

Linear stationary iterative methods

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Let $A \in \mathbb{R}^{n \times n}$ be invertible and $b \in \mathbb{R}^n$. An **iterative method** for solving $Ax = b$ generates a sequence of iterates $(x^{(k)})_{k=1}^{\infty}$ approximating the exact solution x^* , given an initial guess $x^{(0)}$. We can express this as $x^{(k+1)} := \phi_{k+1}(x^{(0)}, \dots, x^{(k)}; A, b)$ for some functions ϕ_k ; if these functions are eventually independent of k , the method is said to be **stationary**.

We will consider stationary iterative methods of the form $x^{(k+1)} := Gx^{(k)} + f$, called **(first-degree) linear stationary iterative methods**. Such methods are typically derived from a **splitting** $A = M - N$, where M is an invertible matrix that “approximates” A but is easier to solve linear systems with. Specifically, since $Mx^* = Nx^* + b$, we take $G := M^{-1}N = I - M^{-1}A$ and $f = M^{-1}b$ so that the exact solution is a fixed point of the iteration. Equivalently, we can view $x^{(k+1)} = x^{(k)} + M^{-1}(b - Ax^{(k)})$ as a correction of $x^{(k)}$ based on the **residual** $r^{(k)} := b - Ax^{(k)}$. We also note that the **error** $e^{(k)} := x^* - x^{(k)}$ satisfies $e^{(k+1)} = Ge^{(k)}$, so the convergence of a splitting method depends on the properties of its **iteration matrix** G .

Splitting methods

Let L , D , and U denote the *strictly* lower, diagonal, and *strictly* upper triangular parts of A . The splittings for some basic linear stationary iterative methods are as follows.

Method	M
Jacobi	D
ω -Jacobi ($\omega \neq 0$)	$\frac{1}{\omega}D$
Gauss-Seidel	$L + D$
ω -Gauss-Seidel/successive overrelaxation (SOR) ($\omega \neq 0$)	$L + \frac{1}{\omega}D$
Symmetric successive overrelaxation (SSOR) ($\omega \neq 0, 2$)	$\frac{\omega}{2-\omega}(L + \frac{1}{\omega}D)D^{-1}(U + \frac{1}{\omega}D)$
Richardson ($\alpha \neq 0$)	$\frac{1}{\alpha}I$

The parameter ω is known as the **relaxation/damping** parameter and arises from taking $x^{(k+1)} = (1 - \omega)x^{(k)} + \omega\hat{x}^{(k+1)}$, where $\hat{x}^{(k+1)}$ denotes the result of applying the corresponding nonparametrized ($\omega = 1$) method to $x^{(k)}$. If $\omega < 1$, the method is said to be **underrelaxed/underdamped**; if $\omega > 1$, it is said to be **overrelaxed/overdamped**.

The SSOR method arises from performing a “forward” ω -Gauss-Seidel step with $M = L + \frac{1}{\omega}D$ followed by a “backward” ω -Gauss-Seidel step with $M = U + \frac{1}{\omega}D$.

Convergence theorems

Clearly, since $e^{(k)} = G^k e^{(0)}$, if $\|G\| < 1$ for some operator norm, then the method **converges** in the sense that $x^{(k)} \rightarrow x^*$ for all $x^{(0)}$.¹ More generally, we see that the method is convergent if and only if $G^k \rightarrow 0$, which in turn depends on the **spectral radius** $\rho(G)$ of G , the maximum of the absolute values of its eigenvalues when regarded as a complex matrix.²

Namely, if (λ, v) is an eigenpair of G with $\|v\| = 1$, then $|\lambda|^k = |\lambda^k| = \|G^k v\| \leq \|G^k\|$ for the induced operator norm, so $\rho(G)^k = \rho(G^k) \leq \|G^k\|$. Thus, if $G^k \rightarrow 0$, then $\rho(G) < 1$. In fact, the converse is also true.

Let $G \in \mathbb{C}^{n \times n}$. Then $G^k \rightarrow 0$ if and only if $\rho(G) < 1$ (such a matrix is called **convergent**).

