In problem 6, I assume that $a < b$ was intended (though the result is trivial when $a \geq b$). I also assume that you have proved basic facts about metric spaces in the lectures (e.g., convergent sequences are Cauchy and bounded, subsequences of a convergent sequence converge to the limit of the sequence, closed balls are closed sets, etc.).

1. (a) If $x \in X$, then $x \in \bigcap_{\lambda \in \Lambda} C_\lambda$ if and only if $x \in C_\lambda$ for all $\lambda \in \Lambda$, so $x \notin \bigcap_{\lambda \in \Lambda} C_\lambda$ if and only if $x \notin C_\lambda$ for some $\lambda \in \Lambda$. This is precisely what was to be shown since, by definition, $X \setminus \bigcap_{\lambda \in \Lambda} C_\lambda = \{x \in X : x \notin \bigcap_{\lambda \in \Lambda} C_\lambda\}$ and $\bigcup_{\lambda \in \Lambda} X \setminus C_\lambda = \{x \in X : \exists \lambda \in \Lambda\ (x \in X \setminus C_\lambda)\} = \{x \in X : \exists \lambda \in \Lambda\ (x \notin C_\lambda)\}$.

   (b) By part (a) with the C_λ's replaced by their relative complements in X, we have $X \setminus \bigcap_{\lambda \in \Lambda}(X \setminus C_\lambda) = \bigcup_{\lambda \in \Lambda} C_\lambda$. Taking the relative complement of both sides in X then yields $\bigcap_{\lambda \in \Lambda} X \setminus C_\lambda = X \setminus \bigcup_{\lambda \in \Lambda} C_\lambda$.

2. If $x \in \liminf A_n := \bigcup_{n \geq 1} \bigcap_{j \geq n} A_j$, then $x \in \bigcap_{j \geq N} A_j$ for some $N \geq 1$, which is to say that $x \in A_j$ for all $j \geq N$. Now for any $n \geq 1$, we have $x \in A_J$ for $J := \max\{N, n\}$ since $J \geq N$; consequently, $x \in \bigcup_{j \geq n} A_j$ since $J \geq n$. Hence $x \in \bigcap_{n \geq 1} \bigcup_{j \geq n} A_j =: \limsup A_n$.

3. By parts (a) and (b) of problem 1, we have $X \setminus \limsup A_n = X \setminus (\bigcap_{n \geq 1} \bigcup_{j \geq n} A_j) = \bigcup_{n \geq 1}(X \setminus \bigcup_{j \geq n} A_j) = \bigcup_{n \geq 1} \bigcap_{j \geq n}(X \setminus A_j) = \liminf(X \setminus A_n)$.

4. Suppose that there exists a $y_0 \in Y$ and an $A > 0$ for which $d(y, y_0) \leq A$ for all $y \in Y$. Then for any $y_1, y_2 \in Y$, we have $d(y_1, y_2) \leq d(y_1, y_0) + d(y_0, y_2) \leq A + A$, so Y must be bounded (take $M = 2A$ in the given definition of boundedness).

5. If Y is empty, it is trivially bounded and closed in X, so in what follows we will assume that Y is nonempty.

   (a) Suppose to the contrary that Y is unbounded and fix a $y_0 \in Y$. Then by problem 4, there exists a sequence $\{y_n\}_{n=1}^\infty \subseteq Y$ such that $d(y_n, y_0) > n$ for each n. As Y is sequentially compact, there exists a subsequence $\{y_{n_k}\}_{k=1}^\infty$ of $\{y_n\}_{n=1}^\infty$ converging in Y. However, $\{y_{n_k}\}_{k=1}^\infty$ is unbounded since $d(y_{n_k}, y_0) > n_k \geq k$, which is a contradiction since any convergent sequence in a metric space must be bounded. We conclude that Y must be bounded.

   (b) Suppose that $\{y_n\}$ is a sequence in Y converging to some $x \in X$. As Y is sequentially compact, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ converging to some $y \in Y$. Since any subsequence of $\{y_n\}$ also converges to x, we must have $x = y \in Y$, which means that Y is closed in X.

6. Note that $Y = [a, b]$ is bounded (put $y_0 = a$ and $A = b - a$ in problem 4) and closed (it is the closed ball of radius $\frac{b-a}{2}$ centred at $\frac{a+b}{2}$). Thus, if $\{y_n\}$ is a sequence in Y, it is bounded, and therefore has a subsequence converging in $\mathbb{R}$ by the Bolzano–Weierstrass theorem. Since Y is closed, the limit of this subsequence must lie in Y, which shows that Y is sequentially compact.

7. Clearly, Y is closed (it is the closed ball of radius 1 centred at 0) and bounded (put $y_0 = 0$ and $A = 1$ in problem 4). Now for each $n \in \mathbb{N}$, let $y^{(n)}$ be the sequence $\{y_i^{(n)}\}_{i=1}^\infty$ with $y_i^{(n)} := 1$ if $i = n$ and $y_i^{(n)} := 0$ otherwise. Then $\{y^{(n)}\}_{n=1}^\infty \subseteq Y$ since $d(y^{(n)}, 0) = 1$ for each n. But $d(y^{(n)}, y^{(n')}) = \sqrt{2}$ for all $n \neq n'$ (because $|y_i^{(n')} - y_i^{(n)}|$ is 1 if $i = n$ or $i = n'$ and is 0 otherwise), so no subsequence of $\{y^{(n)}\}_{n=1}^\infty$ can be Cauchy, and hence no such subsequence can be convergent. Therefore Y is not sequentially compact.