Def: A seq (q_n) in $\mathbb{Q}$ is cauchy if $\forall \varepsilon \in \mathbb{Q}_{>0}$, $\exists N \in \mathbb{Z}_{>0}$ $\forall m,n \geq N$ $(|q_m - q_n| < \varepsilon)$

Prop: If q_n is conv. then it is cauchy

For seq. in $\mathbb{Q}$ — In fact, we can show that cauchy sq. are bd.

convergent $\Rightarrow$ Cauchy

$\Rightarrow$ Bounded

convergent $\Rightarrow$ Cauchy $\Rightarrow$ Bounded

$\not\Leftarrow$ eg. decimal approx. of $\sqrt{2}$ $\qquad$ $\not\Leftarrow$ eg. $q_n = (-1)^n$

$\boxed{\text{Ex}}$ Consider the seq. of dec. approx. of $\sqrt{2}$

1.4, 1.41, 1.412, 1.4142 ...

$q_1 = 1 + \frac{4}{10}$ $\qquad$ $q_3 = 1 + \frac{4}{10} + \frac{1}{100} + \frac{2}{1000}$ ...

$q_2 = 1 + \frac{4}{10} + \frac{1}{100}$

More precisely, we choose

$q_n = \max\{\frac{a}{10^n} : a \in \mathbb{Z} \text{ and } (\frac{a}{10^n})^2 \leq 2\}$.

Hence, $q_n^2 \leq 2 < (q_n + \frac{1}{10^n})^2$. This seq. can't converge in $\mathbb{Q}$.

Proof: Indeed, sps $q_n \to q$. Then $q^2 \leq 2$. On the other hand, $(q_n + \frac{1}{10^n})^2 = q_n^2 + 2q_n \cdot \frac{1}{10^n} + (\frac{1}{10^n})^2 \to q^2$ b/c $\frac{1}{10^n} \to 0$ (note that $\frac{1}{10^n} < \frac{1}{n}$) so $2 \leq q^2$. But this would mean that $q^2 = 2$, which we know is impossible.

However, this sequence is cauchy. Note that $q_n \leq q_{n+1} \leq q_n + \frac{9}{10^{n+1}}$. If $m,n \geq N$, we may assume WLOG that $m \geq n$. Then $|q_m - q_n| \overset{m \geq n}{=} q_m - q_n \overset{m \geq N}{\leq} q_m - q_N \leq \frac{9}{10^{N+1}} + \frac{9}{10^{N+2}} + \dots + \frac{9}{10^m} = \frac{9}{10^{N+1}}(1 + \frac{1}{10} + \dots + \frac{1}{10^m})$

$= \frac{9}{10^{N+1}} \cdot \frac{1 - (\frac{1}{10})^{m+1}}{1 - \frac{1}{10}} < \frac{1}{10^N} \leq \frac{1}{N}$

Now, given an $\varepsilon \in \mathbb{Q}_{>0}$, $\exists$ an $N \in \mathbb{Z}$ s.t. $N > \frac{1}{\varepsilon}$ (By the A.P.), and if $m,n \geq N$ we have $|q_m - q_n| < \frac{1}{N} < \varepsilon$. This shows that the seq. is cauchy.

What abt. Cauchy seq. of real #'s? Do they also conv. in $\mathbb{R}$?

Theorem: (Cauchy completeness of $\mathbb{R}$) If $(x_n)_{n=1}^{\infty}$ is a cauchy seq. of <u>real</u> numbers in $\mathbb{R}$, then it is conv.

Prop: (Density of $\mathbb{Q}$ in $\mathbb{R}$) $\forall x \in \mathbb{R}$, $\forall \varepsilon \in \mathbb{R}_{>0}$, $\exists$ $q \in \mathbb{Q}$ $(|x - q| < \varepsilon)$

Proof (of thm): Let $(x_n)_{n=1}^{\infty}$ be a Cauchy seq. of real numbers in $\mathbb{R}$. For each n, let $y_n \in \mathbb{Q}$ be a rational number s.t. $|x_n - y_n| < \frac{1}{n}$ (using density of $\mathbb{Q}$). We claim that y_n is cauchy.