Proof. It remains to show that $G^k \rightarrow 0$ if $\rho(G) < 1$. Let UTU^* be a Schur factorization of G and let D and N denote the diagonal and strictly upper triangular parts of T . Since the product of a diagonal matrix and a strictly upper triangular matrix is strictly upper triangular, and the product of n (or more) strictly upper triangular $n \times n$ matrices is zero, for all $k \geq n$, we have

$$\|G^k\|_2 = \|(D + N)^k\|_2 \leq \sum_{j=0}^{n-1} \binom{k}{j} \|D\|_2^{k-j} \|N\|_2^j = \sum_{j=0}^{n-1} \binom{k}{j} \rho(G)^{k-j} \|N\|_2^j \rightarrow 0. \quad \blacksquare$$

Using this fact, we can also prove a well-known formula for the spectral radius.³

Gelfand's formula

Let $G \in \mathbb{C}^{n \times n}$ and $\|\cdot\|$ be an operator norm. Then $\rho(G) = \lim_{k \rightarrow \infty} \|G^k\|^{1/k}$.

Proof. We previously saw that $\rho(G) \leq \|G^k\|^{1/k}$ for all k , so $\rho(G) \leq \liminf_{k \rightarrow \infty} \|G^k\|^{1/k}$. On the other hand, if $\varepsilon > 0$ is arbitrary and $\hat{G} := \frac{G}{\rho(G) + \varepsilon}$, then $\rho(\hat{G}) < 1$, so by the preceding result, $\|\hat{G}^k\| \leq 1$ for all sufficiently large k , which is to say that $\|G^k\| \leq (\rho(G) + \varepsilon)^k$. Hence we also have $\limsup_{k \rightarrow \infty} \|G^k\|^{1/k} \leq \rho(G) + \varepsilon$. \blacksquare

As $\|e^{(k)}\| \leq \|G^k\| \|e^{(0)}\| \approx \rho(G)^k \|e^{(0)}\|$ for large k , the rate of convergence can often be estimated using the spectral radius of the iteration matrix. More precisely, a direct computation shows that if G is diagonalizable and has a unique dominant eigenvalue, then $\frac{\|e^{(k+1)}\|}{\|e^{(k)}\|} \sim \rho(G)$, provided that $e^{(0)}$ has a nonzero component in the corresponding eigenspace.

General matrices

If the Jacobi method converges, then the ω -Jacobi method converges for $\omega \in (0, 1]$.

Proof. For the ω -Jacobi method, we have $G = (1 - \omega)I + \omega G_J$, where G_J denotes the iteration matrix for the Jacobi method. Hence $\rho(G) \leq (1 - \omega) + \omega \rho(G_J) < 1$. \blacksquare

If the ω -Gauss-Seidel method converges, then $\omega \in (0, 2)$.

Proof. For the ω -Gauss-Seidel method, we have $G = (L + \frac{1}{\omega}D)^{-1}((\frac{1}{\omega} - 1)D - U)$, so $\det(G) = \det(\frac{1}{\omega}D)^{-1} \det((\frac{1}{\omega} - 1)D) = \det((1 - \omega)I) = (1 - \omega)^n$. On the other hand, the determinant of G is the product of its eigenvalues, so $|1 - \omega|^n \leq \rho(G)^n < 1$. \blacksquare

If the SSOR method converges, then $\omega \in (0, 2)$.

Proof. For the SSOR method, we have $G = G_b G_f$, where G_f and G_b denote the iteration matrices for the forward and backward ω -Gauss–Seidel methods. Arguing as in the preceding proof, we obtain $|1 - \omega|^{2n} \leq \rho(G)^n < 1$. ■

Symmetric positive definite matrices

If A is symmetric positive definite (SPD), then A is invertible and all the splitting methods above are applicable since its diagonal entries must be positive. Recall also that symmetric matrices are partially ordered by the Loewner order \prec in which $A \prec B$ if and only if $B - A$ is SPD, and that an SPD matrix A defines an inner product $\langle x, y \rangle_A := \langle Ax, y \rangle_2$.

If A is SPD and $A \prec M + M^\top$, then $\|G\|_A < 1$.

Proof. Let x be a vector with $\|x\|_A = 1$ such that $\|G\|_A = \|Gx\|_A$ (which exists by the extreme value theorem) and let $y := M^{-1}Ax$. Then

$$\begin{aligned} \|G\|_A^2 &= \|x - y\|_A^2 \\ &= 1 - \langle x, y \rangle_A - \langle y, x \rangle_A + \langle y, y \rangle_A \\ &= 1 - \langle (M + M^\top - A)y, y \rangle_2 < 1. \quad \blacksquare \end{aligned}$$

Convergence for SPD matrices

Let A be SPD.

- If $A \prec \frac{2}{\omega}D$, then the ω -Jacobi method converges.
- If $\omega \in (0, 2)$, then the ω -Gauss–Seidel method converges.
- If $\omega \in (0, 2)$, then the SSOR method converges.
- If $\alpha \in (0, \frac{2}{\rho(A)})$, then the Richardson method converges.

