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# ON A QUESTION OF SLAMAN AND STEEL

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Abstract. We consider an old question of Slaman and Steel: whether Turing equivalence is an increasing union of Borel equivalence relations none of which contain a uniformly computable infinite sequence. We show this question is deeply connected to problems surrounding Martin's conjecture, and also in countable Borel equivalence relations. In particular, if Slaman and Steel's question has a positive answer, it implies there is a universal countable Borel equivalence which is not uniformly universal, and that there is a $(\equiv_T, \equiv_m)$-invariant function which is not uniformly invariant on any pointed perfect set.

## 1. Introduction

This paper is a contribution to the study of problems surrounding Martin's conjecture on Turing invariant functions and countable Borel equivalence relations. Our central focus is an old open question of Slaman and Steel which they posed [SS] in reaction to their proof in the same paper that Turing equivalence is not hyperfinite. The question they asked is whether Turing equivalence can be expressed as a union of Borel equivalence relations $E_n$ where $E_n \subseteq E_{n+1}$ for all $n$ and so that no $E_n$-class $[x]_{E_n}$ contains an infinite sequence of reals uniformly computable from $x$. While this seems to be a very specific question about computability, we show (Theorem 3.5) that it is equivalent to a much more general question of whether every countable Borel equivalence relation is what we call hyper-Borel-finite (see Definition 3.1).

This question of Slaman and Steel has been completely unstudied since the 1988 paper where it was posed, and it remains open. However, we show that it is deeply connected to problems in both Borel equivalence relations, and problems surrounding Martin's conjecture. In particular, we show (Corollary 5.2.(1)) that if Slaman and Steel's question has a positive answer, then there is a Borel invariant function from Turing equivalence to many-one equivalence which is not uniformly invariant on any pointed perfect set. (In Section 2 we discuss some open problems concerning invariant functions from Turing equivalence to many-one equivalence which are suggested by Kihara-Montalbán's recent work [KM]). We also show (Corollary 5.2.(2)) that if Slaman and Steel's question has a positive answer, then many-one equivalence on $2^\omega$ is a universal countable Borel equivalence relation. Since many-one equivalence on $2^\omega$ is not uniformly universal [M, Theorem 1.5.(5)], this implies that if Question 3.4 has a positive answer, the conjecture of the second author that every universal countable Borel equivalence relation is uniformly universal ([M, Conjecture 1.1]) is false.

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Our main construction is given in Theorem 5.1. This is the first result constructing a non-uniform function between degree structures in computability theory from any sort of hypothesis.

Suppose we want to construct a counterexample to part I of Martin's conjecture. That is, we want to build a Turing invariant function $f\colon 2^\omega \to 2^\omega$ such that the Turing degree of $f$ is not constant on a cone, and $f(x) \not\geq_T x$ on a cone. An obvious strategy is to build $f$ is countably many stages. At stage $n$, we determine some partial information about $f(x)$ in order to diagonalize against $f(x)$ computing $x$ via the $n$th Turing reduction. At stage $n$ we also specify how to "code" $f(y)$ into $f(x)$ for some of the $y$ such that $y \equiv_T x$ (to ensure that at the end of the construction, $f$ is Turing invariant). Now consider the relation $E_n$ where $x \mathrel{E_n} y$ if $f(y)$ has been coded into $f(x)$ by the $n$th stage of the construction. Clearly $E_n$ is an equivalence relation, $E_n \subseteq E_{n+1}$ for all $n$, and Turing equivalence is the union of these equivalence relations: $\equiv_T = \bigcup_n E_n$.

A huge problem in attempts to construct counterexamples to Martin's conjecture is that we know essentially nothing about the ways in which Turing equivalence can be written as an increasing union, apart from Slaman and Steel's original theorem that Turing equivalence is not hyperfinite. In particular, it is open whether every way of writing Turing equivalence as an increasing union $\equiv_T = \bigcup_n E_n$ must be trivial in the sense that there is some $n$ and some pointed perfect set $P$ where $E_n$ is already equal to Turing equivalence, i.e. $E_n \restriction P = (\equiv_T \restriction P)$ (see Conjecture 6.1). If Conjecture 6.1 is true, attempts to build counterexamples to Martin's conjecture in the way indicated above seem hopeless.

