $x \parallel y$.
- Case B. $y \prec z$, $x \parallel y$, and $x \parallel z$.
- Case C. x, y, z are pairwise incomparable.
- Case D. $x \prec z$, $x \parallel y$, and $y \parallel z$.

Cases A and B were treated in [BFT95, Thms. 4.1 and 4.2], where it was shown that

$$\mathbf{P}[f(y) < f(x)] \geq \frac{1}{3}.$$

Together with Lemma 2.1, this gives

$$\frac{1}{3} \leq \mathbf{P}[f(y) < f(x)] \leq 1 - e^{-1} < \frac{2}{3}.$$

Thus, in either case, $\delta(P; x, y) \geq 1/3$. In Case C, Brightwell, Felsner, and Trotter [BFT95, Sec. 6] established the bound $\delta(P) \geq C_{\text{BFT}}$, which is insufficient for the conclusion of Theorem 1.3. Nevertheless, their argument yields the following slightly stronger bound.

Lemma 2.3 (*Case C of BFT* [BFT95, §6]). *Let (x, y, z) be a BFT triple whose elements are pairwise incomparable. Then*

$$\max \{ \delta(P; x, y), \delta(P; y, z) \} > 0.2764.$$

We include the details to verify that their argument yields the bound 0.2764 in Appendix A.

2.2. Case D of Brightwell–Felsner–Trotter analysis. Our treatment of Case D requires new arguments involving the range $\pi(P)$. We develop these in Sections 3 and 4, building on the following quantitative analysis of Case D. Let (x, y, z) be a BFT triple in Case D, i.e. $h(x) \leq h(y) \leq h(z) \leq h(x) + 2$ and

$$x \prec z, \quad x \parallel y, \quad y \parallel z.$$

For $i, j \in \mathbb{Z}$, let

$$p(i, j) := \mathbf{P}[f(y) - f(x) = i, f(z) - f(y) = j].$$

We also write

$$p := \mathbf{P}[f(y) < f(x)], \quad p' := \mathbf{P}[f(z) < f(y)], \quad S := p + p',$$

and set

$$a_1 := p(1, 1), \quad a_2 := p(1, 2) + p(2, 1), \quad a_3 := p(2, 2).$$

The inequalities in [BFT95, (5.2), (5.5), (5.7)–(5.8)] give

$$(2.1) \quad 2a_1 + a_2 \leq S, \quad 2a_2 + 2a_3 \leq S,$$

$$(2.2) \quad 4a_1a_3 \leq a_2^2, \quad 7 \leq 9S + 7a_1 + 3a_2 + 3a_3.$$

In particular, the quadratic inequality in (2.2) follows from the special case of the cross-product inequality proved in [BFT95, Thm. 3.2], and the general case of the inequality [BFT95, Conj. 3.1] remains a challenging open problem. Let

$$(2.3) \quad R := S - 2a_1 - a_2.$$

By (2.1), we have $0 \leq R \leq S$. The next lemma shows that a lower bound for R gives an improvement on the BFT bound.

Lemma 2.4 (*Case D of BFT*). *With the notation above, suppose additionally that $\delta(P) \leq 0.2764$. Then*

$$(2.4) \quad \max \{ \delta(P; x, y), \delta(P; y, z) \} \geq C_{\text{BFT}} + \frac{R}{5 + 3\sqrt{5}}.$$

Moreover,

$$(2.5) \quad R \geq \mathbf{P}[f(x) - f(y) \geq 2] + \mathbf{P}[f(y) - f(z) \geq 2].$$

In [BFT95, §5], only the weaker bound obtained by dropping the term involving R was used, whereas we will need to use the full strength of Lemma 2.4 in the proof of Theorem 1.3. The proof of this lemma closely follows the argument in [BFT95, §5].

Proof. First note that Corollary 2.2 implies $p, p' \leq 0.2764 < 1/2$. Since $x \parallel y$ and $y \parallel z$, we have $S > 0$. Set

$$b_i := \frac{a_i}{S} \quad (i = 1, 2, 3), \quad s := 1 - \frac{R}{S}.$$

Then $s \in [0, 1]$, and (2.1)–(2.3) imply

$$(2.6) \quad 2b_1 + b_2 = s, \quad b_2 + b_3 \leq \frac{1}{2}, \quad 4b_1b_3 \leq b_2^2.$$

Let $F(s)$ (for $s \in [0, 1]$) be given by

$$F(s) := \max \left\{ \frac{7}{2}s, \frac{7}{2}\sqrt{s^2 + \frac{1}{4}} - \frac{1}{4} \right\}.$$

Claim 2.5. *We have*

$$(2.7) \quad 7b_1 + 3b_2 + 3b_3 \leq F(s).$$

Proof. If $b_1 = 0$, then the left side of (2.7) is at most $3/2$, which is at most $F(s)$. When $b_1 = 0$, the constraint $b_2 + b_3 \leq 1/2$ gives

$$7b_1 + 3b_2 + 3b_3 = 3(b_2 + b_3) \leq \frac{3}{2}.$$

On the other hand, by the definition of $F(s)$,

$$F(s) \geq \frac{7}{2}\sqrt{s^2 + \frac{1}{4}} - \frac{1}{4} \geq \frac{7}{2} \cdot \frac{1}{2} - \frac{1}{4} = \frac{3}{2},$$

where the second inequality uses $s^2 \geq 0$. Combining these bounds proves (2.7) when $b_1 = 0$.

Otherwise, write $u := b_1 > 0$. By (2.6), we have $u \leq s/2$ and

$$b_2 + b_3 \leq b_2 + \frac{b_2^2}{4u} = \frac{s^2}{4u} - u.$$

It follows that

$$7b_1 + 3b_2 + 3b_3 \leq 7u + 3 \min \left\{ \frac{1}{2}, \frac{s^2}{4u} - u \right\}.$$

The two expressions inside the minimum are equal when u is equal to

$$u_0 := \frac{\sqrt{1 + 4s^2} - 1}{4}.$$

For $0 < u \leq u_0$, the right side is $7u + 3/2$, which is increasing in u . For $u_0 \leq u \leq s/2$, it is $4u + 3s^2/(4u)$, which is convex in u and hence is maximized at an endpoint. Evaluating at $u = u_0$ and $u = s/2$ proves (2.7). $\square$

Now, note that the function F is convex on $[0, 1]$, with

$$F(0) = \frac{3}{2}, \quad F(1) = \frac{7\sqrt{5} - 1}{4} =: c.$$

Thus F lies below the chord joining these two endpoint values, so

$$F(s) \leq c - \left(c - \frac{3}{2}\right)(1 - s).$$

Combining this with (2.2) and (2.7), we obtain

$$\begin{aligned} 7 &\leq 9S + S(7b_1 + 3b_2 + 3b_3) \leq S(9 + F(s)) \\ &\leq (9 + c)S - \left(c - \frac{3}{2}\right)R. \end{aligned}$$

Since

$$\frac{7}{2(9 + c)} = C_{\text{BFT}}, \quad \frac{c - 3/2}{2(9 + c)} = \frac{1}{5 + 3\sqrt{5}},$$

it follows that

$$\frac{S}{2} \geq C_{\text{BFT}} + \frac{R}{5 + 3\sqrt{5}}.$$

The assumption that $p, p' \leq 1/2$ gives

$$\max\{\delta(P; x, y), \delta(P; y, z)\} = \max\{p, p'\} \geq \frac{S}{2},$$

proving (2.4).

To prove (2.5), note that swapping adjacent incomparable elements gives

$$p(-1, j) = p(1, j-1), \quad p(j, -1) = p(j-1, 1) \quad (j \geq 2).$$

Consequently,

$$2a_1 + a_2 = p(-1, 2) + p(-1, 3) + p(2, -1) + p(3, -1).$$

The four events on the right are disjoint and are contained in the two events defining S (i.e. $\{f(x) - f(y) \geq 1\}$ and $\{f(y) - f(z) \geq 1\}$). These two events are themselves disjoint, since $x \prec z$. Subtracting the four events from these two events immediately implies (2.5). $\square$

3. POSETS OF SMALL RANGE

Let $D \geq 2$ throughout this section. In this section, we start using the assumption $\pi(P) \leq D$ from Theorem 1.3. Recall that the *range* $\pi(P)$ is the maximum, over $x \in X$, of the number of elements incomparable with x .

We begin with the following simple lemma. Let $S \subseteq X$ be a subset. We say that an event E is *S -dependent* if it is specified by a fixed collection of comparisons among elements of S , that is,

$$E = \bigcap_{i=1}^k \{f(u_i) < f(v_i)\},$$

where $u_i, v_i \in S$ for all $i \in [k]$. Write

$$q_0 := 1, \quad q_m := (D+1)^{-m(D+1)} \quad \text{for } m \geq 1,$$

and note that

$$\eta_D = \frac{(D+1)^{-4(D+1)}}{4096} = \frac{(q_2)^2}{4096}.$$

Lemma 3.1. *Let P be a finite poset satisfying $\pi(P) \leq D$, and let $S \subseteq X$. Let E be an S -dependent event with $\mathbf{P}[E] > 0$. Then*

$$\mathbf{P}[E] \geq q_m,$$

where $m := |S|$.