Given an $\varepsilon \in \mathbb{Q}$, $\exists$ an $N \in \mathbb{Z}_{>0}$ s.t. $\frac{1}{N} < \frac{\varepsilon}{3}$ and $|x_m - x_n| < \frac{\varepsilon}{3}$ for $m,n \geq N$. Then if $n,m \geq N$, then $|y_m - y_n| \leq \underbrace{|y_m - x_m|}_{\text{small cause density}} + \underbrace{|x_m - x_n|}_{\text{small cause Cauchy}} + |x_n - y_n| < \frac{1}{m} + \frac{\varepsilon}{3} + \frac{1}{n}$

$\leq \frac{1}{N} + \frac{\varepsilon}{3} + \frac{1}{N} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$

$= \varepsilon$. This shows that $(y_n)_{n=1}^{\infty}$ is Cauchy.

Hence, by the previous prop. we know that $y_n \to y$ for some $y \in \mathbb{R}$. We conclude $x_n = y_n + (x_n - y_n) \to y + 0 = y$

Real Numbers:

Let us say that the Cauchy seq. (q_n) and (r_n) in $\mathbb{Q}$ are related if $q_n - r_n \to 0$ and write $(q_n) \sim (r_n)$.

Prop: $\sim$ is an eq. relation

Proof: Exercise

Def: Real numbers are the set $\mathbb{R}$ of equiv. classes of this relation.

Def: $[(q_n)_{n=1}^{\infty}] \overset{\mathbb{R}}{+} [(r_n)_{n=1}^{\infty}] := [(q_n \overset{\mathbb{Q}}{+} r_n)_{n=1}^{\infty}]$

$[(q_n)_{n=1}^{\infty}] \cdot_{\mathbb{R}} [(r_n)_{n=1}^{\infty}] := [(q_n \cdot_{\mathbb{Q}} r_n)_{n=1}^{\infty}]$

We need to check that this is well defined, and that $q_n + r_n$ and $q_n \cdot r_n$ are Cauchy sequences if q_n & r_n are.

Prop: Let (q_n), (r_n) be Cauchy seq. in $\mathbb{Q}$. Then $(q_n + r_n)$ is a cauchy seq. in $\mathbb{Q}$ and is well defined in $\mathbb{R}$

Proof:

a) The sum is cauchy: Let $\varepsilon \in \mathbb{Q}_{>0}$. Since q_n is cauchy, $\exists N \in \mathbb{Z}_{>0}$ such that $|q_n - q_m| < \frac{\varepsilon}{4}$ for $n,m \geq N$. Similarly for r_n, $\exists M \in \mathbb{Z}_{>0}$ s.t. $|r_n - r_m| < \frac{\varepsilon}{4}$ for $n,m \geq M$. Sps $W = \max\{N, M\}$ for $m,n \geq W$, $\forall n \geq W$ $|q_n + r_n - (q_m + r_m)| < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2} < \varepsilon$.

↳ $\leq |q_n - q_m| + |r_n - r_m|$

b) Well defined: Does not depend on particular representatives.

Sps we have $q_n \sim q_n'$, $r_n \sim r_n'$. By def. of $\sim$, $q_n - q_n' = 0$, and $r_n - r_n' = 0$. By limit laws, $q_n - q_n' + r_n - r_n' = 0$, so $(q_n + r_n) - (q_n' + r_n') = 0$.

Prop: Let (q_n), (r_n) be Cauchy seq. in $\mathbb{Q}$. Then $\cdot$ is well defined in $\mathbb{Q}$.

Def: $|[(q_n)]| := [(|q_n|)]$

$\mathbb{R}_{>0} := \{x \in \mathbb{R} : |x| = x\}$

$x \leq y$ is def. to mean that $y - x \in \mathbb{R}_{>0}$

$\mathbb{Q} \subseteq \mathbb{R}$ in the sense that $[(q, q \dots)]$ corresponds to the rational # $q \in \mathbb{Q}$.

