PROJECT DESCRIPTION

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My research focuses on computational algebraic topology. I study both the general, algebraic techniques and the specific applications of these to spaces and spectra of interest. In this proposal, I will briefly provide some background to contextualize my research, then I will describe previous and on-going projects. I will outline the questions I intend to study with the support of the NSF, and then I will finish by discussing the broader impact of this project.

1. Background

Algebraic topology and algebraic geometry have been closely connected. Beginning with work of Morava and Quillen, the influence of algebraic geometry on algebraic topology has been understood in the context of formal groups and the chromatic filtration [42, 44]. This eventually led to the Hopkins-Miller theorem, the theory of topological modular forms, and Lurie's derived algebraic geometry [49, 24, 36, 39]. The theory of algebraic K-theory provides another connection, as both varieties and commutative ring spectra have well-defined algebraic K-theories. These groups are very difficult to compute, and both topology and geometry have developed techniques to determine them. This connection was further explored by Voevodsky in his construction of the motivic homotopy category.

The filtration of formal groups in characteristic p by those of height at most n lifts topologically to the chromatic filtration of stable homotopy. In this, any finite spectrum admits a resolution by its $E(n)$-localizations, where $E(n)$ is a spectrum naturally associated to formal groups of height at most n, and these localizations play the role of the aforementioned filtered pieces [37]. The filtration quotients for this tower are the localizations with respect to Morava $K(n)$, a spectrum whose associated formal group, F_n, has height exactly n. Even understanding the filtration quotients for the sphere spectrum is a difficult undertaking, with little known beyond heights 1 and 2.

In the 1970s, Morava showed that the E_2 term for the Adams-Novikov spectral sequence for the homotopy groups of the $K(n)$-local sphere, $L_{K(n)}(S^0)$, is computable using only F_n [42]. If $\mathbb{G}_n$ is the automorphisms group of F_n, extended by the Galois group $\mathrm{Aut}_{\mathbb{F}_p}(\mathbb{F}_{p^n})$, and if E_{n*} is the Lubin-Tate ring corepresenting deformations of F_n [38], then the E_2 term is the continuous cohomology of $\mathbb{G}_n$ with coefficients in E_{n*} [42, 18]. If $p-1$ divides n, then $\mathbb{G}_n$ has infinite cohomological dimension coming from finite p-primary subgroups [30]. Thus the cohomology of these subgroups with coefficients in E_{n*} provides an approximation to $\pi_* L_{K(n)} S^0$. For heights 1 and 2, the homotopy of the $K(n)$-local sphere can actually be computed from a finite resolution involving these approximations [45, 8, 23].

The Hopkins-Miller theorem shows that these algebraic approximations of the $K(n)$-local sphere can be rigidified to spectra [35, 49]. A consequence of Landweber's exact functor theorem is the existence of a spectrum E_n which carries the universal deformation of F_n. It was shown that E_n is a highly structured ring spectrum, and Hopkins and Miller showed that the group $\mathbb{G}_{\mathrm{n}}$ acts on this spectrum via maps of highly structured ring spectra. The real power of this approach is that one can take homotopy fixed points with respect to finite subgroups G sitting inside $\mathbb{G}_{\mathrm{n}}$, getting new highly structured ring spectra: the "higher real K-theories" $EO_n(G)$ of Hopkins and Miller. The Adams-Novikov E_2 term for these spectra is exactly the cohomology of G with coefficients in E_{n*} considered above. Even so, except for a very limited number of cases ($n = 1, 2$ at all primes, [34]), their homotopy groups are still unknown. The part of the homotopy of EO_{p-1} in higher Adams-Novikov filtration is also known at p, but even here the zero line is largely a mystery [26]. The key missing step is a good understanding of the cohomology of the p-torsion part of maximal finite subgroups with coefficients in the homotopy of the Lubin-Tate spectra.

The chromatic approach has been used by Rognes to attempt to understand an equally mysterious object: $K(S^0)$, the algebraic K-theory of the sphere spectrum. Algebraic K-theory is functorial in the ring, so the hope is that $K(S^0)$ can be understood from K applied to the chromatic tower. One of Rognes current programs is to use a kind of Galois descent and the Hopkins-Miller theorem to understand $K(L_{K(n)} S^0)$ from $K(E_n)$. However, even these easier K-spectra are difficult to understand, so one approach is to consider possibly more tractable spectra like $BP\langle n\rangle$. If these are sufficiently structured, then the topological Hochschild homology (THH) of them can be computed [13, 14, 3], and from this, one can attempt to apply the Bökstedt-Hsiang-Madsen topological cyclic homology machinery to approximate K [12]. The $K(n)$-localization of $BP\langle n\rangle$ is essentially E_n, and since this is some sort of "nilpotent completion" and localization, the algebraic K-theories should be related.