Proof. Let Q be obtained from P by adjoining the comparisons specified by E and taking the transitive closure. Since $\mathbf{P}[E] > 0$, these comparisons are consistent with the partial order of P , so Q is a well-defined poset. Set

$$T := S \cup \{v \in X : v \parallel s \text{ for some } s \in S\}, \quad U := X \setminus T.$$

The assumption on the range gives

$$|T| \leq \sum_{s \in S} (1 + \pi(s)) \leq m(D+1).$$

We denote the strict partial orders of P and Q by $\prec_P$ and $\prec_Q$, respectively. We first show that, for all $u, v \in U$,

$$u \prec_Q v \implies u \prec_P v.$$

Indeed, if $u \prec_P v$, there is nothing to prove. Otherwise, a path witnessing $u \prec_Q v$ must use an added comparison from E , so there exists $s \in S$ such that

$$u \prec_Q s \prec_Q v.$$

By the definition of U , both u and v are comparable with s in P . Since Q extends P , these comparisons must have the same orientations in both posets. Hence

$$u \prec_P s \prec_P v,$$

which implies $u \prec_P v$, as desired. In particular, this implies that

$$(3.1) \quad Q[U] = P[U],$$

where $P[U]$ denotes the subposet induced by U , with the order inherited from $\prec_P$.

For a finite poset R , let $e(R)$ denote its number of linear extensions. It follows from (3.1) that

$$(3.2) \quad e(Q) \geq e(Q[U]) = e(P[U]).$$

We next give an upper bound for $e(P)$. For $t \in T$, let

$$d(t) := |\{v \in X : v \prec_P t\}|.$$

In any linear extension g of P , we have

$$d(t) + 1 \leq g(t) \leq d(t) + 1 + \pi(t).$$

Thus $g(t)$ has at most $\pi(t) + 1 \leq D + 1$ possible values. Once the relative order on U is fixed, the positions $g(t)$ for $t \in T$ determine the entire linear extension, as the elements of U fill the remaining positions in their prescribed order. Consequently,

$$(3.3) \quad e(P) \leq e(P[U]) (D + 1)^{|T|}.$$

Combining (3.2) and (3.3), we therefore obtain

$$\mathbf{P}[E] = \frac{e(Q)}{e(P)} \geq (D + 1)^{-|T|} \geq (D + 1)^{-m(D+1)} = q_m,$$

as desired. $\square$

We now combine Lemma 3.1 with Lemma 2.4 to describe the structure of BFT triples when $\delta(P)$ is close to C_{BFT} . Let $G(P)$ denote the *incomparability graph* of P , whose vertex set is X and whose edges join incomparable elements. An edge is a *bridge* if deleting it increases the number of connected components. For subsets $V, W \subseteq X$, we write $V \prec W$ if every element of V precedes every element of W in P . For two distinct elements $x, y \in X$, we say that y *covers* x , denoted by $x \lessdot y$, if $x \prec y$ and no element $w \in X$ satisfies $x \prec w \prec y$. We use the same notation when either set consists of a single element, identifying that singleton with its element.

Lemma 3.2 (*Structure of BFT triples*). *Let P be a finite poset satisfying*

$$\pi(P) \leq D \quad \text{and} \quad \delta(P) \leq C_{\text{BFT}} + \eta_D,$$

and let (x, y, z) be a BFT triple. Then the following statements hold:

- (i) *The triple (x, y, z) is in Case D;*
- (ii) *z covers x ;*
- (iii) *y is incomparable only with x and z ;*
- (iv) *The edges xy and yz are bridges of $G(P)$;*

(v) *Writing*

$$L := \{w \in X : w \prec y\}, \quad U := \{w \in X : y \prec w\},$$

we have the following partition of the ground set

$$(3.4) \quad X = L \sqcup \{x, y, z\} \sqcup U,$$

and

$$(3.5) \quad L \prec z, \quad x \prec U, \quad L \prec U.$$

Intuitively, Lemma 3.2 states that, under its hypotheses, every BFT triple induces a copy of the $(2+1)$ poset and interacts with the rest of the poset in a highly constrained way. Here, $(2+1)$ -*poset* is the disjoint union of a two-element chain and a singleton. Equivalently, it consists of three elements x, y, z with $x \prec z$ and y incomparable to both x and z . Since the $(2+1)$ poset itself has balance constant $1/3$, such triples are natural places to look for well-balanced pairs.

Proof. We start by proving (i). Since $D \geq 2$, we have

$$\delta(P) \leq C_{\text{BFT}} + \eta_D \leq C_{\text{BFT}} + \frac{3^{-12}}{4096} < 0.2764.$$

Cases A and B, including their duals, give a pair with balance constant at least $1/3$, contradicting $\delta(P) < 0.2764$. Case C is excluded by Lemma 2.3. Thus the triple is in Case D, proving (i).

To prove (ii), suppose to the contrary that there exists w satisfying $x \prec w \prec z$. Then every linear extension g of P satisfies

$$g(z) - g(x) \geq 2.$$

Since $x \parallel y$ and $y \parallel z$, the event $\{f(x) < f(y) < f(z)\}$ has positive probability (recall that f is the uniform random linear extension of P). On this event, both w and y lie between x and z , so the gap is at least three. Taking expectations gives

$$h(z) - h(x) = \mathbf{E}[f(z) - f(x)] > 2,$$

contradicting the definition of a BFT triple. Thus $x \prec z$, proving (ii).

For (iii), we first establish two consequences of the bound on $\delta(P)$. By Lemma 2.4,

$$C_{\text{BFT}} + \frac{R}{5 + 3\sqrt{5}} \leq \delta(P) \leq C_{\text{BFT}} + \eta_D.$$

Consequently,

$$(3.6) \quad R \leq (5 + 3\sqrt{5})\eta_D = \frac{5 + 3\sqrt{5}}{4096(D+1)^{D+1}} q_3 < q_3.$$

Suppose that some linear extension g satisfies $g(y) < g(w) < g(x)$ for an element $w \in X \setminus \{x, y\}$. Then the event

$$E := \{f(y) < f(w) < f(x)\}$$

is $\{x, y, w\}$ -dependent and has positive probability. Lemma 3.1 and (2.5) give

$$q_3 \leq \mathbf{P}[E] \leq \mathbf{P}[f(x) - f(y) \geq 2] \leq R,$$

contradicting (3.6). Thus every linear extension g satisfies $g(y) < g(x)$ only if $g(x) = g(y) + 1$. The same argument applies to the pair y, z , so

$$(3.7) \quad \begin{aligned} g(y) < g(x) &\implies g(x) = g(y) + 1, \\ g(z) < g(y) &\implies g(y) = g(z) + 1. \end{aligned}$$

The first condition in (3.7) implies that, for every $w \in X \setminus \{x, y\}$,

$$(3.8) \quad w \prec y \quad \text{or} \quad x \prec w.$$

Indeed, if neither relation holds, then there would be a linear extension g with $g(y) < g(w) < g(x)$, contradicting (3.7). Similarly, the second condition implies that, for every $w \in X \setminus \{y, z\}$,

$$(3.9) \quad w \prec z \quad \text{or} \quad y \prec w.$$

If an element $w \notin \{x, y, z\}$ were incomparable with y , then (3.8) and (3.9) would give $x \prec w \prec z$. This contradicts (ii). Thus y is incomparable only with x and z , proving (iii).

We now prove (v). The partition in (3.4) follows from (iii). For $w \in L$, equation (3.9) gives $w \prec z$. For $w \in U$, equation (3.8) gives $x \prec w$. Also, $L \prec U$ follows by transitivity through y . This proves (3.5), and hence (v).

Finally, these relations, together with $x \prec z$, give

$$L \cup \{x\} \prec U \cup \{z\}.$$

Hence there are no incomparability edges between these two sets. By (iii), the only edges incident with y are xy and yz . Deleting xy therefore separates $L \cup \{x\}$ from its complement, while deleting yz separates $U \cup \{z\}$ from its complement. Thus both edges are bridges, proving (iv). $\square$

4. PROOF OF THEOREM 1.3

In this section, we prove Theorem 1.3. Throughout this section, let P be a finite nonchain poset satisfying $\pi(P) \leq D$. We may assume without loss of generality that the incomparability graph $G(P)$ is connected. Indeed, the conclusion for P follows from that for the subposet induced by any connected component with at least two vertices, since comparison probabilities within a component are unchanged. We will also assume that P has $n \geq 3$ elements, as otherwise $\delta(P) = 1/2$.

4.1. The strategy to choose the right BFT triple. Our starting point is the approach introduced by Kahn–Saks [KS84], and refined by Brightwell–Felsner–Trotter [BFT95]. We list the elements of P as $v_1, \dots, v_n$ in nondecreasing order of height,

$$h(v_1) \leq \dots \leq h(v_n),$$

breaking ties arbitrarily. The strategy of Brightwell–Felsner–Trotter is to find three consecutive elements (v_k, v_{k+1}, v_{k+2}) that form a BFT triple. Applying the arguments outlined in Section 2 to this triple then yields $\delta(P) \geq C_{\text{BFT}} = \frac{5-\sqrt{5}}{10}$. Our strategy builds on the BFT approach by choosing the BFT triple *more carefully*.