Moreover, if the eigenvalues of A are $\lambda_1 \geq \dots \geq \lambda_n > 0$, then $\rho(G)$ is minimized when $\alpha = \alpha^* := \frac{2}{\lambda_1 + \lambda_n}$, in which case $\|e^{(k+1)}\|_2 \leq (1 - \frac{2}{\kappa+1})\|e^{(k)}\|_2$, where κ is the 2-norm condition number of A .

Proof. The convergence statements follow immediately from the preceding result.

For the Richardson method, $\rho(G(\alpha)) = \max_i |1 - \alpha\lambda_i| = \max\{1 - \alpha\lambda_n, \alpha\lambda_1 - 1\}$, so $\rho(G(\alpha)) = 1 - \alpha\lambda_n \geq \rho(G(\alpha^*))$ when $\alpha \leq \alpha^*$ and $\rho(G(\alpha)) = \alpha\lambda_1 - 1 \geq \rho(G(\alpha^*))$ when $\alpha \geq \alpha^*$. Finally, since G and A are normal, we have $\|G(\alpha^*)\|_2 = \rho(G(\alpha^*)) = 1 - \frac{2}{\kappa+1}$. ■

Diagonally dominant matrices

- A is **weakly diagonally dominant (WDD)** if every row i is WDD: $|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$.
- A is **strictly diagonally dominant (SDD)** if every row i is SDD: $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$.
- The **directed graph** of A is $\mathcal{G}_A = (V, E)$ with $V = \{1, \dots, n\}$ and $(i, j) \in E$ if and only if $a_{ij} \neq 0$.
 - A is **irreducible** if \mathcal{G}_A is strongly connected.
- A is **irreducibly diagonally dominant (IDD)** if it is irreducible, WDD, and some row is SDD.
- A is **weakly chained diagonally dominant (WCDD)** if it is WDD and for every row i , there exists an SDD row j with a path from i to j in \mathcal{G}_A . (Thus, SDD and IDD matrices are both WCDD.)

If A is WCDD, then A is invertible.

Proof. Suppose for the sake of contradiction that there exists an $x \in \ker(A)$ with $\|x\|_\infty = 1$, and let i_1 be such that $|x_{i_1}| = 1$. Let $(x_{i_1}, \dots, x_{i_k})$ be a path in \mathcal{G}_A such that row i_k is SDD. Since $\sum_j a_{i_1 j} x_j = 0$, we have

$$|a_{i_1 i_1}| = |-a_{i_1 i_1} x_{i_1}| \leq \sum_{j \neq i_1} |a_{i_1 j}| |x_j| \leq \sum_{j \neq i_1} |a_{i_1 j}|,$$

so row i_1 is not SDD. However, since it is WDD, equality must hold throughout, which implies that $|x_{i_2}| = 1$ because $a_{i_1 i_2} \neq 0$. Iterating this argument, we ultimately deduce that row i_k is not SDD, which is a contradiction. ■

As a result, if A is WCDD, then all the splitting methods above are applicable since its diagonal entries must be nonzero (otherwise, it would have a zero row and fail to be invertible).

We also note that this immediately implies the **Levy-Desplanques theorem**: if A is SDD, then A is invertible. This, in turn, is equivalent to the **Gershgorin circle theorem**: if λ is an eigenvalue of A , then $|\lambda - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| =: r_i$ for some i (in other words, $\lambda \in B_{r_i}(a_{ii})$ for some i). Similarly, if A is IDD, then A is invertible; or equivalently, if A is irreducible and λ is an eigenvalue of A such that $|\lambda - a_{ii}| \geq r_i$ for every i , then $|\lambda - a_{ii}| = r_i$ for every i .

Convergence for WCDD matrices

If A is WCDD, then the ω -Jacobi and ω -Gauss-Seidel methods converge for $\omega \in (0, 1]$.

Proof. If $|\lambda| \geq 1$, then $|\frac{\lambda-1}{\omega} + 1| \geq |\lambda|$, so $(\lambda - 1)M + A$ has the same WDD/SDD rows and directed graph as A . Hence $\lambda I - G = M^{-1}((\lambda - 1)M + A)$ is invertible for such λ , so $\rho(G) < 1$. ■

Consistently ordered matrices

In this section, we *assume* that the diagonal entries of A are nonzero and consider the iteration matrices $G_J := -D^{-1}(L + U)$ and $G_{GS}(\omega) := (L + \frac{1}{\omega}D)^{-1}(\frac{1-\omega}{\omega}D - U)$ for the Jacobi and ω -Gauss-Seidel methods, respectively.