Rognes has conjectured that a spectrum like $BP\langle n\rangle$, which as a given chromatic type, undergoes a chromatic red shift upon application of $K(-)$. In other words, if E is type n, then Rognes conjectured that $K(E)$ admits a v_{n+1}-self map. This has computationally been verified for low heights by Ausoni and Rognes, working with ku and the Adams summand ℓ and showing that the $V(1)$-homology of the algebraic K-theory has a v_2-self map [5, 4]. However, integral computations are currently beyond our reach, and little is known for spectra like ko and tmf, spectra which better approximate the sphere.

Algebraic K-theory also provides a link to the motivic homotopy. Though this field has had tremendous success, culminating in Voevodsky's Fields Medal for solving the Milnor conjecture [51], there are few computations, even working over $\mathbb{R}$ or $\mathbb{C}$. Working over $\mathbb{C}$, there are also close connections to classical stable homotopy theory, and one can recover the classical Adams spectral sequence from the motivic one by inverting a single homotopy class. This endows the classical Adams spectral sequence with a third grading, arising from the motivic weight, and we can then use both spectral sequences to deduce differentials, extensions, and patterns in the other.

2. Previous Research

2.1. The eo_{p-1} homology of $B\Sigma_p$. The $H\mathbb{F}_p$ based Adams spectral sequence in the category of eo_{p-1}-modules has a simple E_2-term [6], computable in terms of the Hopf algebra

$$\pi_*(H\mathbb{F}_p \underset{eo_{p-1}}{\wedge} H\mathbb{F}_p).$$

Andre Henriques and I determined the structure of this Hopf algebra. For $p = 2$, this recovers the classical "change of rings" result for computing ko-homology, and for $p = 3$, this provides a technique for computing 3-local tmf-homology [31]. For larger primes, this presupposes the existence of such a spectrum.

I used this spectral sequence to compute the p-local eo_{p-1} homology of the classifying space $B\Sigma_p$. The problem is analogous in many ways to Mahowald's computation of the ko-homology of $\mathbb{R}P^\infty$, and my computational approach mirrors Mahowald's by applying judicious choices of filtration to the homology of $B\Sigma_p$.

2.2. The topological Hochschild homology of ℓ and ko. THH is the starting point for computing the algebraic K-theory of structured ring spectra. Given a commutative S-algebra A (in the sense of [21]), we can define $THH^S(A)$ to be the derived smash product of A with itself as A-bimodules:

$$THH^S(A) = A \underset{A\wedge A}{\wedge} A.$$

Bökstedt discovered a spectral sequence converging to $H_*THH^S(A)$ with E_1 term given by the Hochschild homology of H_*A:

$$E_1 = HH(H_*A),$$

and he computed this spectral sequence for $A = H\mathbb{F}_p$ and $A = H\mathbb{Z}$ [13, 14]. McClure and Staffeldt used Bökstedt's computation to find the $V(0)$ and $V(1)$-homology of $THH(\ell)$ [41], and these computations were expanded upon by Angeltveit and Rognes [3].

Using recent advances in the theory of structured ring spectra, Vigleik Angeltveit, Tyler Lawson, and I recast their computations in terms of topological Hochschild homology with coefficients in a bimodule and extend their results to p-local statements about the homotopy of $THH(\ell)$ [2]. The quotient maps of the ℓ-bimodules $k(1) = \ell/p$, $H\mathbb{Z} = \ell/v_1$, and $H\mathbb{F}_p = \ell/(p, v_1)$ give rise to a family of Bockstein spectral sequences for replacing p and v_1. These could be run to determine the structure of the homotopy groups.

We found that as an ℓ_*-module, the homotopy splits

$$THH_*(\ell) = \ell_* \oplus \Sigma^{2p-1} F \oplus T,$$

where F is an ℓ_*-submodule of $\ell_* \otimes \mathbb{Q}$ and where T is an infinitely generated torsion ℓ-module. This module can be compactly represented as

$$T = \bigoplus_{n \geq 0} \bigoplus_{k=1}^{p-1} \Sigma^{2kp^{n+2}+2(p-1)} T_n,$$

where T_0 is $\ell_*/(p, v_1^p)$, and T_n is built out of p copies of T_{n-1} with the first and last attached by a tower of v_1-multiplications.

Since ku is an E_∞ ko-algebra, we were able to apply similar methods to also compute the 2-local homotopy of $THH(ko)$. Here there is a similar square of bimodules, though we must also consider the effect of reduction modulo η. We find that $THH(ko; ku)$ is very similar to $THH(ku)$, and the η-Bockstein spectral sequence is quite computable, though less easy to describe. As a shocking consequence, we found that apart from the ko_*-summand of $THH(ko)$, the class η^2 acts as zero in the homotopy.

