Thm (Cauchy completeness of $\mathbb{R}$). If (x_n) is a Cauchy seq. in $\mathbb{R}$, then it is convergent.

For seq. in $\mathbb{Q}$	For seq. in $\mathbb{R}$
$\text{conv} \underset{\not\Leftarrow}{\Rightarrow} \text{cauchy} \underset{\not\Leftarrow}{\Rightarrow} \text{bounded}$	$\text{conv} \underset{\substack{\Leftarrow \\ \text{completeness}}}{\Rightarrow} \text{cauchy} \underset{\not\Leftarrow}{\Rightarrow} \text{bounded}$

We now know that the seq. of decimal expansions defined last class converges to some $x \in \mathbb{R}_{>0}$ w/. $x^2 = 2$

Rmk If $y \in \mathbb{R}_{>0}$ and $y^2 = 2$, then $x^2 - y^2 = 0$, so $(x-y)(x+y) = 0$. Hence $x = y$. In other words, there is a unique nonnegative sequare root of 2, which we will denote by $\sqrt{2}$ or 2.

SUPREMA AND INFIMA

Def. Let $S \subseteq X$ (where X is $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$). If there exists a $u \in X$ s.t. $s \leq u$ for all $s \in S$, then S is said to be BOUNDED ABOVE and u is called an UPPER BOUND of S. If, in addition, $u \in S$, then u is called a MAXIMUM or GREATEST element of S.

Similary. we can define **BOUNDED BELOW**, **LOWER BOUND** and **MINIMUM / LEAST** elt.

Ex. Let $X = \mathbb{Z}$ and $S = \{1, 2, 4\}$. Then 4.5.6.7 etc are UBs of S, but only 4 is a max

Rmk If S has a max, then it is unique.

Pf. If $s \le u$ and $s \le u'$ for all $s \in S$ and $u, u' \in S$, then $u' \le u$ because $u' \in S$ and u is an UB of S and similarly $u \le u'$ so $u = u$

We write $u = \max(S)$. if it exists. Clearly, if S is non-$\emptyset$ and finite, $\max(S)$ exists.

Ex. $X = \mathbb{Z}$, $S = \{1, 2, 3, 4 \ldots\}$, S has no max, nor even any UBs

$X = \mathbb{Q}$, $S = \{1 - \frac{1}{n} : n \in \mathbb{Z}_{>0}\} = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4} \ldots\}$, S is bounded above by 1, but S has no max.

Def Let $S \subseteq X$. If u is an UB of S and $u \le u'$ for all UBs u' of S, then u is called a **LEAST UPPER BOUND** or **SUPREMUM** of S.

Ex. (cont'd) 1 is a LUB of S because 1 is an UB, and if there were an UB $u < 1$, then we would have $1 - \frac{1}{n} \le u$ for all $n \in \mathbb{Z}_{>0}$. Taking the limit as $n \to \infty$, we get $1 \le u$, which is a contradiction. In other words, all UBs u must be ≥ 1.

Rmk LUBs are unique if they exist. If u is the LUB of S, we write $u = \sup(S)$ (or $\text{lub}(S)$). If $\max(S)$ exists, then $\sup(S) = \max(S)$

Pf. (a) If u is LUB then $u \le u'$, since u is the LUB and u' is a UB

Similarly, $u' \le u$, since u' is LUB and u is UB, so $u = u'$

(b) By def. $\max(S)$ is an UB of S and since $\max(S) \in S$, we have $\max(S) \le u'$ for all UBs u' of S by def of UB

Hence, by def of LUB, $\sup(S) = \max(S)$

But even subsets of $\mathbb{Q}$ can fail to have LUBs, despile being non-$\emptyset$ and bounded above

Ex. $S = \{q \in \mathbb{Q} : q^2 \le 2\}$ has no sup in $\mathbb{Q}$.

(Recall the seq. of decimal approx. from last class. It satisfied

$$\underbrace{q_n}_{\in S}{}^2 \le 2 < (\underbrace{q_n + \tfrac{1}{10^n}}_{=r_n \quad \text{UBs of } S})^2 \text{ for all } n \in \mathbb{Z}_{>0}$$

By def', $q_n \in S$ for all n. If $q \in S$, then $q^2 \le 2 < r_n^2$, so $q \le r_n$ for all n. (because if $q > r_n$, then $q^2 > r_n^2 > 2$)

Notice that $0 \le 2 - q_n^2 < 2\underbrace{q_n}_{\le 2 \text{ for ex.}} \cdot \frac{1}{10^n} + (\frac{1}{10^n})^2 \le 4 \cdot \frac{1}{10^n} + (\frac{1}{10^n})^2 \to 0$

Hence $q_n^2 \to 2$

This also implies tha $r_n^2 = q_n^2 + 2q_n \cdot \frac{1}{10^n} + (\frac{1}{10^n})^2 \to 2$ as well.