In the authors' opinion, understanding how Turing equivalence may be expressed as an increasing union, and Slaman and Steel's Question 3.4 seem to be a vital steps towards understanding Martin's conjecture. If Question 3.4 has a positive answer, one can hope to improve on the construction in Theorem 5.1 to give a counterexample to Martin's conjecture. If Question 3.4 has a negative answer, perhaps Conjecture 6.1 is true, and there is no nontrivial way of "approximating" Turing equivalence from below.

1.1. **Preliminaries.** Our conventions and notation are largely standard. For background on Martin's conjecture, see [MSS]. For a recent survey of the field of countable Borel equivalence relations, see [K19].

We use lowercase $x, y, z$ to denote elements of $2^\omega$, and $f, g$ for functions on $2^\omega$. If $x \in 2^\omega$, we use $\overline{x}$ to denote the real obtained by flipping all the bits of $x$ (or the complement of $x$, viewing $x$ as a subset of $\omega$). If $f\colon 2^\omega \to 2^\omega$, we similarly use $\overline{f}$ to denote the function where $\overline{f}(x) = \overline{f(x)}$ for all $x$. If $A \subseteq \omega$ and $x \in 2^\omega$, we let $x \restriction A$ denote the restriction of the function $x$ to $A$. Equivalently, viewing elements of $2^\omega$ as subsets of $\omega$, $x \restriction A$ is $x \cap A$.

Fix a computable bijection $\langle \cdot, \cdot \rangle \colon \omega^2 \to \omega$. If $x \in 2^\omega$, the *$j$th column of $x \in 2^\omega$* is $\{n \colon \langle j, n \rangle \in x\}$.

## 2. Versions of Martin's conjecture for invariant functions from Turing to many-one degrees

In [KM], Kihara and Montalbán study uniformly degree invariant functions from Turing degrees to many-one degrees. One of our main results is that if Slaman and Steel's question has a positive answer, then there is a degree invariant function from Turing degrees to many-one degrees which is not uniformly Turing invariant on any pointed perfect set. In this section, we briefly discuss some open problems around such functions which are suggested by Kihara-Montalbán's work.

Recall a function $f\colon 2^\omega \to 2^\omega$ is $(\equiv_T, \equiv_m)$-*invariant* if $x \equiv_T y$ implies $f(x) \equiv_m f(y)$. (In the terminology of Borel equivalence relations, we would say $f$ is a *homomorphism* from $\equiv_T$ to $\equiv_m$.) A function $f\colon 2^\omega \to 2^\omega$ is *uniformly* $(\equiv_T, \equiv_m)$-*invariant* if there is a function $u\colon \omega^2 \to \omega^2$ so that if $x \equiv_T y$ via the programs $(i, j)$, then $f(x) \equiv_m f(y)$ via the programs $u(i, j)$. If $c \in 2^\omega$, then $x \leq_m^c y$ if there is a function $\rho\colon \omega \to \omega$ computable from $c$ so that $x(n) = y(\rho(n))$ for all $n$. If $f, g\colon 2^\omega \to 2^\omega$, then we write $f \leq_m^\triangledown g$ if there is a Turing cone of $x$ with base $c$ so that $f(x) \leq_m^c g(x)$.

Kihara and Montalbán show that uniformly $(\equiv_T, \equiv_m)$-invariant functions are well-quasi-ordered by $\leq_m^\triangledown$ and are in bijective correspondence with Wadge degrees via a simply defined map which they give [KM]. It follows from this bijection with Wadge degrees that the smallest uniformly $(\equiv_T, \equiv_m)$-invariant functions which are not constant on a cone are the Turing jump: $x \mapsto x'$ and its complement $x \mapsto \overline{x'}$, which are easily seen to correspond to the maps associated to universal open and closed sets; the lowest nontrivial classes in the Wadge hierarchy.

Implicit in Kihara-Montalbán's work are obvious analogues of Martin's conjecture [SS, Conjecture I, II] and Steel's conjecture [SS, Conjecture III] for $(\equiv_T, \equiv_m)$-invariant functions. We state these conjectures:

**Conjecture 2.1** (Martin's conjecture for $(\equiv_T, \equiv_m)$-invariant functions)**.** *Assume* $\mathsf{AD} + \mathsf{DC}$*. Then*