As an application of this computation, using a result of Blumberg-Cohen-Schlichtkrull concerning the THH of Thom spectra, we have found a conceptually easier proof that ku and ko are not Thom spectra. If we assume that these are the Thom spectra of a triple loop map, then there is not finite complex whose ko or ku-homology is the homotopy of the first few ko or ku-cells of THH. This non-existence can be shown directly with the Adams spectral sequence.

2.3. **The 5-local homotopy of eo_4.** Using the obvious connective version of the Hopf algebroid found by Hopkins-Gorbounov-Mahowald [25], I computed the Adams-Novikov E_2-term for the homotopy groups of EO_4 and its conjectural connective cover eo_4 [32]. The method was similar to that employed by Bauer in his description of the homotopy of tmf [7], as I used a series of Bockstein spectral sequences to build up the Adams-Novikov E_2-term from simpler, classically known cohomology computations.

Using work of Hopkins and Miller [26], I also computed the Adams-Novikov differentials and solve the various multiplicative extension problems for eo_4, for $(eo_4)/5$, and for $(eo_4)/(5, v_1)$. In particular, one sees immediately a result of Hopkins that everything in the image of J in the homotopy groups of spheres maps to zero under the Hurewicz homomorphism. Moreover, one seems quite explicitly Gross-Hopkins and Mahowald-Rezk dualities in the homotopy [33, 40], again mirroring the situation with tmf at the prime 3.

This computation has the final application of producing full homotopy ring of $eo_4[\Delta^{-1}]$ and of EO_4. In particular, we find as a ring the zero line of the Adams-Novikov spectral sequence for EO_4, resolving a difficult problem in invariant theory. These methods apply to all of the spectra EO_{p-1}, though not in a sufficiently practical way to provide a complete, prime independent, description. Pending the computation of an analogous Hopf algebroid, these methods should yield insight also into the Adams-Novikov zero line for $EO_{f(p-1)*}$.

2.4. **The existence of a v_2^{32} self map on $S^0/(2, v_1^4)$.** One of the early successes of tmf was the determination of a minimal v_2-self map on the generalized Smith-Toda complex $M(1,4)$, the 4 cell complex which is the cone on both 2 and v_1^4.

The Periodicity Theorem of Devinatz-Hopkins-Smith shows that some power of v_2 occurs as a self-map on this finite spectrum [20], though it does not provide a lower bound for the required exponent. This power is relevant computationally: it determines the periodicity of the family of v_2-periodic elements.

Correcting work of Mahowald and Davis [17], Mike Hopkins and Mark Mahowald sketched an argument showing that v_2^{32} is the smallest surviving power of v_2 in the Adams spectral sequence for $M(1,4)$, Mark Behrens and I completed the argument, filling in necessary details [9]. At its heart, the argument is one about the survival of the class v_2^{32} in the Adams spectral sequence for $M(1,4)$. Since v_2^{32} is in the 192-stem, this is well beyond the range accessible to standard computation.

To surmount this problem, we use a modified Adams spectral sequence and an algebraic tmf resolution. For the former, we note that we can view the Adams spectral sequence as taking input from the derived category of comodules over the dual Steenrod algebra. In other words, if we take the cone on a map of Adams filtration s, then we can modify the Adams resolution so that the new classes in the Adams E_2-term occur in filtrations at least $s-1$. This allows for much easier identifications of the possible targets for a differential on v_2^{32}.

Having narrowed the scope of the problem using the modified Adams spectral sequence, we can explicitly identify all classes in the range using an algebraic tmf-resolution. The topological basis is simple: form a double complex that is simultaneously the $H\mathbb{F}_2$-based Adams resolution and the tmf-based Adams resolution. The effect computationally is that we can compute Ext over the Steenrod algebra out of Ext over the tensor powers of the cohomology of tmf. Since this cohomology is the quotient of Steenrod algebra by the subHopf algebra $\mathcal{A}(2)$ generated by Sq^1, Sq^2, and Sq^4, it therefore suffices by a change-of-rings argument to understand $\mathrm{Ext}_{\mathcal{A}(2)}$ of the tensor powers of $H^*(tmf)$. We show that as an $\mathcal{A}(2)$-module $H^*(tmf)$ breaks up into the sum of the cohomologies of the bo Brown-Gitler spectra, and using vanishing line arguments, we show that in the desired range, there are very few summands and tensor factors which can contribute.

From this point on, the problem is essentially solved by the computation of the homotopy groups of tmf by Hopkins and Mahowald [34]. The class v_2^8 is in the Hurewicz image, and this class supports a differential in tmf. Careful analysis allows us both to determine the differential on v_2^{16} and to show that there are no possible targets for a differential on v_2^{32}. Since this map is again detected in tmf, we also conclude that it is a non-bounding permanent cycle.