To see why the choice of triple matters, consider finite restrictions of the infinite *Fibonacci poset* $\mathcal{F}$ defined in Section 1.3: Let m be a positive even integer, and let $\mathcal{F}_m$ be the poset with ground set $\{x_{-m}, x_{-m+1}, \dots, x_{m-1}, x_m\}$, with $x_i \prec x_j$ if and only if $j - i \geq 2$. The consecutive triples centered at odd indices are all BFT triples, but they need not contain equally well-balanced pairs. Indeed, writing F_k for the Fibonacci numbers, with $F_0 = 0$ and $F_1 = 1$, a direct count gives

$$\delta(\mathcal{F}_m; x_i, x_{i+1}) = \frac{F_{m+i+1}F_{m-i}}{F_{2m+2}}.$$

It follows from direct calculation that, at the center ($i = 0$),

$$\delta(\mathcal{F}_m; x_0, x_1) \longrightarrow \frac{5 - \sqrt{5}}{10} \approx 0.276393,$$

whereas at the boundary ($i = m - 1$),

$$\delta(\mathcal{F}_m; x_{m-1}, x_m) \longrightarrow \frac{3 - \sqrt{5}}{2} \approx 0.381966.$$

Thus, in this case, the best-balanced pairs lie near the boundary, while choosing pairs near the center is suboptimal. As $m \rightarrow \infty$, the boundary recedes to infinity, while the balance of every adjacent pair at a fixed index converges to C_{BFT} . The best-balanced pairs near the boundary are therefore lost in the limit. This illustrates why the $1/3$ – $2/3$ conjecture can fail for infinite posets and suggests how to improve the bound for finite posets: we must find a way to distinguish these better balanced pairs from the suboptimal ones.

To identify promising BFT triples, we introduce the *displacement* of an element from its position in the height order. For each $i \in [n]$, let

$$d_i := h(v_i) - i, \quad M := \max_{i \in [n]} |d_i|.$$

Note that in the Fibonacci example, we have

$$|h(x_i) - (m + i + 1)| = \frac{F_{2|i|}}{F_{2m+2}}.$$

Thus the displacement at every fixed index tends to zero as $m \rightarrow \infty$, while its maximum absolute value is attained at the boundary and remains bounded away from zero. This suggests that the displacement function may help locate the better-balanced pairs that are lost in the infinite-volume limit. The next two lemmas make this observation rigorous.

Lemma 4.1 (*Choosing a BFT triple*). *Suppose that $G(P)$ is connected and $n \geq 3$. Then there exists $k \in [n - 2]$ such that (v_k, v_{k+1}, v_{k+2}) is a BFT triple and*

$$\max\{|d_k|, |d_{k+2}|\} = M.$$

Proof. Since the positions in a linear extension are distinct integers in $[n]$, we have

$$d_1 \geq 0, \quad d_n \leq 0, \quad d_1 + d_2 \geq 0, \quad d_{n-1} + d_n \leq 0.$$

It follows that either $d_i = M$ for some $i \leq n - 2$, or $d_j = -M$ for some $j \geq 3$. Set $k = i$ in the first case and $k = j - 2$ in the second. In either case,

$$h(v_{k+2}) - h(v_k) = 2 + d_{k+2} - d_k \leq 2,$$

and the displacement of one of the two endpoints has absolute value M .

It remains to show that these three elements do not form a chain. Write $x := v_k$, $y := v_{k+1}$, and $z := v_{k+2}$. If they form a chain, then $x \prec y \prec z$, and every linear extension g satisfies $g(z) - g(x) \geq 2$. The bound on the mean span would therefore force $g(z) - g(x) = 2$ in every linear extension.

Suppose that an element $w \notin \{x, y, z\}$ is incomparable with at least one of x, y, z . Then neither $w \prec x$ nor $z \prec w$ can hold. So there exists a linear extension g with $g(x) < g(w) < g(z)$. This implies that both w and y would lie between x and z , giving $g(z) - g(x) \geq 3$, a contradiction. Hence the triple has no incomparability edges either internally or to its complement. This contradicts the connectedness of $G(P)$ and proves the lemma. $\square$

4.2. A lower bound for the balance constant. Fix the triple (x, y, z) from Lemma 4.1. The next estimate uses the structural conclusions of Lemma 3.2 to obtain a lower bound for the balance of this triple in terms of M . Note that, to find the inequalities used in the proof of this lemma, we directed an AI assistant to explore the underlying algebraic optimization problem. This approach is similar in spirit to the treatment of Case D in [BFT95, §5]. As in that work, all the required identities and inequalities are justified explicitly below.

Lemma 4.2. *Let P be a finite poset satisfying*

$$\pi(P) \leq D \quad \text{and} \quad \delta(P) \leq C_{\text{BFT}} + \eta_D,$$

and let (x, y, z) be a BFT triple chosen as in Lemma 4.1. Then

$$(4.1) \quad \max\{\delta(P; x, y), \delta(P; y, z)\} \geq C_{\text{BFT}} + \frac{M^2}{441}.$$

Proof. By Lemma 3.2, the triple (x, y, z) satisfies all the structural conclusions stated there. Let L, U be as in Lemma 3.2, and write

$$L' := L \cup \{x\}, \quad U' := U \cup \{z\}, \quad m := |L'|.$$

Since $L \prec y \prec U$ and x, y, z are consecutive in the height order, every element of L occurs before x in that order, and every element of U occurs after z . Thus $x = v_m$, $y = v_{m+1}$, and $z = v_{m+2}$.

Let $f_{L'}$ and $f_{U'}$ be independent uniform linear extensions of $P[L']$ and $P[U']$, respectively. Set

$$r := \mathbf{P}[f_{L'}(x) = m], \quad s := \mathbf{P}[f_{U'}(z) = 1], \quad v := r + s, \quad Z := 1 + r + s.$$

By (3.5) and $x \prec z$, we have

$$L' \prec U'.$$

Note that each pair of linear extensions of $P[L']$ and $P[U']$ gives exactly one extension of P in which x precedes y and y precedes z : concatenate the two extensions and insert y between them. There is one additional extension with y preceding x precisely when x is last in the left extension, and one with z preceding y precisely when z is first in the right extension. These three possibilities exhaust the linear extensions of P . Consequently,

$$(4.2) \quad e(P) = e(P[L'])e(P[U'])Z,$$

and

$$(4.3) \quad p := \mathbf{P}[f(y) < f(x)] = \frac{r}{Z}, \quad p' := \mathbf{P}[f(z) < f(y)] = \frac{s}{Z}.$$

Since $r, s \in [0, 1]$, both probabilities are at most $1/2$. It then follows that

$$(4.4) \quad \delta(P; x, y) = p, \quad \delta(P; y, z) = p'.$$

Now, let

$$t := \mathbf{E}[m - f_{L'}(x)], \quad u := \mathbf{E}[f_{U'}(z) - 1].$$

These are the expected numbers of elements after x in the extension of $P[L']$ and before z in the extension of $P[U']$, respectively. We now compare these quantities with the corresponding expectations under a uniformly random linear extension f of P . Let

$$\alpha := \mathbf{E}[|\{w \in L : f(x) < f(w)\}|], \quad \beta := \mathbf{E}[|\{w \in U : f(w) < f(z)\}|]$$

be the expected numbers of elements of L after x and of U before z , respectively. We now show that

$$\alpha = \frac{(1+s)t}{Z}, \quad \beta = \frac{(1+r)u}{Z}.$$

To prove the first identity, fix a linear extension $g_{L'}$ of $P[L']$. In every full extension of P inducing $g_{L'}$, the number of elements of L after x is $m - g_{L'}(x)$. If $g_{L'}(x) < m$, then y can be inserted between the two blocks, or immediately after z when the extension of $P[U']$ begins with z . These give $(1+s)e(P[U'])$ full extensions. If $g_{L'}(x) = m$, the number being counted is zero. Averaging over all full extensions therefore gives

$$\begin{aligned} \alpha &= \frac{e(P[U'])}{e(P)} \left[(1+s) \sum_{g_{L'}(x) < m} (m - g_{L'}(x)) + (2+s) \underbrace{\sum_{g_{L'}(x) = m} (m - g_{L'}(x))}_{=0} \right] \\ &= \frac{(1+s)e(P[U'])}{e(P)} \sum_{g_{L'}} (m - g_{L'}(x)) \\ &= \frac{(1+s)e(P[U'])}{e(P)} e(P[L']) t = \frac{(1+s)t}{Z}, \end{aligned}$$

where the sum runs over all linear extensions $g_{L'}$ of $P[L']$. The second identity is proved analogously.