I. *If $f\colon 2^\omega \to 2^\omega$ is $(\equiv_T, \equiv_m)$-invariant and the many-one degree $[f(x)]_m$ of $f$ is not constant on a Turing cone of $x$, then $f \geq_m^\triangledown j$, or $f \geq_m^\triangledown \overline{j}$, where $j(x) = x'$ is the Turing jump.*
II. *If $f, g\colon 2^\omega \to 2^\omega$ are $(\equiv_T, \equiv_m)$-invariant, then $f \geq_m^\triangledown g$ or $\overline{g} \geq_m^\triangledown f$. Furthermore, the order $\leq_m^\triangledown$ well-quasi-orders the functions on $2^\omega$ that are $(\equiv_T, \equiv_m)$-invariant.*

**Conjecture 2.2** (Steel's conjecture for $(\equiv_T, \equiv_m)$-invariant functions)**.** *Suppose* $\mathsf{AD}{+}\mathsf{DC}$*, and suppose $f\colon 2^\omega \to 2^\omega$ is $(\equiv_T, \equiv_m)$-invariant. Then there is a uniformly $(\equiv_T, \equiv_m)$-invariant function $g$ so that $f \equiv_m^\triangledown g$.*

Conjecture 2.2 implies Conjecture 2.1 by Kihara-Montalbán's work in [KM].

There is an important relationship between Turing invariant functions and $(\equiv_T, \equiv_m)$-invariant functions. Since $x \leq_T y$ if and only if $x' \leq_m y'$, any Turing invariant function can be turned into a $(\equiv_T, \equiv_m)$-invariant function by applying the Turing jump. However, because of the parameter $c$ in the definition of $\leq_m^\triangledown$, it is not true that if $f' \geq_m^\triangledown g'$, then $f(x) \geq_T g(x)$ on a Turing cone of $x$. In particular, we do not know whether Conjecture 2.1 and Conjecture 2.2 imply Martin's conjecture and Steel's conjecture. However, if we strengthen Conjecture 2.2 to use the relation "$\leq_m$ on a cone" rather than $\leq_m^\triangledown$, then we do obtain a strengthening of Steel's conjecture [SS, Conjecture III].

**Conjecture 2.3.** *Suppose* $\mathsf{AD}$*, and suppose $f$ is $(\equiv_T, \equiv_m)$-invariant. Then there is a uniformly $(\equiv_T, \equiv_m)$-invariant function $g$ so that $f(x) \equiv_m g(x)$ on a Turing cone of $x$.*

A standard argument (see the first footnote in [MSS]) shows that if $f$ is $(\equiv_T, \equiv_m)$-invariant, then $f(x) \equiv_m g(x)$ on a cone for some uniformly $(\equiv_T, \equiv_m)$-invariant function $g$ if and only if $f$ is itself uniformly $(\equiv_T, \equiv_m)$-invariant on a pointed perfect set.

**Proposition 2.4.** *Conjecture 2.3 implies Steel's conjecture,* [SS, Conjecture III]*.*

*Proof.* Suppose $f\colon 2^\omega \to 2^\omega$ is Turing invariant. Then by Conjecture 2.3, the map $x \mapsto f(x)'$, is uniformly $(\equiv_T, \equiv_m)$-invariant on a pointed perfect set. Hence $f$ is uniformly Turing invariant on the same pointed perfect set. □

Kihara and Montalbán's work is more generally stated for functions to the space $\mathcal{Q}^\omega$, where $\mathcal{Q}$ is a better-quasi-order. One can more generally ask about the analogues of the above conjectures for functions to $\mathcal{Q}^\omega$. We have the following observation due to Kihara-Montalbán that the relation $\leq_m^\triangledown$ cannot be replaced with "$\leq_m$ on a cone" in their work when $\mathcal{Q} \neq 2$:

**Proposition 2.5** (Kihara-Montalbán, private communication)**.** *Suppose* $\mathsf{AD}$*. Then the* $(\equiv_T, \equiv_m)$*-invariant functions from* $2^\omega$ *to* $3^\omega$ *which are not constant on a cone are not well-quasi-ordered by the relation "*$\leq_m$ *on a cone".*

*Proof.* By [M, Theorem 3.6], many-one reducibility on $3^\omega$ is a uniformly universal countable Borel equivalence relation. Letting $=_\mathbb{R}$ denote equality on the real numbers, there is hence a uniform Borel reduction $f\colon 2^\omega \times \mathbb{R} \to 3^\omega$ from $\equiv_T \times =_\mathbb{R}$ to many-one reducibility on $3^\omega$. For each $y \in \mathbb{R}$, the function $f_y(x) = f(x, y)$ is thus a uniformly $(\equiv_T, \equiv_m)$-invariant function. Note that if $y \neq y'$, then $f_y(x)$ and $f_{y'}(x)$ are not $\equiv_m$-equivalent on a cone of $x$, nor are they constant on a cone (since $f$ is a Borel reduction).