3. Current Research

3.1. **The action of finite subgroups of $\mathbb{G}_n$ on E_{n*}.** Hopkins conjectured that for finite subgroups G of $\mathbb{G}_n$, E_{qf*} is equivariantly isomorphic to a much more readily describable G-algebra. This is a natural extension of Hopkins and Miller's early work on the subject, where they showed using formal group techniques that this is true for $f = 1$ [26]. Recent work with Mike Hopkins and Doug Ravenel provides a solution to the problem, using the theory of formal A-modules (where A is $\mathbb{Z}_p[\zeta]$), the theory of crystals, and elementary Tate cohomology. Elementary obstruction theory allows us to reduce to the case of showing that Hopkins' conjecture is true for $G = \mathbb{Z}/p$, and I will sketch our argument in this case.

Formal A-modules have a deformation theory similar to that of formal groups, and there is a natural $\mathbb{Z}/p$-equivariant forgetful map from E_{qf*} to the ring E_f representing deformations of F_n in formal A-modules. As a $\mathbb{Z}/p$-module, E_f is particularly simple: everything in degree $(-2k)$ is in the ζ^k eigenspace. This ring also admits a natural, surjective map from R, the symmetric algebra on f copies of $\bar{\rho}$ (appropriately localized and completed), where $\bar{\rho}$ is the torsion-free quotient of the regular representation by the trivial representation. An equivalent form of Hopkins' conjecture is that the Tate cohomology of R is the same as the Tate cohomology of E_{fq*}.

Here the theory of crystals provides a key step. While it is not *a priori* clear that we can equivariantly lift the formal A-module over E_f to one over the R, using crystals we can easily produce a lift over a well-behaved quotient. This is actually sufficient: the kernel of the natural surjective map from E_{fq*} to this quotient has computable Tate cohomology. Combining this with the long exact sequence in Tate cohomology associated to this short exact sequence of modules yields the result.

3.2. **Computation of the homotopy groups of $EO_{f(p-1)}(\mathbb{Z}/p)$.** Our description of the group action gives the Adams-Novikov E_2-term for the homotopy of EO_{fq}. The computation of the Adams-Novikov differentials is more involved, drawing on both the E_∞-structure of EO_{fq} and on Ravenel's "method of infinite descent" for computing homotopy groups [47, 48].

Since E_{fq} is an E_∞ ring spectrum, given any map $u\colon S^{2k} \to E$, we can form a $\mathbb{Z}/p$-equivariant map

$$Nu = u \dots g^{p-1}(u)\colon S^{2kp} \to E.$$

As an equivariant spectrum the source of this map is $S^{k\rho}$, where ρ is the complex regular representation. Since this map is equivariant, it induces a map of homotopy fixed point spectra and of homotopy fixed point spectral sequences. The source of these maps can be identified with the Spanier-Whitehead dual of a Thom spectrum over $B\mathbb{Z}/p$, together with its cellular filtration, and the differentials are related to attaching maps. Naturality then allows us to conclude a great number of differentials in the corresponding homotopy fixed point spectral sequence for EO_{fq}. This was Hopkins and Miller's original argument for height $p-1$. Standard power operation techniques tell us the remaining differentials, once we are able to identify the target. Ravenel's "method of descent" allows us to explicitly identify the targets.

The method of descent is built from the filtration of MU by the Thom spectra $X(i)$ which appear in the proofs of the Nilpotence and Periodicity theorems [20]. Ravenel has shown that just as MU splits into a wedge of copies of BP, the spectra $X(i)$ split into a wedge of copies of spectra $T(i)$ whose BP homology is $BP_*[t_1, \dots t_i] \subset BP_*BP$ [47, 48]. The spectra $T(i)$ have a filtration by $T(i-1)$-module spectra for which the associated graded is $T(i-1)[t_i]$. For $i = 1$, the attaching maps are exactly the classes we needed identified in the homotopy fixed point spectral sequence. Using similar techniques to the norm arguments (with the key fact that these spectra are not E_∞), we find short differentials in the homotopy fixed point spectral sequence for $T(i)_* EO_{fq}$ and use these to inductively produce all of the desired longer differentials.

Analysis of the $T(i)$-homology of E_{fq} has two additional pay-offs. Firstly, for a range of values of f (on the order of $f < p+2$), we can use a cohomology version of the method of descent to understand the action of $\mathbb{Z}/p$ on E_{fq}. The key fact, which follows easily from Devinatz and Hopkins' original work [19], is that as a $\mathbb{Z}/p$-module, $E_{fq*}T(f)$ is the symmetric algebra on f copies of the regular representation of $\mathbb{Z}/p$, localized by inverting an additive trace and then completed. For f in this range, simple degree arguments show that the cohomology descent spectral sequence collapses at each stage. This in particular quickly provides a description of the group action. Secondly, the equivariant description of $E_{fq*}T(f)$ extends in an obvious way to larger groups: if $\mathbb{Z}/p^k$ is a subgroup of $\mathbb{G}_{\mathrm{n}}$, then $E_{fq*}T(f)$ is the symmetric algebra on f/p^{k-1} copies of the regular representation of $\mathbb{Z}/p^k$, localized and completed. This fact should be helpful in some of the further applications.

