MATH 131A September 5, 2024

Lecture 13: Taylor's theorem and the integral

Instructor: Nicholas Hu *Notetaker: Nicholas Hu*

We begin by recalling the mean value theorem from last lecture.

Theorem 13.1 (Mean value theorem)**.** *Let f be continuous on $[a, b]$ and suppose that f is differentiable on (a, b). Then there exists an $x \in (a, b)$ such that $f'(x) = \frac{f(b)-f(a)}{b-a}$.*

13.1 Taylor's theorem

Let us consider a continuous function f on $[a, x]$ and suppose that f is differentiable on (a, x). If $p(t) := f(a)$ for all $t \in [a, x]$, then the conclusion of the mean value theorem can be rephrased as

$$f(x) = p(x) + f'(x_1)(x-a) = f(a) + f'(x_1)(x-a)$$

for some $x_1 \in (a, x)$. Thus, we can regard p as a polynomial approximation to f (of degree 0) based on the value of f at a, and $f'(x_1)(x-a)$ as the error in this approximation at a given $x > a$.

Now suppose that we wish to construct a polynomial approximation to f of the form $p(t) := c_0 + c_1(t-a)$ (i.e., of degree 1) based on the values of f and f' at a (assuming that f is differentiable). Imposing the conditions $p(a) = f(a)$ and $p'(a) = f'(a)$, we obtain $c_0 = f(a)$ and $c_1 = f'(a)$. Next, given an $x > a$, let C be such that $f(x) = p(x) + q(x)$, where $q(t) := C(t-a)^2$. By construction, $g := f - (p+q)$ satisfies $g(a) = g(x) = 0$ as well as $g'(a) = 0$. Hence, by Rolle's theorem, we have $g'(x_1) = 0$ for some $x_1 \in (a, x)$, and by Rolle's theorem again, we have $g''(x_2) = 0$ for some $x_2 \in (a, x_1)$. Therefore $f''(x_2) = (p+q)''(x_2) = 2C$, so $C = \frac{1}{2}f''(x_2)$. We conclude that

$$f(x) = p(x) + \frac{f''(x_2)}{2}(x-a)^2 = f(a) + f'(a)(x-a) + \frac{f''(x_2)}{2}(x-a)^2$$

for some $x_2 \in (a, x)$.

This process can be continued to construct a polynomial approximation to f of the form $p(t) := c_0 + c_1(t-a) + c_2(t-a)^2$ (assuming that f is differentiable sufficiently many times). Imposing the conditions $p(a) = f(a)$, $p'(a) = f'(a)$, and $p''(a) = f''(a)$, we obtain

$$f(x) = p(x) + \frac{f'''(x_3)}{6}(x-a)^3 = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(x_3)}{6}(x-a)^3$$

for some $x_3 \in (a, x)$. Generalizing this, we arrive at the following theorem.

Theorem 13.2 (Taylor)**.** *Let f be k times continuously differentiable on the closed interval between a and x, and suppose that $f^{(k+1)}$ exists on the open interval between a and x (allowing both $a < x$ and $x < a$). Then*

$$f(x) = \sum_{j=0}^{k} \frac{f^{(j)}(a)}{j!}(x-a)^j + \frac{f^{(k+1)}(x_{k+1})}{(k+1)!}(x-a)^{k+1}$$

for some x_{k+1} strictly between a and x.

Proof. Exercise. □

LaTeX template courtesy of the UC Berkeley EECS department.

13.2 The integral

Definition 13.3. A **partition** of $[a,b]$ is a set $P = \{x_0, x_1, \dots, x_n\} \subseteq [a,b]$ with $a = x_0 < x_1 < \cdots < x_n = b$.

Definition 13.4. Let $f : [a,b] \to \mathbb{R}$ be a *bounded* function and $P = \{x_i\}_{i=0}^n$ be a partition of $[a,b]$. The **upper (Darboux) sum** of f with respect to P is

$$U(f,P) := \sum_{i=1}^{n} M_i \cdot \Delta x_i, \quad \text{where } M_i := \sup_{x \in [x_{i-1}, x_i]} f(x) \text{ and } \Delta x_i := x_i - x_{i-1}.$$

Similarly, the **lower (Darboux) sum** of f with respect to P is

$$L(f,P) := \sum_{i=1}^{n} m_i \cdot \Delta x_i, \quad \text{where } m_i := \inf_{x \in [x_{i-1}, x_i]} f(x) \text{ and } \Delta x_i = x_i - x_{i-1}.$$

The **upper (Darboux) integral** of f over $[a,b]$ is

$$\overline{\int_a^b} f(x)\,dx := \inf\{U(f,P) : P \text{ is a partition of } [a,b]\}.$$

Likewise, the **lower (Darboux) integral** of f over $[a,b]$ is

$$\underline{\int_a^b} f(x)\,dx := \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}.$$

If $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = I$, we say that f is **(Darboux) integrable** and that its **(Darboux) integral** over $[a,b]$ is

$$\int_a^b f(x)\,dx := I.$$

We will see that the lower and upper integrals are analogous to the limit inferior and superior of a sequence, respectively, and that the integral is analogous to the limit of a sequence.