Ex: $\mathbb{Q}$: $\frac{1}{2} \longmapsto$ $\mathbb{R}$: $[(\frac{1}{2}, \frac{1}{2} \dots)]$ $\qquad$ $\mathbb{Z}$: $2 \longmapsto$ $\mathbb{Q}$: $[(2, 2 \dots)]$

Remark: If $x = [(q_n)]$ for some seq. w/ $q_n \geq 0$ for all n, then $x \geq 0$. The same is true if $q_n \geq 0$ for all $n \geq N$ (for some $N \in \mathbb{Z}_{>0}$) bc if $\tilde{q}_n \begin{cases} q_n, & n \geq N \\ 0, & n < N \end{cases}$ then $\tilde{q}_n \geq 0$ and $q_n \sim \tilde{q}_n$

Prop: (Archimedian Property of $\mathbb{R}$) $\forall x \in \mathbb{R}$, $\exists N \in \mathbb{Z}_{>0}$ s.t. $(N > x)$

Proof: Let $x = [(q_n)]$. Then, since q_n is cauchy, $\exists$ an $r \in \mathbb{Q}_{>0}$ s.t. $|q_n| < r$ for all n. By the A.P. for $\mathbb{Q}$, $\exists$ an $N \in \mathbb{Z}_{>0}$ s.t. $r < N$. We claim $x < N$.

To prove this, we show that $x \leq N$ and $x \neq N$.

First, $N - x = [(N)_n] \cdot [(q_n)_{n=1}^{\infty}] = [(N - q_n)_{n=1}^{\infty}]$ (sequence of just N) and $|N - x| = [(|N - q_n|)_{n=1}^{\infty}]$.

Leave on hold, the got it wrong? Proof posted online. TBD.

We can now define conv. seq. Cauchy seq... in $\mathbb{R}$.

↳ If $|\dots| < \varepsilon$ for all $\varepsilon \in \mathbb{Q}_{>0}$, then $|\dots| < \varepsilon$ for all $\varepsilon \in \mathbb{R}$ by A.P. in $\mathbb{R}$. bc given $\varepsilon \in \mathbb{R}_{>0}$, $\exists N \in \mathbb{Z}_{>0}$ s.t. $N > \frac{1}{\varepsilon}$, i.e. $\frac{1}{N} < \varepsilon$, and $\frac{1}{N} \in \mathbb{Q}_{>0}$.

Proof: If $(x_n)_{n=1}^{\infty}$ is Cauchy seq. of rational #'s in $\mathbb{R}$, then $x_n \xrightarrow[n \to \infty]{} [(x_n)_{n=1}^{\infty}]$ in $\mathbb{R}$ $\}$ x

eg. class of const. seq. ↳ conv. to eq. class of same #'s

$x_1 = [(x_1, x_1, \dots)]$, $x_2 = [(x_2, x_2 \dots)]$, etc.

↳ $x = [(x_1, x_2, x_3 \dots)]$ converges to x.

("sequence of eq. classes of sequences")

Proof: Given an $\varepsilon \in \mathbb{Q}_{>0}$, since (x_n) is cauchy, $\exists$ $N \in \mathbb{Z}_{>0}$ s.t. $|x_m - x_n| < \frac{\varepsilon}{2}$ for $m,n \geq N$. Then, for $n \geq N$, $|[(x_m)_{m=1}^{\infty}] - [(x_n)_{m=1}^{\infty}]| = [(|x_m - x_n|)_{m=1}^{\infty}]$

(const. seq.)

$\leq \frac{\varepsilon}{2} < \varepsilon$. (less than or equal to)

Warning: If $x = [(q_n)]$ and $q_n > 0$ for all n, this does not imply that $x > 0$.

↳ $\boxed{\text{Ex}}$ $x = [(\frac{1}{n})] = 0$ $\nexists$ positive number