4. Proposed Research Projects

4.1. Computing the homotopy Groups of $EO_{qf}(G)$. For groups with p-torsion subgroup larger than $\mathbb{Z}/p$, even computing the Adams-Novikov E_2-term is difficult. The chief problem is that for $\mathbb{Z}/p^{k+1}$, the full Adams-Novikov zero line for $\mathbb{Z}/p^k$ occurs in higher cohomology. In particular, we have to understand the ring of invariants of the cyclic action of $\mathbb{Z}/p^k$ on

$\mathbb{F}_p[x_1, \ldots, x_{p^k}]$. Techniques of invariant theory should resolve much of this, possibly allowing a complete description similar to that found for simple p-torsion. For the $\mathbb{Z}/p^2$ case, cursory calculations suggest that the full algebra structure is understandable, and we have conjecturally identified elements of order p^2 in the homotopy fixed point spectral sequence. In particular, we appear to see substantially more of the homotopy groups of spheres.

As with much of algebraic topology, the case of $p = 2$ behaves differently. Here we can always completely identify the E_2-page of the homotopy fixed point spectral sequence, provided we restrict attention to the cyclic subgroups. While computing the E_2-term is more tractable, it is still difficult to relate classes therein to actual homotopy classes.

For $p = 2$ and $p = 3$, we have essentially complete stories for height $p(p-1)$. For $p = 2$, the spectrum $EO_2(\mathbb{Z}/4)$ fits into a diagram of spectra with the $K(2)$-localizations of TMF and Mahowald and Rezk's spectrum $TMF_0(3)$. Even with these connections, there were several surprising subtleties in understanding the pattern of differentials. However, in these cases, we can also identify many of the classes and look for patterns involving v_2 phenomena.

In the general case, using transfer arguments, we can show close connections between the Adams-Novikov spectral sequence for $EO_n(\mathbb{Z}/p^k)$ and the one for $EO_n(\mathbb{Z}/p^j)$ for $j < k$, and this allows for the identification of many classes and many permanent cycles. This also allows us to produce differentials on many classes to conclude that other classes are permanent cycles. The norm based geometric differentials used for the $\mathbb{Z}/p$ action work equally well here, though the attaching maps in the underlying Thom spectra over $B\mathbb{Z}/p^k$ are more difficult to understand.

4.2. **Non-existence of Smith-Toda complexes & the Odd Primary Kervaire Problem.** Mike Hopkins, Doug Ravenel, and I plan to use the computations with EO_{fq} to strengthen Nave's non-existence results for Smith-Toda complexes [43] and to generalize Ravenel's work on the odd primary Kervaire problem [46]. The two problems are closely interrelated, as both involve identifying Adams-Novikov elements in the homotopy of EO_{fq}.

Nave used Hopkins and Miller's computation of the differential in the Adams-Novikov spectral sequence for EO_q to show that the Smith-Toda complex $V((p+1)/2)$ does not exist at p. Since $EO_{fq}(G)$ serves in some sense as a better approximation to the sphere as f increases (and seems to do so in a quite strong sense as the p-torsion of G increases), there is some hope that mirroring Nave's methods will produce stricter non-existence results.

Careful analysis of the Adams-Novikov E_2-term allows us to related classes on the zero line to powers of the BP_* classes v_i. This in turn will allow us to compute various stages of an algebraic Atiyah-Hirzebruch spectral sequence computing the Adams-Novikov E_2-term for the EO_{fq}-homology of $V(i)$. Since these spectral sequences are still quite sparse, our hope is that purely algebraic methods will produce all of the differentials from the differentials for the homotopy of EO_{fq}, just as in my computation of the $V(0)$ and $V(1)$-homologies of eo_4.

The above analysis of $EO_{p(p-1)}(\mathbb{Z}/p^2)$ at $p = 3$ plays a key role in our analysis of the odd primary Kervaire problem. Ravenel's original approach can be recast into a statement about the images of the β family in the Adams-Novikov spectral sequence for EO_{p-1}. At $p = 3$, Ravenel's argument breaks down, in part due to the non-existence of a v_2-self map of $V(1)$, so we must use a spectrum that captures more of the stable stems. The computations at height $p(p-1)$ indicate that $EO_{p(p-1)}$ does see enough of the stable stems to distinguish between the β elements at 3.