Remark. If P is a partition of $[a,b]$, then clearly

$$\inf_{x \in [a,b]} f(x) \cdot (b-a) \le L(f,P) \le U(f,P) \le \sup_{x \in [a,b]} f(x) \cdot (b-a).$$

In particular, this implies that the lower and upper integrals always exist.

Example 13.5. Let $f(x) := x$ on $[0,1]$ and $P := \{x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{3}{4}, x_3 = 1\}$. Then $U(f,P) = \frac{1}{4} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} = \frac{11}{16}$ and $L(f,P) = 0 \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{4} = \frac{5}{16}$.

Proposition 13.6. *Let P and P' be partitions of $[a,b]$. If $P \subseteq P'$ (in which case P' is called a* ***refinement*** *of P), then $U(f,P) \ge U(f,P')$ and $L(f,P) \le L(f,P')$.*

Proof. Suppose first that $P = \{x_i\}_{i=0}^n$ and that $P' = P \cup \{x_*\}$, where $x_{i-1} < x_* < x_i$ for some $1 \le i \le n$.

Then

$$\begin{aligned} U(f,P) &= M_1(x_1-x_0)+\cdots+M_i(x_i-x_{i-1})+\cdots+M_n(x_n-x_{n-1}) \\ &= M_1(x_1-x_0)+\cdots+M_i(x_*-x_{i-1})+ \\ &\quad M_i(x_i-x_*)+\cdots+M_n(x_n-x_{n-1}) \\ &\geq M_1(x_1-x_0)+\cdots+\sup_{x\in[x_{i-1},x_*]} f(x)\cdot(x_*-x_{i-1})+ \\ &\quad \sup_{x\in[x_*,x_i]} f(x)\cdot(x_i-x_*)+\cdots+M_n(x_n-x_{n-1}) \\ &= U(f,P'). \end{aligned}$$

If P' contains more than one point not in P, this argument can be iterated. The argument for lower sums is similar. $\square$

Proposition 13.7. *Let $f:[a,b]\to\mathbb{R}$ be bounded. Then*

$$\underline{\int_a^b} f(x)\,dx \leq \overline{\int_a^b} f(x)\,dx.$$

Proof. Let P and Q be partitions of $[a,b]$. Then $P\cup Q$ is partition of $[a,b]$ (called the **common refinement** of P and Q) that contains both P and Q. Hence $L(f,P)\leq L(f,P\cup Q)\leq U(f,P\cup Q)\leq U(f,Q)$. Taking the supremum over P followed by the infimum over Q, we obtain the result. $\square$

Theorem 13.8. *Let $f:[a,b]\to\mathbb{R}$ be bounded. Then f is integrable if and only if for all $\varepsilon>0$, there exists a partition P of $[a,b]$ such that $U(f,P)-L(f,P)<\varepsilon$.*

Proof. Suppose that f is integrable. Given an $\varepsilon>0$, there exists a partition P_L of $[a,b]$ such that $\overline{\int_a^b} f(x)\,dx - \frac{\varepsilon}{2} < L(f,P_L)$ and a partition P_U of $[a,b]$ such that $U(f,P_U) < \overline{\int_a^b} f(x)\,dx + \frac{\varepsilon}{2}$ Then $U(f,P_L\cup P_U)-L(f,P_L\cup P_U) < \left(\overline{\int_a^b} f(x)\,dx+\frac{\varepsilon}{2}\right)-\left(\underline{\int_a^b} f(x)\,dx-\frac{\varepsilon}{2}\right)=\varepsilon$.

Conversely, if for any given $\varepsilon>0$, there exists a partition P of $[a,b]$ such that $U(f,P)-L(f,P)<\varepsilon$, then we have $0\leq \overline{\int_a^b} f(x)\,dx-\underline{\int_a^b} f(x)\,dx \leq U(f,P)-L(f,P)<\varepsilon$ for this partition, which shows that f is integrable as ε can be chosen arbitrarily. $\square$