Therefore $\sup(S)$ can't exist in $\mathbb{Q}$ because say s were the sup. Then

if $s^2 < 2$, then s could not be an UB:

if $q_n \le s$ for all n, then by taking the lim. as $n \to \infty$, we would get $2 \le s^2$, which is a contradiction.

if $s^2 > 2$, then s could not be the LUB:

if $s \le r_n$ for all n, then we would get $s^2 \le 2$, which is contra.

($s^2=2$ is impossible in $\mathbb{Q}$)

Thm (Dedekind completeness of $\mathbb{R}$) If $A \subseteq \mathbb{R}$ is non-$\emptyset$ and bounded above, then $\sup(A)$ exists (in $\mathbb{R}$)

Pf. Let $a_1 \in A$ (such as elt. exists since $A \neq \emptyset$) and b_1 be an UB of A (which exists since A is bounded above)

Consider $c_1 := \frac{a_1+b_1}{2}$.

If c_1 is an UB for A, define $a_2 := a_1$ and $b_2 := b_1$

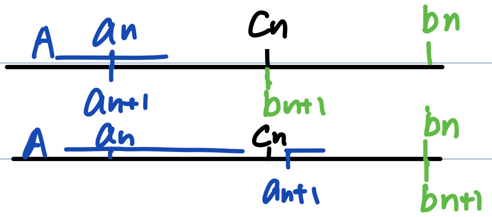
Otherwise, if c_1 is NOT an UB for A, there must be an $a_2 \in A$ with $c_1 < a_2$. Define $b_2 := b_1$

In either case, $a_2 \in A$ and b_2 is an UB for A. Repeat to define $a_3, b_3, a_4, b_4 \dots$

In general, given a_n and b_n for some $n \in \mathbb{Z}_{>0}$, let $c_n := \frac{a_n+b_n}{2}$

If c_n is an UB : $a_{n+1} := a_n$, $b_{n+1} := c_n$

If c_n is not an UB : $\exists\, a_{n+1} \in A$ $(a_{n+1} > c_n)$, $b_{n+1} := b_n$



Claim For all n, (i) $a_n \in A$, b_n is an UB (of A), which implies $a_n \le b_n$

(ii) $b_n - a_n \le \frac{b_1-a_1}{2^{n-1}}$

(iii) $0 \le b_n - b_{n+1} \le \frac{b_n-a_n}{2}$ $\left(\overset{(ii)}{\le} \frac{b_1-a_1}{2^n} \right)$

Pf. exercise

Claim $(b_n)_{n=1}^{\infty}$ is cauchy

Pf. Consider $|b_m - b_n|$. WLOG $m \le n$. Then $|b_m - b_n| = b_m - b_n =$ $(b_m - b_{m+1}) + (b_{m+1} - b_{m+2}) + \cdots + (b_{n-1} - b_n) \le \frac{b_1 - a_1}{2^m} + \frac{b_1 - a_1}{2^{m+1}} + \cdots + \frac{b_1 - a_1}{2^{n-1}}$ $\le \frac{b_1 - a_1}{2^{m-1}}$. As we did last class, we can use this to show that the seq. is cauchy. (Check!)

Hence by Cauchy completeness of $\mathbb{R}$, $b_n \to b$ for some $b \in \mathbb{R}$

Now if $a \in A$, then $a \le b_n$ for all n by (i), so $a \le b$.

Therefore b is an UB for A.

On the other hand, $0 \le b_n - a_n \le \frac{b_1 - a_1}{2^{n-1}} \to 0$, so $a_n = \overset{\nearrow b}{b_n} - (\overset{\nearrow 0}{b_n - a_n}) \to b$

Because of this, b is the UB of A.

Indeed, suppose that b' were an UB of A with $b' < b$, and let $\varepsilon = b - b' > 0$. Then there exists an $N \in \mathbb{Z}_{>0}$ s.t. for all $n \ge N$, we have $|b - a_n| < \varepsilon$. ie, $b - \varepsilon < a_n < b + \varepsilon$. But then $a_N > b - \varepsilon = b'$ which contradicts the assumption that b' is an UB of A (since $a_N \in A$). We conclude that $b = \sup(A)$.

Similarly, if $A \subseteq \mathbb{R}$ is non-$\emptyset$ and bd. below, then $\inf(A)$ exists.

INFIMUM means GREATEST LOWER BOUND

Prop. Let $A, B \subseteq \mathbb{R}$ be non-$\emptyset$ and bd. above. If $A \subseteq B$, then $\sup(A) \le \sup(B$