4.3. **Applications of geometric models of EO_n.** Behrens and Lawson have a program which focuses on the moduli stack of abelian varieties with nice properties [10]. On certain classes of Shimura varieties, Lurie's derived Artin representability produces a sheaf of E_∞ ring spectra with desirable properties mirroring the local properties of the Shimura varieties. In particular, the $K(n)$-localizations for appropriate values of n are closely related to $EO_n(G)$ for various G related to the underlying abelian varieties. Behrens and Lawson have also produced an analogue to the image of J spectrum, generalizing Behrens' $Q(2)$ spectrum for TMF [8].

Computing with Behrens and Lawson's sheaf, in particular finding the homotopy groups of the global sections, requires knowing the coherent cohomology of the Shimura variety. In general, even H^0 is not known. This makes computation almost impossible and forces consideration of other avenues. Behrens and Lawson have shown that understanding the action of finite subgroups of the Morava stabilizer group provides a way to understand the $K(n)$-local homotopy and then to try to understand some of the global structure.

Behrens and Lawson's sheaf also has very nice applications to the existence of appropriate connective models for $EO_n(G)$. If one assumes certain, often mild conditions, then the Shimura stack on which the sheaf is defined is actually compact. While this in general will not force the global sections to be connective, Serre duality suggests that there will be a gap in the homotopy, similar to the gap present in the L_2-localization of tmf. This sort of gap would allow one to simply take the connective cover without losing much information, producing a nice model for $eo_n(G)$. Cursory computations at $n = 4$ suggest that the resulting object has π_0 strictly larger than $\mathbb{Z}_p$, making these slightly bigger than we might hope.

4.4. **The geometry of TR.** One of the primary approaches to computing the algebraic K-theory of an E_∞ ring spectrum is to work up the TR-tower of Hesselholt and Madsen [28], computing eventually the Bökstedt-Hsiang-Madsen TC [12]. This approach uses the natural action of S^1 on $THH(R)$.

Working localized at a prime p, Hesselholt and Madsen define groups

$$TR_q^n(R;p) = \pi_q\big(THH(R)^{C_{p^{n-1}}}\big),$$

where $C_{p^{n-1}}$ is the cyclic subgroup of S^1 of order p^{n-1}. These groups fit together into complicated diagrams with maps arising as the restrictions and transfers associated to the inclusions $C_{p^{n-1}} \to C_{p^n}$. This structure provides a great deal of rigidity which greatly facilitates computations. Even so, these groups are in general quite difficult to compute, since we take honest, rather than homotopy, fixed points. However, these geometric fixed points fit into fiber squares with the more homotopy invariant structures: the homotopy fixed point spectrum $THH(R)^{h\mathbb{Z}/p^k}$ and the Tate spectrum $THH(R)^{t\mathbb{Z}/p^k}$.

Many of the arguments establishing the values of these groups are subtle and difficult to understand, relying on (sometimes miraculous) theorems which establish strong co-connectivity results that allow us to conclude the behavior of the general case from a few introductory cases. Even in simple cases like $R = H\mathbb{Z}/p^2$, these methods have yet to yield complete results [15]. Work with Vigleik Angeltveit and Tyler Lawson seeks to use unstable homotopy, equivariant homotopy, and geometry to better understand the constituent homotopy fixed point and Tate spectral sequences.

If R is an E_∞ ring spectrum, then $THH(R)$ is an E_∞ R-algebra. The methods Hopkins, Ravenel, and I employ to compute the differentials for EO_{fq} are universal: given any E_∞ ring spectrum on which $\mathbb{Z}/p^k$ acts via E_∞ self-maps, we can produce differentials from a norm argument. In particular, if β is the periodicity generator in $H^2(\mathbb{Z}/p^k; R_0)$ coming from the unit, then the norm argument produces a family of differentials which are determined by the attaching maps in $B\mathbb{Z}/p^k$. Appropriately interpreted, this allows us to immediately reproduce many of the classical differentials for $R = H\mathbb{F}_p$ or $H\mathbb{Z}_p$. There is therefore strong evidence that applying the same techniques will give differentials in more complicated settings.

The second approach is to analyze the geometric fixed point spectrum directly. The map to the homotopy fixed points is a kind of power operation construction, and approaching the direct geometry of the geometric fixed points (the ultimate object of study) should make more transparent some of the results which heretofore are mysterious. Moreover, in the cases considered, R is a complex orientable spectrum. This allows us to tie questions about TR and the fixed point spectra considered to formal groups and level structures in the formal group. In particular, MU itself provides a universal example for complex orientable spectra, and we are focusing extensively on this case.