Thus, the relation on Borel functions "$\leq_m$ on a cone" cannot be a well-quasi-order on the Borel uniformly $(\equiv_T, \equiv_m)$-invariant functions from $2^\omega \to 3^\omega$, since then it would therefore give a well-quasi-order of $\mathbb{R}$. □

In fact, it is easy to see from the proof of [M, Theorem 3.6] that for all $y, y'$ and all $z$, $f_y(z) \not\geq_m f_{y'}(z)$. So all the functions $f_y$ constructed above are incomparable under $\leq_m$.

It is an open question whether the relation $\leq_m^\triangledown$ can be replaced with "$\leq_m$ on a cone" in Kihara-Montalbán's theorem on the space $2^\omega$.

**Question 2.6.** *Assume* $\mathsf{AD} + \mathsf{DC}$*. Is there is an isomorphism between the Wadge degrees and the degrees of the uniformly* $(\equiv_T, \equiv_m)$*-invariant functions under the relation "*$\leq_m$ *on a cone"? If* $f$ *is uniformly* $(\equiv_T, \equiv_m)$*-invariant and the many-one degree* $[f(x)]_m$ *of* $f$ *is not constant on a Turing cone of* $x$*, then is* $f(x) \geq_m j(x)$ *on a cone, or* $f(x) \geq_m \overline{j(x)}$ *on a cone, where* $j(x) = x'$ *is the Turing jump?*

## 3. Slaman and Steel's Question

The following notion is essentially due to Slaman and Steel:

**Definition 3.1** ([SS])**.** Suppose $(f_i)_{i\in\omega}$ is a countable sequence of Borel functions $f_i\colon X \to X^\omega$. Say that a countable Borel equivalence relation $F$ on $X$ is $(f_i)_{i\in\omega}$-*finite* if there is no infinite set $\{f_i(x)(j) \colon j \in \omega\}$ such that $\{f_i(x)(j) \colon j \in \omega\} \subseteq [x]_F$. That is, no $f_i(x)$ is a sequence of infinitely many different elements in the $F$-class of $x$. Say that $E$ is *hyper-*$(f_i)_{i\in\omega}$*-finite* if there is an increasing sequence $F_0 \subseteq F_1 \subseteq \ldots$ of Borel subequivalence relations of $E$ such that $F_n$ is $(f_i)_{i\in\omega}$-finite for every $n$, and $\bigcup_n F_n = E$. Finally, say that $E$ is *hyper-Borel-finite* if for every countable collection of Borel functions $(f_i)_{i\in\omega}$ where $f_i\colon X \to X^\omega$, $E$ is *hyper-*$(f_i)_{i\in\omega}$*-finite.*

Here we can think of each set $\{f_i(x)\}_{i\in\omega}$ as being a potential witnesses that some $F$-class is infinite, which we would like to avoid.

Clearly every hyperfinite Borel equivalence relation is hyper-Borel-finite. It is an open problem to characterize the hyper-Borel-finite equivalence relations.

**Question 3.2.** *Is there a non-hyperfinite countable Borel equivalence relation that is hyper-Borel-finite?*

**Question 3.3.** *Is every countable Borel equivalence relation hyper-Borel-finite?*

Slaman and Steel consider the special case of this definition where the function $f_i\colon 2^\omega \to (2^\omega)^\omega$ is the $i$th Turing reduction $\Phi_i(x)$:

$$f_i(x)(j) = \begin{cases} \text{the } j\text{th column of } \Phi_i(x) & \text{if } \Phi_i(x) \text{ is total} \\ x & \text{otherwise.} \end{cases}$$

We say that $\equiv_T$ is *hyper-recursively-finite* if $\equiv_T$ is hyper-$(f_i)_{i\in\omega}$-finite for the above functions $(f_i)_{i\in\omega}$. Slaman and Steel posed the question of whether $\equiv_T$ is hyper-recursively-finite in [SS, Question 6], though in the setting of $\mathsf{AD}$ rather than just for Borel functions. We work in the Borel setting because it makes the statements of some of our theorems more straightforward, however all the arguments of the paper can be adapted to the setting of $\mathsf{AD}$ as usual.