Counting the elements before each of x, y, z now gives

$$\begin{aligned} h(x) &= m + p - \alpha, \\ h(y) &= m + 1 + p' - p, \\ h(z) &= m + 2 + \beta - p'. \end{aligned} \tag{4.5}$$

Since $h(z) - h(x) \leq 2$, we obtain

$$\alpha + \beta \leq p + p' = \frac{v}{1+v}. \tag{4.6}$$

Also, whenever x is not last in $f_{L'}$, at least one element follows it. Thus $t \geq 1 - r$, and similarly $u \geq 1 - s$. It follows that

$$\alpha + \beta \geq \frac{(1+s)(1-r) + (1+r)(1-s)}{Z} = \frac{2-2rs}{1+v}.$$

Combining these inequalities gives

$$v + 2rs \geq 2. \tag{4.7}$$

Now, write

$$a := \max \{ \delta(P; x, y), \delta(P; y, z) \} = \max \{ p, p' \},$$

and

$$\rho := \frac{\sqrt{5}-1}{2}, \quad V(a) := \frac{2a}{1-2a}.$$

Now, since $v = r + s$, we have

$$v^2 - 4rs = (r-s)^2 \geq 0.$$

Hence $rs \leq v^2/4$. Together with (4.7), this gives $v \geq 2\rho$. On the other hand,

$$\frac{v}{1+v} = p + p' \leq 2a,$$

so

$$2\rho \leq v \leq V(a).$$

As V is increasing and $V(C_{\text{BFT}}) = 2\rho$, this implies $a \geq C_{\text{BFT}}$. Set $\varepsilon := a - C_{\text{BFT}}$ (note that this ε is local to this proof and is not to be confused with ε in Theorem 1.2). Since

$a \leq \delta(P) \leq C_{\text{BFT}} + \eta_D$, we have $a < 0.2764$ and $0 \leq \varepsilon \leq \eta_D < 10^{-5}$. For $C_{\text{BFT}} \leq a \leq 0.2764$, we have

$$V'(a) = \frac{2}{(1-2a)^2} \leq \frac{2}{(1-2 \cdot 0.2764)^2} < 11, \quad V(a) \leq V(0.2764) = \frac{691}{559} < 1.237.$$

Hence

$$(4.8) \quad 0 \leq v - 2\rho \leq 11\varepsilon.$$

Moreover, by (4.7),

$$(r-s)^2 = v^2 - 4rs \leq v^2 + 2v - 4 = (v-2\rho)(v+2\rho+2) \leq 50\varepsilon.$$

Together with (4.8), this yields

$$(4.9) \quad \begin{aligned} \max\{|r-\rho|, |s-\rho|\} &= \max\left\{\left|\frac{v-2\rho+(r-s)}{2}\right|, \left|\frac{v-2\rho-(r-s)}{2}\right|\right\} \\ &\leq \frac{|v-2\rho|+|r-s|}{2} \leq \frac{11\varepsilon+\sqrt{50\varepsilon}}{2} \leq 5\sqrt{\varepsilon}. \end{aligned}$$

To control the remaining contributions to the heights, define

$$\alpha_* := \alpha - \frac{(1+s)(1-r)}{Z}, \quad \beta_* := \beta - \frac{(1+r)(1-s)}{Z}.$$

Both quantities are nonnegative, since $t \geq 1-r$ and $u \geq 1-s$. By (4.6),

$$\begin{aligned} 0 \leq \alpha_* + \beta_* &\leq \frac{v-2+2rs}{1+v} \leq \frac{v-2+v^2/2}{1+v} \\ &= \frac{(v-2\rho)(1+(v+2\rho)/2)}{1+v} \\ &\leq v-2\rho \leq 11\varepsilon. \end{aligned}$$

Here the penultimate inequality uses $v \geq 2\rho$.

Since $x = v_m$ and $z = v_{m+2}$, the height identities (4.5) give

$$(4.10) \quad \begin{aligned} d_m &= \frac{r-(1+s)(1-r)}{Z} - \alpha_*, \\ d_{m+2} &= \frac{(1+r)(1-s)-s}{Z} + \beta_*. \end{aligned}$$

Now, let $F(r, s) := r - (1+s)(1-r)$ and $G(r, s) := (1+r)(1-s) - s$. Both functions vanish at $r = s = \rho$. For F , the absolute values of its partial derivatives with respect to r, s are at most 3, 1, respectively, on $[0, 1]^2$; for G , they are at most 1, 3. Since $F(\rho, \rho) = 0$, it then follows from the mean value theorem and the triangle inequality that

$$|F(r, s)| \leq 3|r-\rho| + |s-\rho| \leq 3 \cdot 5\sqrt{\varepsilon} + 5\sqrt{\varepsilon} = 20\sqrt{\varepsilon},$$

where the second inequality is due to (4.9). The same argument gives $|G(r, s)| \leq 20\sqrt{\varepsilon}$. Also note that $\alpha_*, \beta_* \geq 0$ and $\alpha_* + \beta_* \leq 11\varepsilon$, so each is at most 11ε . Applying these inequalities to (4.10) and using $Z \geq 1$,

$$\begin{aligned} |d_m| &\leq \frac{|r-(1+s)(1-r)|}{Z} + \alpha_* \leq 20\sqrt{\varepsilon} + 11\varepsilon, \\ |d_{m+2}| &\leq \frac{|(1+r)(1-s)-s|}{Z} + \beta_* \leq 20\sqrt{\varepsilon} + 11\varepsilon. \end{aligned}$$

By our choice of the triple in Lemma 4.1,

$$M = \max\{|d_m|, |d_{m+2}|\} \leq 21\sqrt{\varepsilon}.$$

Therefore

$$\max\{\delta(P; x, y), \delta(P; y, z)\} = a = C_{\text{BFT}} + \varepsilon \geq C_{\text{BFT}} + \frac{M^2}{441},$$

as desired. $\square$

4.3. Completion of the proof. Since $G(P)$ is connected, some element v_j is incomparable with v_1 . The event $f(v_j) < f(v_1)$ has positive probability and involves only two elements. Lemma 3.1 therefore gives

$$(4.11) \quad M \geq d_1 = \sum_{j=2}^n \mathbf{P}[f(v_j) < f(v_1)] \geq q_2.$$

Recall that we assume $\delta(P) \leq C_{\text{BFT}} + \eta_D$, as otherwise the result is immediate. Choose (x, y, z) as in Lemma 4.1. Lemma 4.2 and (4.11) therefore give

$$\begin{aligned} \delta(P) &\geq \max\{\delta(P; x, y), \delta(P; y, z)\} \\ &\geq C_{\text{BFT}} + \frac{M^2}{441} \geq C_{\text{BFT}} + \frac{q_2^2}{441} > C_{\text{BFT}} + \eta_D, \end{aligned}$$

which proves the theorem. $\square$

5. PRELIMINARIES FOR THEOREM 1.4

In this section we review results in the literature that will be used in the proof of Theorem 1.4.

5.1. The order polytope and log-concave random variables. Let $P = (X, \prec)$ be a finite poset with n elements. The *order polytope* of P is

$$\mathcal{O}(P) := \{(z_x)_{x \in X} \in [0, 1]^X : z_x \leq z_y \text{ whenever } x \prec y\}.$$

Let $\mathbf{F} = (F_x)_{x \in X}$ be chosen uniformly from $\mathcal{O}(P)$. The random point $\mathbf{F}$ can be coupled with a uniformly random linear extension f of P by requiring that, almost surely,

$$F_x < F_y \iff f(x) < f(y) \quad (x, y \in X).$$

Note that the induced linear extension is uniform because the simplices corresponding to the linear extensions of P have equal volume. In particular, the comparisons among the coordinates of $\mathbf{F}$ have the same probabilities as the corresponding comparisons in f . Writing

$$Z_x := (n+1)F_x, \quad \Delta(x, y) := |h(x) - h(y)|, \quad d(x, y) := \sqrt{\text{Var}(Z_x - Z_y)},$$

we have $\mathbf{E}[Z_x] = \mathbf{E}[f(x)] = h(x)$, the height function of P . We will use the following folklore result that is a consequence of the Brunn–Minkowski inequality, and for which the earliest written reference we have found is [AK25a, Cor. 5.1(a)].

Lemma 5.1 (*Log-concave density* [AK25a, Cor. 5.1(a)]). *For distinct $x, y \in X$, the random variable $Z_x - Z_y$ has a nondegenerate log-concave density.*

The following two elementary estimates follow from the results of Lovász and Vempala [LV07, Lems. 5.4, 5.5(a), and 5.7]. We include a short derivation here, as our formulation differs slightly from the cited statements.

Lemma 5.2 (*Log-concave estimates*). *For every random variable V with a log-concave density, mean μ , and standard deviation $\sigma > 0$,*

$$(5.1) \quad \min\{\mathbf{P}[V < 0], \mathbf{P}[V > 0]\} \geq e^{-1} - \frac{|\mu|}{\sigma},$$

and

$$(5.2) \quad \mathbf{P}[|V - \mu| > t\sigma] \leq e^{-t+1} \quad (t \geq 0).$$

Proof. For (5.1), set $W = (V - \mu)/\sigma$, and let g be its density function. Applying [LV07, Lem. 5.4] to W and $-W$ gives

$$\min\{\mathbf{P}[W < 0], \mathbf{P}[W > 0]\} \geq e^{-1}.$$

Since W has mean zero and variance one, [LV07, Lem. 5.5(a)] gives $g \leq 1$. Thus moving the threshold from 0 to $-\mu/\sigma$ changes either probability by at most

$$\int_{\min\{0, -\mu/\sigma\}}^{\max\{0, -\mu/\sigma\}} g(s) ds \leq \frac{|\mu|}{\sigma}.$$

This implies

$$\min\{\mathbf{P}[V < 0], \mathbf{P}[V > 0]\} \geq e^{-1} - \frac{|\mu|}{\sigma},$$

as desired.