- A has **property $A_{q,r}$** ($q, r \geq 1$) if there exists a partition $\{S_k\}_{k=1}^p$ of $\{1, \dots, n\}$ with $p = q + r$ such that if $a_{ij} \neq 0$ for some $i \neq j$ and $i \in S_k$, then either $i \in S_1 \cup \dots \cup S_q$ and $j \in S_{k+r}$, or $i \in S_{q+1} \cup \dots \cup S_p$ and $j \in S_{k-q}$.
 - A has **property A_p** (or is **p -cyclic**) ($p \geq 2$) if it has property $A_{1,p-1}$. (Property A_2 is usually called “property A ” and is equivalent to the directed graph of A being bipartite, ignoring loops.)
- A is **(q, r) -consistently ordered** ($q, r \geq 1$) if there exists a partition $\{S_k\}_{k=1}^p$ of $\{1, \dots, n\}$ (not necessarily with $p = q + r$) such that if $a_{ij} \neq 0$ and $i \in S_k$, then $i \in S_1 \cup \dots \cup S_{p-r}$ and $j \in S_{k+r}$ if $i < j$, or $i \in S_{q+1} \cup \dots \cup S_p$ and $j \in S_{k-q}$ if $i > j$. (A $(1, p-1)$ -consistently ordered matrix is sometimes called “consistently ordered p -cyclic”.)

If A has property $A_{q,r}$, then it has a **(q, r) -ordering vector**: a $\gamma \in \mathbb{Z}^n$ such that if $a_{ij} \neq 0$ for some $i \neq j$, then $\gamma_j - \gamma_i = r$ or $\gamma_j - \gamma_i = -q$; and for each $k \in \{1, \dots, p := q + r\}$, there exists an i such that $\gamma_i \equiv k \pmod{p}$. Conversely, if A has such a vector, then A has property $A_{q,r}$.

If A is (q, r) -consistently ordered, then it has a **compatible (q, r) -ordering vector**: a (q, r) -ordering vector γ such that if $a_{ij} \neq 0$ for some $i \neq j$, then $\gamma_j - \gamma_i = r$ or $\gamma_j - \gamma_i = -q$ according as $i < j$ or $i > j$. Conversely, if A has such a vector, then A is (q, r) -consistently ordered.

Proof. If A has property $A_{q,r}$ and $\{S_k\}_{k=1}^p$ is as in the definition of this property, take $\gamma_i := k$, where k is such that $i \in S_k$. Conversely, if A has a (q, r) -ordering vector γ , take $S_k := \{i : \gamma_i \equiv k \pmod{p}\}$, where $p = q + r$. The same constructions apply in the consistently ordered case. ■

This proof also shows that we may assume without loss of generality that $p = q + r$ in the definition of a (q, r) -consistently ordered matrix. We also note that A has property $A_{q,r}$ if and only if it can be *symmetrically permuted* (that is, conjugated by a permutation matrix) to a (q, r) -consistently ordered matrix, since the former property is invariant under symmetric permutations.

If A is (q, r) -consistently ordered, then $\det(\lambda D + \alpha^{-q}L + \alpha^r U)$ is independent of α (for all λ).

Proof. Suppose that σ is a permutation of $\{1, \dots, n\}$ with $a_{\sigma(i),i} \neq 0$ for all i . Then

$$\prod_{i=1}^n (\lambda D + \alpha^{-q}L + \alpha^r U)_{\sigma(i),i} = \lambda^{n-(n_L+n_U)} (\alpha^{-q})^{n_L} (\alpha^r)^{n_U} \prod_{i=1}^n a_{\sigma(i),i},$$

where $n_L := \#\{i : \sigma(i) > i\}$ and $n_U := \#\{i : \sigma(i) < i\}$. Now if γ is a compatible (q, r) -ordering vector for A , then $\sum_{\sigma(i) > i} \gamma_i - \gamma_{\sigma(i)} = -qn_L$ and $\sum_{\sigma(i) < i} \gamma_i - \gamma_{\sigma(i)} = rn_U$, so $-qn_L + rn_U = \sum_{\sigma(i) \neq i} \gamma_i - \gamma_{\sigma(i)} = \sum_{\sigma(i) \neq i} \gamma_i - \sum_{j \neq \sigma^{-1}(j)} \gamma_j = 0$. Hence any product of the form above – of which the determinant is a sum – is independent of α . ■

Suppose that A is (q, r) -consistently ordered and let $p := q + r$. Then λ is an eigenvalue of $G_{GS}(\omega)$ if and only if

$$(\lambda + \omega - 1)^p = \lambda^r \omega^p \mu^p$$

for some eigenvalue μ of G_J .