**Question 3.4** ([SS])**.** *Is $\equiv_T$ hyper-recursively-finite?*

This problem about $\equiv_T$ is equivalent to the more general problem of whether every countable Borel equivalence relation is hyper-Borel-finite. This self-strengthening property of hyper-recursive-finiteness of $\equiv_T$ will be an essential ingredient in our proof of Theorem 5.1.

**Theorem 3.5.** *The following are equivalent:*

(1) *$\equiv_T$ is hyper-recursively-finite.*
(2) *Every countable Borel equivalence relation $E$ is hyper-Borel-finite.*

*Proof.* (1) is a special case of (2), and is hence implied by it. We prove that (1) implies (2). Fix a witness $F_0 \subseteq F_1 \subseteq \ldots$ that $\equiv_T$ is recursively finite. We wish to show that every countable Borel equivalence relation $E$ is hyper-Borel-finite. We may assume that $E$ is a countable Borel equivalence relation on $2^\omega$. We may further suppose that $E$ is $\Delta^1_1$ and $(f_i)_{i\in\omega}$ is uniformly $\Delta^1_1$; our proof relativizes.

Since $E$ is a $\Delta^1_1$ relation with countable vertical sections, and $(f_i)_{i\in\omega}$ is uniformly $\Delta^1_1$, there is some computable ordinal notation $\alpha$ such that for all $x \in 2^\omega$ and for all $y \mathrel{E} x$, $x^{(\alpha)} \geq_T y$ , and $x^{(\alpha)} \geq_T \bigoplus_{i\in\omega} f_i(x)$. Now if we let $\beta = \omega\cdot\alpha$, then $x^{(\alpha)} \geq_T y$ implies $x^{(\beta)} \geq_T y^{(\beta)}$. Hence, if $x \mathrel{E} y$, then $x^{(\beta)} \equiv_T y^{(\beta)}$, and $x^{(\beta)} \geq_T \left(\bigoplus_{i\in\omega} f_i(x)\right)^{(\beta)}$. Note that the function $x \mapsto x^{(\beta)}$ is injective.

Define $E_k$ by

$$x \mathrel{E_k} y \iff xEy \wedge x^{(\beta)} F_k y^{(\beta)}.$$

We claim that $(E_k)_{k\in\omega}$ witness that $E$ is hyper-$(f_i)$-finite. Suppose not. Then there exists $E_k$, $x$ and $i$ such that $\{f_i(x)(j)\colon j\in\omega\}$ is infinite and $x \mathrel{E_k} f_i(x)(j)$ for all $j \in \omega$. This implies $x^{(\beta)} \mathrel{F_k} (f_i(x)(j))^{(\beta)}$ for all $j$ by definition of $E_k$. Now the sequence $\left((f_i(x)(j))^{(\beta)}\right)_{j\in\omega}$ is uniformly recursive in $x^{(\beta)}$ since $x^{(\beta)} \geq_T (f_i(x))^{(\beta)}$. The set $\left\{(f_i(x)(j))^{(\beta)}\colon j\in\omega\right\}$ is still infinite since the jump operator $x \mapsto x^{(\beta)}$ is injective. This contradicts that $(F_k)_{k\in\omega}$ is a witness that $\equiv_T$ is hyper-recursively-finite. □

The key in the above proof is that given any countable Borel equivalence $E$ on $X$ and Borel functions $(f_i)$ from $X \to X^\omega$, we can find an injective Borel homomorphism $h$ from $E$ to $\equiv_T$ so that the image of each $f_i$ under $h$ is a computable function. Similar theorems to Theorem 3.5 are true for for other weakly universal countable Borel equivalence relations, and collections of "universal" functions with respect to them. For example, let $E_\infty$ be the orbit equivalence relation of the shift action of the free group $\mathbb{F}_\omega = \langle \gamma_{i,j} \rangle_{i,j \in \omega}$ on $\omega^{\mathbb{F}_\omega}$ (so we are indexing the generators of $\mathbb{F}_\omega$ by elements of $\omega^2$). Let $f_i(x)(j) = (\gamma_{i,j} \cdot x)$. Then $E_\infty$ is hyper-$(f_i)$-finite if and only if every countable Borel equivalence relation is hyper-Borel-finite.

Boykin and Jackson have introduced the class of Borel bounded equivalence relations [BJ]. For these equivalence relations it is an open problem whether there is some non-hyperfinite Borel bounded equivalence relation, and also whether all Borel equivalence relations are Borel bounded. Similarly both these problems are open for the hyper-Borel-finite Borel equivalence relations. We pose the question of whether there is a relationship between $E$ being hyper-Borel-finite and being Borel bounded.