For (5.2), we apply [LV07, Lem. 5.7] to $(V - \mu)/\sigma$ to get the stated conclusion for $t > 1$. For the case $t \in [0, 1]$, the bound is trivial since $e^{1-t} \geq 1$. $\square$

5.2. Three auxiliary inequalities. For each $x \in X$, let

$$Q_x := \max_{y \prec x} F_y, \quad R_x := \min_{x \prec y} F_y,$$

where an empty maximum is 0 and an empty minimum is 1. Let

$$a_x := (n+1)\mathbf{E}[F_x - Q_x] = \frac{n+1}{2}\mathbf{E}[R_x - Q_x],$$

where the second equality follows because, conditional on all coordinates other than F_x , the random variable F_x is uniform on $[Q_x, R_x]$, so $F_x - Q_x$ has conditional mean $(R_x - Q_x)/2$. The number $2a_x$ is known as the *discrete window parameter* $\text{win}(x)$ in the work of Aires and Kahn [AK25a, (6)].

The following lemma follows from the estimates in [AK25a, Eq. (24), Thm. 2.9]. We include a short derivation here, as our formulation differs slightly from the cited statements.

Lemma 5.3 (*Window estimates*). *For distinct $x, y \in X$ and every antichain $B \subseteq X$,*

$$(5.3) \quad a_x \geq 1, \quad d(x, y) \geq \frac{a_x}{\sqrt{3}}, \quad \sum_{x \in B} a_x \geq \frac{|B|^2}{2}.$$

Proof. The coupling between $\mathbf{F}$ and f gives

$$a_x = \mathbf{E}[f(x) - \max_{y \prec x} f(y)] \geq 1,$$

where the empty maximum is 0. This proves the first claim. For the second claim, we use the same conditioning argument applied in [AK25a, Eq. (24)]. Let $\mathcal{G} = \sigma(F_z : z \neq x)$. Conditional on $\mathcal{G}$, the variable F_x is uniform on $[Q_x, R_x]$, and hence

$$\text{Var}(Z_x - Z_y \mid \mathcal{G}) = (n+1)^2 \text{Var}(F_x \mid \mathcal{G}) = \frac{(n+1)^2}{12} (R_x - Q_x)^2.$$

The law of total variance now gives

$$\begin{aligned} d(x, y)^2 &= \text{Var}(Z_x - Z_y) = \mathbf{E}[\text{Var}(Z_x - Z_y \mid \mathcal{G})] + \text{Var}(\mathbf{E}[Z_x - Z_y \mid \mathcal{G}]) \\ &\geq \mathbf{E}[\text{Var}(Z_x - Z_y \mid \mathcal{G})] \\ &= \frac{(n+1)^2}{12} \mathbf{E}[(R_x - Q_x)^2] \geq \frac{a_x^2}{3}, \end{aligned}$$

where the last inequality is because of Jensen's inequality. For the third claim, notice that $a_x = \text{win}(x)/2$ in the notation of [AK25a]. Thus [AK25a, Thm. 2.9] gives

$$\sum_{x \in B} a_x \geq \frac{n+1}{2n} |B|^2 \geq \frac{|B|^2}{2},$$

as desired. $\square$

The following lemma is a consequence of the continuous form of Shepp's XYZ inequality [She82, Sec. 2], which in turn is a consequence of the classical FKG inequality [FKG71].

Lemma 5.4 (*Correlation inequality*). *For all $x, y, z \in X$, we have*

$$(5.4) \quad d(x, z)^2 \leq d(x, y)^2 + d(y, z)^2.$$

Proof. The continuous form of Shepp's XYZ inequality [She82] says that the random variables $Z_x - Z_y$ and $Z_z - Z_y$ are positively associated. In particular, they have nonnegative covariance and hence

$$\begin{aligned} d(x, z)^2 &= \text{Var}((Z_x - Z_y) - (Z_z - Z_y)) \\ &\leq \text{Var}(Z_x - Z_y) + \text{Var}(Z_z - Z_y) = d(x, y)^2 + d(y, z)^2, \end{aligned}$$

as desired. $\square$

The next lemma follows from the *Haqi-Kahn ideal inequality*, formerly known as Kahn's ideal conjecture, which was recently proved by Haqi [Haq26, Thm. 4.1]. An *order ideal* is a subset $I \subseteq X$ such that $x \in I$ and $y \prec x$ imply $y \in I$. We write $\max(I) := \max_P(I)$ and $\min(J) := \min_P(J)$ for the sets of maximal elements of $P[I]$ and minimal elements of $P[J]$, respectively.

Lemma 5.5. *Let I be a nonempty proper order ideal of P , and let $J := X \setminus I$. Then*

$$(5.5) \quad \min_{y \in J} h(y) - \max_{x \in I} h(x) \leq |\max(I)| + |\min(J)| - 1.$$

Proof. Haqi-Kahn ideal inequality [Haq26, Thm. 4.1] gives

$$\max_{x \in I} h(x) \geq |I| - |\max(I)| + 1.$$

Applying the same inequality to J in the dual poset gives

$$\min_{y \in J} h(y) = n + 1 - \max_{y \in J} h^*(y) \leq n + 1 - (|J| - |\min(J)| + 1) = |I| + |\min(J)|,$$

where h^* is the height function in the dual poset. Taking the difference of the two inequalities proves the result. $\square$

6. PROOF OF THEOREM 1.4

Fix a constant $\varepsilon > 0$ which we assume to satisfy $\varepsilon < e^{-1}$. Note that this parameter is the deficit from e^{-1} in this section, and is independent of the improvement parameter in Theorem 1.2. We prove the contrapositive of Theorem 1.4. That is to say, we will assume throughout this section that

$$(6.1) \quad \delta(P) \leq e^{-1} - \varepsilon,$$

and we will show that the width of P is bounded in terms of ε^{-1} .

6.1. The proof idea. A useful guiding example is the theorem of Komlós [Kom90], which shows that $\delta(P) \rightarrow 1/2$ as $n \rightarrow \infty$ whenever P has $\Omega(n)$ minimal elements. The guiding principle of our argument is to find a *large antichain* whose elements have closely *clustered average heights*. The connection between small differences of average heights and balanced pairs was already central to the breakthrough work of Kahn–Saks [KS84]. Although a general poset need not have many minimal or maximal elements, we can instead consider antichains of the following form:

$$(6.2) \quad \max(\{y \in X : h(y) \leq h(x)\}) \quad \text{or} \quad \min(\{y \in X : h(y) > h(x)\}),$$

where $x \in X$, and $\min(S)$ and $\max(S)$ denote the sets of minimal and maximal elements of the induced poset $P[S]$, respectively. We call the sets in (6.2) *canopy antichains*.³ There are two reasons for choosing sets of this form.

The first reason comes from Haqi’s recent breakthrough proof of the *Haqi–Kahn ideal inequality* [Haq26]. This inequality guarantees a canopy antichain of size at least half the largest gap between consecutive average heights in P . Using *log-concavity estimates* together with the assumed bound on $\delta(P)$ in (6.1), we then show that this gap is at least a positive constant multiple of the width, where the constant depends only on ε . Thus large width guarantees a *large* antichain of the desired form.

The second reason is that the definition in (6.2) suggests that the heights within a canopy antichain should cluster near $h(x)$. Unfortunately, this need not hold in general, so we modify P by adding relations that remove elements with heights far from $h(x)$ in the canopy antichain. The *correlation inequality* in Lemma 5.4, together with log-concave tail estimates and the assumed bound on $\delta(P)$ in (6.1), shows that these added relations hold with very high probability, so the cost of passing from P to Q is small. The corresponding canopy antichain in Q now has the desired property: its original heights in P now *cluster near $h(x)$* . In particular, the span of its average heights in this antichain is small.

Finally, combining log-concavity with *window estimates* of Aires–Kahn [AK25a], we obtain a precise inequality relating the size of an antichain to the span of its average heights. Together with the clustering behavior observed above, this inequality gives an upper bound on the size of our canopy antichain in terms of ε^{-1} . Since this size is at least a positive constant multiple of the width of P , we obtain the desired upper bound on the width.

We now proceed to the proof of Theorem 1.4.

6.2. Bounding the width of a poset from above. We may assume without loss of generality that P has a unique minimum element and a unique maximum element, since adjoining these elements changes neither $\delta(P)$ nor $w(P)$. We now list the elements of P as $v_1, \dots, v_n$ in nondecreasing order of height,

$$h(v_1) \leq \dots \leq h(v_n),$$

³The name comes from viewing these antichains as canopies over the corresponding order ideals or its dual.

and write

$$\text{gap}(P) := \max\{h(v_1), h(v_{i+1}) - h(v_i) \ (1 \leq i < n), n + 1 - h(v_n)\}.$$

Since P has a minimum element and a maximum element, we have $h(v_1) = n + 1 - h(v_n) = 1$ and so $\text{gap}(P) \geq 1$. We now present two lemmas that will be used throughout this section. In particular, these lemmas give two different upper bounds for the width of a poset.