4.5. **The Algebraic K-theory and THH of EO_n and eo_n.** In a similar vein to the previous, more general application of fixed point machinery, Angeltveit, Lawson, and I hope to compute some algebraic K-groups directly. Ausoni and Rognes have computed the $V(1)$-homotopy of $THH(\ell)$

for $p > 3$ [5], using the Bökstedt-Hsiang-Madsen TC machinery and exploiting the fact that $V(1)$ is a ring spectrum for these values of p. This method breaks down completely for the case $p = 2$, since $V(1)$ does not even exist, making analogous computations for the v_2-periodic part of the homotopy of $K(ku)$ and $K(ko)$ difficult to understand.

Working with a replacement for the spectrum $V(1)$, namely the Thom spectrum $Y(2)$ associated to loops on the third piece of the James filtration for ΩS^3, similar computations to those of the odd primes should be feasible for $p = 2$. These are A_∞ ring spectra, and

$$H_*Y(2) = \mathbb{F}_2[\xi_1, \xi_2].$$

Angeltveit and Rognes computed the homology of $THH(ku)$[3], finding that

$$THH(ku) = E(\lambda_1, \lambda_2) \otimes \mathbb{F}_2[\mu],$$

where $|\lambda_i| = 2^{i+1} - 1$ and $|\mu| = 8$, so a standard change of rings argument shows that

$$Y(2)_*THH(ku) = E(\lambda_1, \lambda_2) \otimes \mathbb{F}_2[\mu] \otimes \mathbb{F}_2[\xi_1^2, \xi_2^2].$$

Moreover, the spectrum $Y(2)$ interpolates between the sphere spectrum and $H\mathbb{F}_p$. This allows us to use the recent results of Bruner and Rognes on power operations [16] to better understand the homotopy fixed point spectral sequences in the TC machine and get a close approximation to the actual values of $TC_*(ku)$ and $TC_*(ko)$, and thereby to the algebraic K-groups.

The primary drawback of this approach is that the rings are too complicated to apply the usual techniques like Tsalidis' Theorem [50]. However, with a better understanding of the geometric underpinnings of the TR approach to algebraic K-theory, this should not be a problem.

The algebraic K-theory of ku and ko sit inside cofiber sequences with more mysterious objects: $K(KU)$ and $K(KO)$ [11]. Blumberg and Mandell showed that

$$K(H\mathbb{Z}) \to K(ku) \to K(KU)$$

and

$$K(H\mathbb{Z}) \to K(ko) \to K(KO)$$

are cofiber sequences. This shows that the algebraic K-theory of ku and ko are essentially those of KU and KO, and our computations would allow us to directly attack the height 1 version of Rognes' program for computing $K(L_{K(n)}S^0)$ by analyzing the action of $\mathbb{Z}_p^\times$ on $K(KU)$.

Much of the story also generalizes nicely to higher chromatic height, assuming appropriately commutative models of $BP\langle n\rangle$ exist. The THH story is very similar:

$$THH(BP\langle n\rangle; H\mathbb{F}_p) = E(\lambda_1, \ldots, \lambda_{n+1}) \otimes \mathbb{F}_p[\mu],$$

where $|\lambda_i| = 2p^i - 1$ and $|\mu| = 2p^{n+1}$.

Computing the homotopy of $THH(BP\langle n\rangle)$ is more difficult. Instead of a commuting square of bimodules, we have a commuting $(n+1)$-dimensional cube, and Angeltveit, Lawson, and I have been able to run the Bockstein spectral sequences recovering $THH_*(BP\langle n\rangle; k(m))$ for $m \leq n$.

Here again, the use of appropriate spectra $Y(n)$ should serve as suitable replacements for $V(n)$. The spectra eo_n, when they exist, should then have computable THH, again looking only at the effect in $Y(n)$. This should provide a technique of approximating the algebraic K-groups while still detecting Rognes' "chromatic red shift".

4.6. **The equivariant homotopy type of $THH(k[x,y]/x^a = y^b)$.** This project is joint with Vigleik Angeltveit and Teena Gerhardt. Hesselholt and Madsen's approach to computing TC relies on identifying the cyclotomic structure of $THH(R)$ [29].

This extra structure allowed them to determine the algebraic K-groups of truncated polynomial algebras over a field and of local fields [27, 29]. Gerhardt used the cyclotomic structure to compute the $R(S^1)$-graded homotopy groups of $THH(\mathbb{F}_p)$ [22], and Angeltveit, Gerhardt, and Hesselholt recently extended the Hesselholt-Madsen results to truncated polynomial rings over $\mathbb{Z}$ [1].

This project's aim is to prove a conjecture of Hesselholt about the equivariant structure of $THH(k[x,y]/x^a - y^b)$, the topological Hochschild homology of the cuspidal curve. This will allow us to bring to bear the Angeltveit-Gerhardt-Hesselholt-Madsen machinery to compute the relevant algebraic K-groups.