**Question 3.6.** *Is every Borel bounded countable Borel equivalence relation hyper-Borel-finite?*

Straightforward measure theoretic and Baire category arguments cannot prove that any countable Borel equivalence relation is not hyper-Borel-finite. This follows for Baire category from generic hyperfiniteness. To analyze hyper-Borel-finiteness in the measure theoretic setting, we first need an easy lemma about functions selecting subsets of a finite set. Below, $\mathrm{Prob}(X)$ indicates the probability of an event $X$.

**Lemma 3.7.** *Supose $(X, \mu)$ is a standard probability space, $k \leq n$, $Y$ is a finite set where $|Y| = n$, and $g\colon X \to [Y]^k$ is any measurable function associating to each $x \in X$ a subset of $Y$ of size $k$. Then there is a set $S \subseteq Y$ with $|S| \leq m$ such that $\mathrm{Prob}(g(x) \cap S \neq \emptyset) \geq 1 - (1 - k/n)^m$.*

*Proof.* If we select $i$ from $Y$ uniformly at random, and $x$ from $X$ at random (wrt $\mu$), then $\mathrm{Prob}(i \notin g(x)) = 1 - k/n$, since $g(x)$ has $k$ elements. So if we pick $m$ elements $i_1, \ldots, i_m$ from $Y$ uniformly at random, and let $S = \{i_1, \ldots, i_m\}$, then $\mathrm{Prob}(S \cap g(x) = \emptyset) = (1 - k/n)^m$. Hence, there must be some fixed set $S = \{i_1, \ldots, i_m\}$ such that $\mathrm{Prob}(g(x) \cap S = \emptyset) \leq (1 - k/n)^m$, and so $\mathrm{Prob}(g(x) \cap S \neq \emptyset) \geq 1 - (1 - k/n)^m$. (It is possible that $|S| < m$ since we may have $i_j = i_l$ for some $j, l$). ☐

Consider the case where $0 \ll k \ll n$, and $m = \lceil \frac{n}{\sqrt{k}} \rceil$. Then we can choose $S$ of size $|S| \leq m$ such that $\mathrm{Prob}(S \cap g(x) \neq \emptyset)$ is close to 1. This is because $(1 - k/n)^{n/k} \approx 1/e$, so $(1 - k/n)^m \approx (1/e)^{\sqrt{k}} \approx 0$.

We now have the following theorem analyzing hyper-Borel-finiteness in the measure theoretic setting:

**Theorem 3.8.** *Suppose $E$ is a countable Borel equivalence relation on a standard Borel space $X$, $(f_i)_{i \in \omega}$ are Borel functions from $X$ to $X^\omega$, and $\mu$ is a Borel probability measure on $X$. Then there is a $\mu$-conull Borel set $B$ so that $E \restriction B$ is hyper-$(f_i)$-finite.*

*Proof.* We claim that for any $\epsilon > 0$, and any single Borel function $f\colon X \to X^\omega$, there is a Borel set $A \subseteq X$ with $\mu(A) > 1 - \epsilon$ such that $E \restriction A$ is $f$-finite. (By

$f$-finite for a single $f$, we mean that no $E \restriction A$-class contains an infinite set of the form $\{f(x)(j) \colon j \in \omega\}$).

The theorem follows easily from this claim. Choose a sequence of positive real numbers $(a_{i,n})_{i,n\in\omega}$ so that $\sum_{i,n} a_{i,n} < \infty$. Then for each $i$ and $n$, let $A_{i,n} \subseteq X$ be a Borel set so that $E \restriction A_{i,n}$ is hyper-$f_i$-finite (just for the single function $f_i$), and $\mu(A_i) > 1 - a_{i,n}$. Then let $B_m = \bigcap_{n\geq m \wedge i\in\omega} A_{i,n}$. Since $B_m \subseteq A_{i,m}$ for every $i$, $E \restriction B_m$ is $(f_i)_{i\in\omega}$-finite (for the entire sequence of $(f_i)_{i\in\omega}$). The $B_m$ are increasing sets. We have $\mu(B_m) > 1-\sum_{n\geq m\wedge i\in\omega} a_{i,n}$, so $\mu(B_m) \to 1$. Let $A = \bigcup_m B_m$. Then $E \restriction A$ is hyper-$(f_i)$-finite as witnessed by $E \restriction B_m$.