Lemma 6.1. *Let x, y be distinct elements of P . Then*

$$(6.3) \quad d(x, y) \leq \frac{\Delta(x, y)}{\varepsilon};$$

$$(6.4) \quad 1 \leq a_x \leq \frac{\sqrt{3}}{\varepsilon} \Delta(x, y).$$

Proof. First note that $Z_x - Z_y$ is a random variable with mean $h(x) - h(y)$, standard deviation $d(x, y)$, and with a log-concave density (Lemma 5.1). Thus (5.1) gives us

$$e^{-1} - \frac{\Delta(x, y)}{d(x, y)} \leq \delta(P; x, y) \leq \delta(P) \leq e^{-1} - \varepsilon.$$

Rearranging, we obtain

$$d(x, y) \leq \frac{\Delta(x, y)}{\varepsilon}.$$

Lemma 5.3 then gives

$$1 \leq a_x \leq \sqrt{3} d(x, y) \leq \frac{\sqrt{3}}{\varepsilon} \Delta(x, y),$$

and the proof is complete. $\square$

In particular, it follows from (6.4) that the heights of any two distinct elements differ by at least $\varepsilon/\sqrt{3}$. Thus the height order is in fact strictly increasing. This also implies that every interval of length $\text{gap}(P)$ contains the heights of at most $1 + (\sqrt{3}/\varepsilon) \text{gap}(P)$ elements. For a nonempty subset $B \subseteq X$, let

$$H(B) := \max_{x \in B} h(x) - \min_{x \in B} h(x),$$

the *span of the average heights* in B .

The following lemma is the main result of this subsection, which gives us (1) an upper bound for the width $w(P)$ in terms of both $\text{gap}(P)$ and ε^{-1} , and (2) an upper bound for the size of an antichain B in terms of its span of the average heights $H(B)$.

Lemma 6.2. *We have*

$$(6.5) \quad w(P) \leq \frac{2\sqrt{3}}{\varepsilon} \text{gap}(P).$$

Moreover, for all $x, y \in X$,

$$(6.6) \quad d(x, y)^2 \leq \frac{\text{gap}(P) \Delta(x, y)}{\varepsilon^2},$$

and every antichain B with $|B| \geq 2$ satisfies

$$(6.7) \quad |B|^2 \leq \frac{4\sqrt{3}}{\varepsilon} H(B).$$

Proof. For each $x \in X$, choose a neighbor y of x in the height order. Applying (6.4) to this pair gives

$$a_x \leq \frac{\sqrt{3}}{\varepsilon} \Delta(x, y) \leq \frac{\sqrt{3}}{\varepsilon} \text{gap}(P).$$

Thus, for an antichain B of size $w(P)$, Lemma 5.3 gives

$$\frac{w(P)^2}{2} \leq \sum_{x \in B} a_x \leq \frac{\sqrt{3}}{\varepsilon} \text{gap}(P) w(P),$$

which proves (6.5).

For (6.6), let $i < j$. Repeated application of Lemma 5.4, followed by (6.3), gives

$$\begin{aligned} d(v_i, v_j)^2 &\leq \sum_{k=i}^{j-1} d(v_k, v_{k+1})^2 \leq \frac{1}{\varepsilon^2} \sum_{k=i}^{j-1} (h(v_{k+1}) - h(v_k))^2 \\ &\leq \frac{\text{gap}(P)}{\varepsilon^2} (h(v_j) - h(v_i)), \end{aligned}$$

as desired.

Finally, let B be an antichain of size $b \geq 2$, and list its elements as $b_1, \dots, b_b$ in increasing order of height. We then have

$$\begin{aligned} \frac{b^2}{2} &\leq \sum_{x \in B} a_x \\ &\leq \frac{\sqrt{3}}{\varepsilon} \sum_{i=1}^{b-1} (h(b_{i+1}) - h(b_i)) + \frac{\sqrt{3}}{\varepsilon} (h(b_b) - h(b_{b-1})) \\ &\leq \frac{2\sqrt{3}}{\varepsilon} H(B), \end{aligned}$$

where the first inequality is due to Lemma 5.3, and the second inequality is by applying (6.4) to each b_i and its successor, using the predecessor for b_b . This proves (6.7). $\square$

Now that we have (6.5), it remains to bound $\text{gap}(P)$ in terms of ε^{-1} .

6.3. Adding beneficial relations to the poset. We may assume that

$$(6.8) \quad \text{gap}(P) \geq \frac{10e(1 + \sqrt{3})}{\varepsilon^3},$$

since otherwise (6.5) already gives the desired bound. In particular, $\text{gap}(P) > 1$, so there are consecutive elements u, v in the height order such that

$$h(v) - h(u) = \text{gap}(P).$$

Set

$$I := \{x \in X : h(x) \leq h(u)\}, \quad J := X \setminus I.$$

Note that I is nonempty since $u \in I$, and is proper because $v \notin I$ (as $h(v) - h(u) > 1$). Let

$$R := \frac{288}{\varepsilon^2}, \quad \ell := R \text{gap}(P) (\log \text{gap}(P))^2.$$

Let Q be obtained from P by adjoining the comparisons

$$(6.9) \quad z \prec_Q u \text{ if } h(z) \leq h(u) - \ell, \quad v \prec_Q z \text{ if } h(z) \geq h(v) + \ell,$$

and taking the transitive closure. Each added comparison is oriented in increasing order of the original heights, so these comparisons are consistent and Q is a poset. Moreover, each added comparison is between two elements of I or between two elements of J , so I remains an order

ideal of Q . We use h_P and h_Q to distinguish the two height functions from now on. We assume that unsubscripted Δ, d, a_x still refer to P . We write $H_P(B) := \max_{x \in B} h_P(x) - \min_{x \in B} h_P(x)$, and $h_P(B) := \{h_P(x) : x \in B\}$.

We would like to work with Q in place of P . The following lemma shows that the cost of making this change is small. Let E be the event that all comparisons in (6.9) hold in the random point $\mathbf{F}$ of $\mathcal{O}(P)$, and let $q := \mathbf{P}[E^c]$.

Lemma 6.3. *We have*

$$(6.10) \quad q \leq \text{gap}(P)^{-4}.$$

Proof. Let x, y be any elements satisfying $h_P(y) > h_P(x)$. Let $V := Z_y - Z_x$. Note that V has mean $\Delta(x, y) = h_P(y) - h_P(x) > 0$ and standard deviation $d(x, y)$. Applying (5.2) with $t = \Delta(x, y)/d(x, y)$ gives

$$\begin{aligned} \mathbf{P}[Z_y < Z_x] &= \mathbf{P}[V < 0] \leq \mathbf{P}[|V - \Delta(x, y)| > \Delta(x, y)] \\ &\leq \exp\left(1 - \frac{\Delta(x, y)}{d(x, y)}\right). \end{aligned}$$

Applying (6.6) then gives us

$$(6.11) \quad \mathbf{P}[Z_y < Z_x] \leq \exp\left(1 - \varepsilon \sqrt{\frac{\Delta(x, y)}{\text{gap}(P)}}\right).$$

Now, by the definition of E and the union bound,

$$q \leq \sum_{h_P(z) \leq h_P(u) - \ell} \mathbf{P}[Z_u < Z_z] + \sum_{h_P(z) \geq h_P(v) + \ell} \mathbf{P}[Z_z < Z_v].$$

We now group the terms in the sums in the following way: For the first family, take $(x, y) := (z, u)$; for the second, take $(x, y) := (v, z)$. Then, group the comparisons in each family according to

$$j \text{ gap}(P) \leq \Delta(x, y) < (j+1) \text{ gap}(P).$$

For each fixed j , this condition places $h_P(z)$ in an interval of length $\text{gap}(P)$. By the separation of the heights in (6.4), such an interval contains at most $1 + (\sqrt{3}/\varepsilon) \text{gap}(P)$ elements. Moreover, (6.11) gives

$$\mathbf{P}[Z_y < Z_x] \leq \exp\left(1 - \varepsilon \sqrt{\frac{\Delta(x, y)}{\text{gap}(P)}}\right) \leq e e^{-\varepsilon \sqrt{j}}.$$

Every added comparison satisfies $\Delta(x, y) \geq \ell$, so its interval index is at least

$$j_0 := \left\lfloor \frac{\ell}{\text{gap}(P)} \right\rfloor.$$

Summing over these intervals and accounting for both families, we obtain

$$q \leq 2 \sum_{j \geq j_0} \left(1 + \frac{\sqrt{3} \text{gap}(P)}{\varepsilon}\right) e e^{-\varepsilon \sqrt{j}} = 2e \left(1 + \frac{\sqrt{3} \text{gap}(P)}{\varepsilon}\right) \sum_{j \geq j_0} e^{-\varepsilon \sqrt{j}}.$$

Comparison with an integral gives

$$\begin{aligned} \sum_{j \geq j_0} e^{-\varepsilon \sqrt{j}} &\leq e^{-\varepsilon \sqrt{j_0}} + \int_{j_0}^{\infty} e^{-\varepsilon \sqrt{t}} dt \\ &= \left(1 + \frac{2\sqrt{j_0}}{\varepsilon} + \frac{2}{\varepsilon^2}\right) e^{-\varepsilon \sqrt{j_0}} \\ &\leq \frac{5}{\varepsilon^2} e^{-\varepsilon \sqrt{j_0}/2}, \end{aligned}$$

where the last inequality uses $\varepsilon < 1$ and $(1+t)e^{-t/2} \leq 2$ for $t \geq 0$. Now note that our largeness assumption gives $R(\log \text{gap}(P))^2 \geq 2$, so $j_0 \geq R(\log \text{gap}(P))^2/2$. Consequently,

$$\sqrt{j_0} \geq \sqrt{R/2} \log \text{gap}(P) = \frac{12}{\varepsilon} \log \text{gap}(P).$$

Since $\text{gap}(P) \geq 1$ and $\varepsilon < 1$, it follows that

$$\begin{aligned} (6.12) \quad q &\leq \frac{10e(1+\sqrt{3})}{\varepsilon^3} \text{gap}(P) e^{-\varepsilon \sqrt{j_0}/2} \\ &\leq \frac{10e(1+\sqrt{3})}{\varepsilon^3} \text{gap}(P)^{-5} \leq \text{gap}(P)^{-4}, \end{aligned}$$

where the last inequality is (6.8). This proves the lemma. $\square$

The next lemma bounds $\text{gap}(P)$ from above in terms of differences of average heights in Q .