Proof. Since D is invertible, μ is an eigenvalue of G_J if and only if $D(\mu I - G_J) = \mu D + L + U$ is singular. Similarly, λ is an eigenvalue of $G_{GS}(\omega)$ if and only if $(L + \frac{1}{\omega}D)(\lambda I - G_{GS}(\omega)) = \frac{\lambda + \omega - 1}{\omega}D + \lambda L + U$ is singular, or equivalently for λ nonzero, $\lambda^{-r/p} \cdot \frac{\lambda + \omega - 1}{\omega}D + (\lambda^{-1/p})^{-q}L + (\lambda^{-1/p})^r U$ is singular. Thus, the result follows from the determinantal invariance property above (note that 0 is an eigenvalue of $G_{GS}(\omega)$ if and only if $\omega = 1$, while 0 is always an eigenvalue of G_J). ■

In the special case of a $(1, p - 1)$ -consistently ordered matrix, we obtain the following corollary.

Convergence for consistently ordered matrices

Suppose that A is $(1, p - 1)$ -consistently ordered. Then $\rho(G_{GS}(1)) = \rho(G_J)^p$. In particular, the Gauss–Seidel method converges if and only if the Jacobi method converges.

Furthermore, in this case, if μ is an eigenvalue of G_J , then so is $\theta\mu$ for each p^{th} root of unity θ , since $\mu D + L + U$ is singular if and only if $\theta(\mu D + \theta^{-1}L + \theta^{p-1}U)$ is.

Optimal Gauss–Seidel relaxation parameter for consistently ordered matrices

Suppose that A is $(1, p - 1)$ -consistently ordered, that the Jacobi method converges, and that the eigenvalues of G_J^p are real and nonnegative. Then the ω -Gauss-Seidel method converges for all $\omega \in (0, \frac{p}{p-1})$, and if ω_* is the unique solution in $(0, \frac{p}{p-1})$ of

$$[\rho(G_J)\omega_*]^p = \left(\frac{p}{p-1}\right)^p (p-1)(\omega_* - 1),$$

then for all $\omega \neq \omega_*$, we have

$$\rho(G_{GS}(\omega_*)) = (p-1)(\omega_* - 1) < \rho(G_{GS}(\omega)).$$

Proof. We sketch the proof for the case $p = 2$; for details and the general case, see [Varga \(1959\)](#).

When A is $(1, 1)$ -consistently ordered, the eigenvalues λ of $G_{GS}(\omega)$ are the zeroes of the equations $(\lambda + \omega - 1)^2 = \lambda\omega^2\mu^2$ for μ in the set of eigenvalues of G_J . For a fixed $\mu \geq 0$, these zeroes are the abscissae of the intersection points of the line $y = \frac{\lambda + \omega - 1}{\omega}$ (which has slope $\frac{1}{\omega}$ and passes through $(1, 1)$) and the curve $y = \pm\lambda^{1/2}\mu$ in the λ - y plane. The larger abscissa is seen to *decrease* as ω increases from 0 until the line and the curve are tangent, beyond which the equation has two conjugate complex zeroes of modulus $\omega - 1$, which *increases* as ω increases. The abscissa of the point of tangency is defined by the equations $\frac{\lambda + \omega - 1}{\omega} = \lambda^{1/2}\mu$ and $\frac{1}{\omega} = \frac{1}{2}\lambda^{-1/2}\mu$, and is maximal when $\mu = \rho(G_J)$. Thus, $[\rho(G_J)\omega_*]^2 = 2^2\lambda$, where $\lambda = \omega_* - 1 = \rho(G_{GS}(\omega_*))$. ■

1. In other words, if $\|G\| < 1$, then $G^k \rightarrow 0$ strongly (because $G^k \rightarrow 0$ in norm). 

2. In other words, $G^k \rightarrow 0$ strongly if and only if $G^k \rightarrow 0$ in norm (because G is an operator on a finite-dimensional space). 

3. This formula remains true if G is a continuous linear operator on a Banach space X . Consequently, in this setting, we still have that $G^k \rightarrow 0$ in norm if and only if $\rho(G) < 1$. These are in turn equivalent to the invertibility of $I - G$ and the convergence of the fixed-point iteration of $x \mapsto Gx + f$ for all $f \in X$ (to $(I - G)^{-1}f$). 