We prove the claim. Fix a Borel function $f \colon X \to X^\omega$. Without loss of generality we may assume that $\{f(x)(j) \colon j \in \omega\}$ is infinite for every $x$. The idea here is to use Lemma 3.7 to find a set $A$ of measure $\mu(A) > 1 - \epsilon$ such that for every $x \in A$, there is some $j$ such that $f(x)(j) \notin A$.

We may first assume by the Borel isomorphism theorem that $X = 2^\omega$. Consider the function $U_l(x) = \{N_s \colon s \in 2^l \wedge (\exists j) f(x)(j) \in N_s\}$. That is, $U_l(x)$ is the collection of basic open neighborhods $N_s$, where $s$ has length $l$, such that $N_s$ contains some element of the sequence $f(x)$. Since the neighborhoods $N_s$ separate points, for every $x$ we have $|U_l(x)| \to \infty$ as $l \to \infty$. Letting $X_{l,k} = \{x \in X \colon |U_l(x)| \geq k\}$, we may choose a sufficiently large $l$ so that $\mu(X_{l,k}) > 1 - \epsilon$.

Now by picking $l \gg k \gg 0$ sufficiently large and applying Lemma 3.7 to the function selecting the least $k$ elements of $U_l(x)$, we can choose a set $S \subseteq \{N_s \colon s \in 2^l\}$ of size $|S| < 2^l/\sqrt{k}$ so that $\mu(\{x \in X_{l,k} \colon U_l(x) \cap S \neq \emptyset\}\})$ is arbitrarily close to $\mu(X_{l,k})$. Note that $\mu(\bigcup S) < \frac{1}{\sqrt{k}}$. Let

$$A = \{x \in X_{l,k} \setminus \bigcup S \colon \exists i f(x)(i) \in \bigcup S\}$$

The claim follows. □

The above proof is trivial in the sense that the subequivalence relations witnessesing hyper-$(f_i)$-finiteness are simply the original equivalence relation restricted to some Borel subset of $X$. This style of witness that an equivalence relation is hyper-Borel-finite cannot work in general to show that an equivalence relation is hyper-Borel-finite. For example, there is no increasing sequence of Borel sets $(A_k)_{k\in\omega}$ such that $2^\omega = \bigcup_k A_k$, and the equivalence relations $\equiv_T \restriction A_k$ witness that $\equiv_T$ is hyper-recursively finite. To see, this, note that some $A_n$ must contain a pointed perfect set, and hence $\equiv_T \restriction A_n$ must contain a uniformly computable infinite sequence.

## 4. Strengthenings of the Kuratowski-Mycielski theorem

Two often used constructions in computability theory are

(1) There is a Borel function $f \colon 2^\omega \to 2^\omega$ so that if $x_0, \ldots, x_n$ are distinct, then $f(x_0), \ldots f(x_n)$ are mutually 1-generic.
(2) There is a Borel function $f \colon 2^\omega \to 2^\omega$ so that for all $x$, $f(x)$ is $x$-generic.

(1) is true since there is a tree of mutual 1-generics (hence $f$ in (1) may be continuous). (2) is true since $x'$ can compute an $x$-generic real uniformly.

It is impossible to have a function $f$ with both properties (1) and (2):

**Proposition 4.1.** *There is no Borel function $f \colon 2^\omega \to 2^\omega$ so that:*

(1) *If $x_0, \ldots, x_n$ are distinct, then $f(x_0), \ldots f(x_n)$ are mutually 1-generic.*
(2) *For all $x$, $f(x)$ is $x$-generic.*

*Proof.* If (2) holds, then $\mathrm{ran}(f)$ is nonmeager. This is true because if $\mathrm{ran}(f)$ is meager, the complement of $\mathrm{ran}(f)$ is comeager and hence it would contain a dense $G_\delta$ set $A$ which is coded by some real $z$. But then $f(z)$ is $z$-generic, and so $f(z) \in A$, so $f(z) \notin \mathrm{ran}(f)$.

Now $\mathrm{ran}(f)$ is $\mathbf{\Sigma}^1_1$ and so it has the Baire property. Since $\mathrm{ran}(f)$ is nonmeager, it is therefore comeager in some basic open set $N_s$. But this implies that $\mathrm{ran}(f)$ contains two elements $f(x_0) \neq f(x_1)$ which are equal mod finite and hence are not mutually 1-generic. □

The point of this section is to prove Lemma 4.2 where we make (1) above compatible with a weakening of (2). Instead of $f(x)$ being $x$-generic, we can make $f(x)$ and $x$ a minimal pair in the Turing degrees. Indeed, we have a slightly stronger statement. Given a countable Borel equivalence relation $E$, we ensure that if $x \mathrel{E} y$, then for every computable set $B \subseteq \omega$, $f(x)$ and $y \upharpoonright B$ form a minimal pair. This lemma will be an essential ingredient in the proof of Theorem 5.1.