Lemma 6.4. *We have*

$$(6.13) \quad \text{gap}(P) \leq 2 \left(\min_{t \in J} h_Q(t) - \max_{s \in I} h_Q(s) \right).$$

Proof. Note that conditioning $\mathbf{F}$ on E makes it uniform on $\mathcal{O}(Q)$. The ground set is unchanged, so the normalization $Z_x = (n+1)F_x$ is the same for both posets. In particular,

$$h_Q(x) = \mathbf{E}[Z_x \mid E].$$

For any square-integrable random variable V , Cauchy–Schwarz gives

$$(6.14) \quad |\mathbf{E}[V \mid E] - \mathbf{E}[V]| = \frac{|\text{Cov}(V, \mathbf{1}_E)|}{1-q} \leq \sqrt{\text{Var}(V)} \sqrt{\frac{q}{1-q}}.$$

Now let $V := Z_t - Z_s$ for $s \in I$ and $t \in J$. Since $s \in I$ and $t \in J$, we have

$$\Delta(s, t) = h_P(t) - h_P(s) \geq h_P(v) - h_P(u) = \text{gap}(P).$$

We then have

$$\begin{aligned} h_Q(t) - h_Q(s) &= \mathbf{E}[V \mid E] \geq \Delta(s, t) - \sqrt{\text{Var}(V)} \sqrt{\frac{q}{1-q}} \\ &\geq \Delta(s, t) - \frac{\sqrt{\text{gap}(P)\Delta(s, t)}}{\varepsilon} \sqrt{\frac{q}{1-q}} \geq \Delta(s, t) \left(1 - \frac{1}{\varepsilon} \sqrt{\frac{q}{1-q}}\right) \\ &\geq \Delta(s, t) \left(1 - \frac{1}{\varepsilon} \frac{1}{\sqrt{\text{gap}(P)^4 - 1}}\right) \geq \frac{\Delta(s, t)}{2} \geq \frac{\text{gap}(P)}{2}, \end{aligned}$$

where the first inequality is due to (6.14), the second inequality is due to (6.6), the third inequality and sixth inequality are because $\Delta(s, t) \geq \text{gap}(P)$, the fourth inequality is because of (6.10), and the fifth inequality is because our largeness assumption (6.8) implies $\text{gap}(P)^4 - 1 \geq 4/\varepsilon^2$. This completes the proof. $\square$

6.4. Putting everything together. Let

$$B_0 := \max_Q(I), \quad B_1 := \min_Q(J).$$

The added comparisons ensure that these two sets lie near the original gap. Indeed, every element $z \in I$ with $h_P(z) \leq h_P(u) - \ell$ lies below u in Q , and hence cannot be maximal in I . The analogous observation applies to J , so

$$\begin{aligned} h_P(B_0) &\subseteq (h_P(u) - \ell, h_P(u)], \\ h_P(B_1) &\subseteq [h_P(v), h_P(v) + \ell). \end{aligned}$$

Both sets are antichains in Q , and therefore also in P . Applying Lemma 5.5 in Q and using (6.13), we obtain

$$|B_0| + |B_1| - 1 \geq \frac{\text{gap}(P)}{2}.$$

Choose the larger of B_0, B_1 , and call it B . Then $|B| \geq \text{gap}(P)/4$ and $H_P(B) \leq \ell$. Since (6.8) gives $\text{gap}(P) > 8$, we have $|B| \geq 2$. Applying (6.7) in the original poset P gives

$$\frac{\text{gap}(P)^2}{16} \leq |B|^2 \leq \frac{4\sqrt{3}}{\varepsilon} H_P(B) \leq \frac{4\sqrt{3}R}{\varepsilon} \text{gap}(P)(\log \text{gap}(P))^2.$$

Consequently,

$$\text{gap}(P) \leq \frac{64\sqrt{3}R}{\varepsilon} (\log \text{gap}(P))^2 = \frac{18432\sqrt{3}}{\varepsilon^3} (\log \text{gap}(P))^2.$$

These facts and the preceding inequality imply

$$\text{gap}(P) \leq \frac{73728\sqrt{3}}{\varepsilon^3} \left(\log \frac{18432\sqrt{3}}{\varepsilon^3} \right)^2.$$

This bounds $\text{gap}(P)$ in terms of ε . Equation (6.5) then bounds $w(P)$ in terms of ε as well,

$$w(P) \leq \frac{2\sqrt{3}}{\varepsilon} \text{gap}(P) \leq \frac{442368}{\varepsilon^4} \left(\log \frac{18432\sqrt{3}}{\varepsilon^3} \right)^2.$$

Taking $\varepsilon = 10^{-100}$ and any integer K exceeding the displayed width bound, finishes the proof of Theorem 1.4. $\square$

7. FINAL REMARKS

7.1. To give a quick summary, the proof of Theorem 1.2 draws on log-concavity arising from deep results in convex geometry [Sta89b, KS84, BFT95], combinatorial counting arguments in the spirit of [CPP25], and Shepp's XYZ correlation inequality [She82]. These are combined with the geometry of order polytopes, the window estimates in [AK25a], probabilistic conditioning, and Haqi's recent proof of the Haqi–Kahn ideal inequality [Haq26].

It is perhaps somewhat surprising to see so many different tools and techniques from across the area come together in the proof of such a small numerical improvement to break the infinite barrier. This brings to mind the following quote by Gian-Carlo Rota, from one of his last publications:

“The solution [...] always comes as an unexpected application of theories that were previously developed without a specific purpose, theories whose effectiveness was at first thought to be highly questionable.” [Rota98].

7.2. The proof of Theorem 1.5 in [AK25b] combines two key ideas: (1) every poset of large range must contain an element x whose position $f(x)$ in a uniformly random linear extension has large variance, and (2) every such element in a poset of bounded width belongs to a relatively balanced pair. The first assertion is by far the harder to establish: an element with large range need not be one whose position has large variance.

Nevertheless, such an element can be found in the union of a maximal pair of sets with no comparabilities between them. The proof relies on a reduction to a special class of posets of width two, followed by a careful analysis of their linear extensions. For the second assertion, repeated applications of the XYZ inequality [She82] show that an element in a poset of bounded width that belongs to no relatively balanced pair must have bounded variance. Both parts make use of various correlation and log-concavity inequalities.

7.3. While the $\frac{1}{3}-\frac{2}{3}$ Conjecture may seem like an ultimate result in the area, the *Kahn–Saks Conjecture* $\delta(P) \rightarrow \frac{1}{2}$ as $w(P) \rightarrow \infty$ resolved in [Air26], is much more natural. Indeed, it was shown by Aigner [Aig85] that for width 2 posets the equality $\delta(P) = \frac{1}{3}$ holds only for a 3-element nonchain poset and posets containing it as a direct sum. See also [Chen18, Sah21] for variations on this result. In fact, one would want to prove the polynomial rate of convergence $\delta(P) \rightarrow \frac{1}{2}$ as the last three authors do in [CPP21a, CPP21b].

7.4. We expect—and sincerely hope—that the bound established in this paper will soon be surpassed. While we would be astonished to see a complete resolution of the $\frac{1}{3}-\frac{2}{3}$ Conjecture, perhaps one could expect an improvement by, say, 0.001, rather than by a microscopic amount in (1.5). Clearly, new tools and fresh ideas are needed to obtain any such bound, especially given how slow the progress has been until this point.

Use of AI. In preparation of the paper, the authors used AI extensively (several different models) in the proofs of several technical lemmas, and as a way to compute the constant in (1.5). We also used it for light editing, reference check and various computations, some of which later needed to be redone by hand. As always, the authors take full responsibility for the content of the paper in its entirety.

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APPENDIX A. PROOF OF LEMMA 2.3

We reproduce the proof of Brightwell–Felsner–Trotter [BFT95, §6], making the constant 0.2764 explicit for use in this paper. The proof in fact yields the stronger lower bound $67/242 \approx 0.27686$ for Lemma 2.3, but this improvement is not needed for the purposes of this paper.