4.7. **Computations in motivic stable homotopy.** Recent work of Dugger and Isaksen has introduced me to the motivic Steenrod algebra and motivic Adams spectral sequence. At this point, my interest is intensely exploratory, with three main foci.

Building on my experience from my other projects, I have conjecturally computed the homotopy of $THH(H\mathbb{F}_p)$, working over an algebraically closed field of characteristic zero. The computation relied on several fairly large suppositions and yielded surprisingly difficult answers. From these, we can reproduce the classical computations of Bökstedt through standard "motivic-to-classical" legerdemain. My first proposal is to understand to what extent to Bökstedt spectral sequence works in the motivic context and to understand its convergence.

Using standard techniques of stable homotopy and Voevodsky's description of the dual Steenrod algebra over $\mathbb{R}$ [52], I found a Bockstein spectral sequence which computes the Adams E_2-term over $\mathbb{R}$ from the Adams E_2-term over $\mathbb{C}$. This seems to be some sort of descent spectral sequence associated to the Galois cover $Spec(\mathbb{C}) \to Spec(\mathbb{R})$. I used this to quickly compute the aforementioned Adams E_2-term for ko over $\mathbb{R}$. Dugger has applied this method to compute $\pi_{0,0}$ and $\pi_{-1,-1}$ of the $H\mathbb{F}_2$-nilpotent completion of the sphere, verifying in these cases the close connection with $\mathbb{Z}/2$-equivariant homotopy theory. Recently, Dan Isaksen and I applied these techniques to compute the entire zero, one, and two lines of the Adams E_2-term working over $\mathbb{R}$. Dugger, Isaksen, and I plan to further explore this approach, computing more homotopy groups of spheres and tying to understand $\mathbb{Z}/2$-equivariant phenomena from a motivic perspective.

With Dan Dugger, I also computed the Adams E_2-terms for motivic analogues of tmf at 2 over $\mathbb{C}$ and of ko at 2 over $\mathbb{R}$ and $\mathbb{C}$. These also yield surprising and sometimes shocking complicated results. I am interested in understanding the Adams differentials in this context. Working over $\mathbb{C}$, there is a natural comparison map to the classical case. However, convergence is not clear in this context, and over $\mathbb{R}$, the situation becomes significantly more murky. Understanding this (and if there are motivic spectra whose homotopy is computed by these spectral sequences) is the second goal. In particular, even understanding the existence of a connective Hermitian K-theory spectrum and of a connective K-theory spectrum whose cohomologies are analogous to the classical ones would be a boon computationally. This would provide an alternative approach to computing the algebraic K-groups of fields using the methods of classical stable homotopy.

Even without the existence of a connective Hermitian K-theory spectrum, the computation of $\mathrm{Ext}_{\mathcal{A}(1)}$ over $\mathbb{R}$ should yield information about the image of J and the v_1-periodic family in the motivic homotopy groups of spheres. There is a motivic analogue of Adams' periodicity operator $P(-)$, and this again detects multiplication by v_1^4. From the canonical map from the Adams E_2-term for the sphere to $\mathrm{Ext}_{\mathcal{A}(1)}$, we learn that the classes $P^i(\eta^k)$ taken together form a complicated, non-nilpotent algebra and that multiplication by Voevodsky's element ρ, the generator of the first Milnor K-group of $\mathbb{R}$ modulo 2, is faithful on these classes.

5. Broader Impact

Computational stable homotopy is a very difficult field for graduate students and young researchers. Many of the most important results and techniques are not in the literature, being instead "folk-theorems". Even techniques in the literature are presented in a different manner than the way they are usually applied. This results in a large initial effort required before any, even basic, computations can be done. The broader impacts of this project seek to address this point exactly.

First, my collaborators come from all different stages of their careers, including many who are starting their careers. Tyler Lawson received his PhD in 2004, Vigleik Angeltveit received his PhD in 2006, and Teena Gerhardt received her PhD in 2007. Additionally, many of my projects have pieces accessible to graduate students. The computational results in motivic homotopy are often approachable by students with basic algebra skills.

Second, I have run working groups and seminars to help introduce graduate students and undergraduates to some of the approaches to computational homotopy theory. This Spring, I will teach a graduate course on spectral sequences in algebraic topology, and I plan to continue with more special-topic working groups covering particular kinds of tools (including operations in spectral sequences, higher product structures and differentials, and geometric constructions).

Third, I plan to build a "Spectral Sequence Wiki", allowing many researchers easy access to a large body of computational examples. The model is the Encyclopedia of Integer Sequences, coupled with slightly more flexibility for user generated content. One of the most striking features of computations in stable homotopy is the repetition of ideas and results across spectral sequences. Having a large, detailed list of examples of Serre, Eilenberg-Moore, Bockstein, and Adams spectral sequences carefully spelled out and running the gamut from elementary to complicated, will render aspects of the subject less esoteric.