We prove Lemma 2.3 by contradiction. Suppose to the contrary that

$$\delta(P; x, y) \leq 0.2764, \quad \delta(P; y, z) \leq 0.2764.$$

Set

$$p := \mathbf{P}[f(y) < f(x)], \quad p' := \mathbf{P}[f(z) < f(y)].$$

Since $h(x) \leq h(y) \leq h(z)$, Lemma 2.1 gives

$$1 - p \geq e^{-1} > 0.2764, \quad 1 - p' \geq e^{-1} > 0.2764.$$

The assumption at the beginning of the proof therefore implies $0 < p, p' \leq 0.2764$.

Let $t, t' \in (0, 1]$ be

$$t := \frac{\mathbf{P}[f(x) = f(y) + 1]}{p}, \quad t' := \frac{\mathbf{P}[f(y) = f(z) + 1]}{p'}.$$

By [BFT95, Lem. 6.2],

$$\mathbf{P}[f(x) - f(y) \geq 2] + \mathbf{P}[f(y) - f(z) \geq 2] \geq \frac{1}{11},$$

which in our notation is equivalent to

$$(A.1) \quad p(1 - t) + p'(1 - t') \geq \frac{1}{11}.$$

We recall the definition of the packed height function from [BFT95, Section 2]. Fix $q \in (0, 0.2764]$ and $u \in (0, 1]$. A pair of nonnegative sequences $(a_i)_{i \geq 1}$ and $(b_i)_{i \geq 1}$ is called *packed* with parameters (q, u) if, for some integer $k \geq 1$,

$$b_i = qu(1 - u)^{i-1} \quad (i \geq 1), \quad a_i = qu(1 + u)^{i-1} \quad (1 \leq i \leq k),$$

and

$$a_{k+1} + a_{k+2} = 1 - q - \sum_{i=1}^k a_i, \quad a_i = 0 \quad (i > k + 2),$$

with one of the following two conditions:

(i) $k \geq 2$, $a_{k+2} = 0$, and

$$\frac{u}{1 + u} \leq \frac{a_{k+1}}{a_k} \leq 1.$$

(ii) $a_{k+1} = a_k + a_{k+2}$ and

$$1 \leq \frac{a_{k+1}}{a_k} \leq 1 + u.$$

Note that these sequences satisfy $\sum_{i \geq 1} b_i = q$ and $b_1/q = u$. For each (q, u) in this range, it was shown in [KS84, Sec. 3] that there is a unique pair of sequences of this form. The *packed height function* is defined by

$$H(q, u) := \sum_{i \geq 1} i(a_i - b_i).$$

The formulas in [BFT95, (2.9)–(2.10)] give

$$(A.2) \quad H(q, u) = \begin{cases} k + 1 - \frac{q(1+u)^{k+1}}{u}, & \text{in case (i),} \\ k + \frac{3}{2} - \frac{q(1+u)^{k-1}(4u^2 + 5u + 2)}{2u}, & \text{in case (ii).} \end{cases}$$

Claim A.1. *For any $q \in (0, 0.2764]$ and $u \in (0, 1]$,*

$$H(q, u) \geq H(q, 1) \geq \frac{5 - 11q}{2}.$$

Proof. The first inequality follows from the monotonicity lemma in [BFT95, Lem. 2.3] (note that we invoke $0.2764 < 1 - 1/\sqrt{2} \approx 0.293$ to use this lemma). We now prove the second inequality. First, for $q \geq 1/5$, the packed sequence at $u = 1$ is in case (ii) with $k = 1$. Thus the formula for H gives

$$(A.3) \quad H(q, 1) = 1 + \frac{3}{2} - \frac{q(4 + 5 + 2)}{2} = \frac{5 - 11q}{2}.$$

Now suppose that $0 < q < 1/5$. At $u = 1$, the definition of a packed sequence gives

$$b_1 = q, \quad b_i = 0 \quad (i \geq 2), \quad a_1 = q, \quad a_2 = 2q.$$

Since $\sum_{i \geq 1} a_i = 1 - q$, the remaining positive mass is

$$\sum_{i \geq 3} a_i = 1 - 2q - a_2.$$

Every term in this remaining mass has index at least three. It thus follows from the two equations above that

$$\begin{aligned} H(q, 1) &= \sum_{i \geq 1} i(a_i - b_i) = \sum_{i \geq 1} i a_i - q \geq q + 2a_2 + 3(1 - 2q - a_2) - q \\ &= 3 - 6q - a_2 = 3 - 8q \geq \frac{5 - 11q}{2}, \end{aligned}$$

where the last inequality uses the assumption that $q < 1/5$. This completes the proof. $\square$

The packing lemma [BFT95, Lem. 2.2] gives

$$(A.4) \quad 2 \geq h(z) - h(x) \geq H(p, t) + H(p', t').$$

It then follows from Claim A.1 that

$$p + p' \geq \frac{6}{11} \quad \text{and thus} \quad \frac{6}{11} - 0.2764 \leq p, p' \leq 0.2764.$$

In particular, both p and p' lie in $(1/5, 3/10)$, so

$$H(p, 1) = \frac{5 - 11p}{2}, \quad H(p', 1) = \frac{5 - 11p'}{2}.$$

For $q \in \{p, p'\}$, let v_q be the positive solution of

$$(1 + v_q)(1 + 2v_q) = \frac{1}{q}.$$

The bounds on q imply $3/5 < v_q < 1$. Suppose first that $t \leq v_p$ or $t' \leq v_{p'}$, say $t \leq v_p$. Writing $v := v_p$, we then have

$$(1 + v)(1 + 2v) = \frac{1}{p}, \quad \text{so} \quad p = \frac{1}{(1 + v)(1 + 2v)}.$$

This implies

$$H(p, v) = 3 - \frac{p(1+v)^3}{v}, \quad H(p, 1) = \frac{5-11p}{2},$$

where the first equality holds at the boundary of case (i) with $k = 2$, and the second is because of (A.3) (recall that $p \geq 1/5$). It then follows from the monotonicity lemma [BFT95, Lem. 2.3] that

$$(A.5) \quad H(p, t) - H(p, 1) \geq H(p, v) - H(p, 1) = \frac{-3v^2 + 6v - 2}{2v(1+v)(1+2v)} \geq \frac{1}{12},$$

where in the last inequality we use the fact that $1/2 \leq v \leq 1$. Consequently,

$$\begin{aligned} h(z) - h(x) &\geq H(p, t) + H(p', t') \\ &\geq H(p, 1) + H(p', 1) + \frac{1}{12} = \frac{5-11p}{2} + \frac{5-11p'}{2} + \frac{1}{12} \\ &\geq 5 - 11 \cdot 0.2764 + \frac{1}{12} > 2, \end{aligned}$$

where the first inequality uses (A.4), the second inequality uses (A.5), and the equality uses (A.3). This contradicts the assumption that (x, y, z) is a BFT triple.

We may therefore assume that $t > v_p$ and $t' > v_{p'}$.

Claim A.2. *Assume that $t > v_p$ and $t' > v_{p'}$. Then*

$$(A.6) \quad H(p, t) - H(p, 1) \geq \frac{1}{2}p(1-t),$$

and likewise for p', t' .

Proof. We start with the case $1/p \leq 1 + 2t + 2t^2$. This hypothesis places (p, t) in packed case (ii) with $k = 1$. Hence

$$H(p, t) = \frac{5}{2} - \frac{p(4t^2 + 5t + 2)}{2t}, \quad H(p, 1) = \frac{5-11p}{2},$$

which in turn implies

$$H(p, t) - H(p, 1) = p(1-t) \left(2 - \frac{1}{t} \right).$$

Moreover, the inequalities $1/p \leq 1 + 2t + 2t^2$ and $p \leq 0.2764$ imply that $(2 - 1/t) > 1/2$. Thus (A.6) holds in this case.

We now deal with the case $1/p > 1 + 2t + 2t^2$. Note that this implies

$$1 + 2t + 2t^2 < \frac{1}{p} < (1+t)(1+2t),$$

where the right inequality is because of the assumption that $t > v_p$. These inequalities then put us in packed case (i) with $k = 2$. The formula for case (i) in (A.2) therefore gives

$$H(p, t) = 3 - \frac{p(1+t)^3}{t}.$$

Set

$$g(t) := \frac{2}{t} - 4 + 5t + 2t^2.$$

A direct calculation gives

$$H(p, t) - H(p, 1) - \frac{1}{2}p(1-t) = \frac{1 - pg(t)}{2}.$$

If $t \geq 2/3$, then

$$1 + 2t + 2t^2 - g(t) = \frac{(1-t)(3t-2)}{t} \geq 0,$$

so $g(t) \leq 1/p$. If $3/5 < t < 2/3$, bounding the terms of $g(t)$ separately gives

$$g(t) < \frac{32}{9} < \frac{1}{0.2764} \leq \frac{1}{p}.$$

This proves (A.6) in both scenarios. The same argument applies to p', t' . $\square$

We now have

$$\begin{aligned} h(z) - h(x) &\geq H(p, 1) + H(p', 1) + \frac{1}{2}(p(1-t) + p'(1-t')) \\ &\geq 5 - \frac{11}{2}(p + p') + \frac{1}{22} \\ &\geq 5 - 11 \cdot 0.2764 + \frac{1}{22} > 2, \end{aligned}$$

where the first inequality is due to (A.4) and (A.6), and the second inequality is due to (A.3) and (A.1). This contradicts the defining inequality for a BFT triple and completes the proof. $\square$

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