**Lemma 4.2.** *Suppose $E$ is a countable Borel equivalence relation on $X$. Then there is a Borel function $f\colon 2^\omega \to 2^\omega$ such that*

1. *If $x_0, \dots, x_n$ are distinct, then $f(x_0), \dots f(x_n)$ are mutually 1-generic.*
2. *For all $x, y, z \in 2^\omega$ such that $x \mathrel{E} y$, and all computable sets $B \subseteq \omega$, if $(y \upharpoonright B) \geq_T z$ and $f(x) \geq_T z$, then $z$ is computable.*

This lemma follows easily from the more general Lemma 4.3

*Proof of Lemma 4.2:* Apply Lemma 4.3 where with $X = Y = 2^\omega$, $C_n \subseteq (2^\omega)^n$ the set of mutually 1-generic $n$-tuples, $S^n$ is the relation $\geq_T$ on $(2^\omega)^n \times 2^\omega$ and $y \mathrel{R} z$ if $y \upharpoonright B \geq_T z$ for a computable set $B$. A standard argument shows that if $y \in 2^\omega$ is such that a nonmeager set of $(x_1, \dots, x_n) \in (2^\omega)^n$ have $y \leq_T (x_1, \dots, x_n)$, then $y$ is computable. □

We now prove the following strengthening of the Kuratowski-Mycielski theorem [K95, Theorem 19.1]. Say that a relation $R \subseteq X \times Y$ has *countable vertical sections* if for all $x \in X$ there are countably many $y \in Y$ such that $x \mathrel{R} y$.

**Lemma 4.3.** *Suppose $E$ is a countable Borel equivalence relation on a Polish space $X$. Let $Y, Z$ be Polish spaces and $R \subseteq X \times Z$ and $S_n \subseteq Y^n \times Z$ be Borel relations with countable vertical sections. Then for any collection $(C_n)_{n\in\omega}$ of comeager sets $C_n \subseteq Y^n$, there is a Borel injection $f\colon X \to Y$ such that*

1. *For all $x_1, \dots, x_n \in X$, $(f(x_1), \dots, f(x_n)) \in C_n$.*
2. *For all $x \in X$ and distinct $x_1, \dots, x_n \in [x]_E$, if $x \mathrel{R} z$ and $(g(x_1), \dots, g(x_n)) \mathrel{S_n} z$, then there is a nonmeager set of $\vec{y} \in Y^n$ such that $\vec{y} \mathrel{S_n} z$.*

Roughly this says that there is a Borel function $g$ so that any finitely many elements of $\mathrm{ran}(g)$ are "mutually generic" (i.e. in $C_n$), and that if $x_1, \dots, x_n \in [x]_E$, then $x$ and $(g(x_1), \dots, g(x_n))$ form a "minimal pair" (with respect to $R$ and $S_n$).

*Proof.* Fix countable bases $\mathcal{B}_X, \mathcal{B}_Y, \mathcal{B}_Z$ of $X, Y$, and $Z$. Also fix a complete metric $d$ generating the topology of $Y$. Say that an *approximation* $p$ of $f$ is a function $p\colon P \to \mathcal{B}_Y$ where $P$ is a Borel partition of $X$ into finitely many Borel sets. Say that an approximation $p'\colon P' \to \mathcal{B}_Y$ refines $p\colon P \to \mathcal{B}_Y$ if $P'$ refines $P$, and if $A' \in P'$ and $A \in P$ are such that $A' \subseteq A$, then $p'(A') \subseteq p(A)$

Suppose that $p_0, p_1, \dots$ is a sequence of approximations where $p_{n+1}$ refines $p_n$,

(a) $\max\{\mathrm{diam}(U)\colon U \in \mathrm{ran}(p_n)\} \to 0$ as $n \to \infty$
(b) for all $n$, there exists $m > n$, so that $A \in \mathrm{dom}(p_n)$, $A' \in \mathrm{dom}(p_m)$ and $A' \subseteq A$ implies $\mathrm{cl}(p_m(A')) \subseteq p_n(A)$, where cl denotes closure.

Then we can associate to this sequence the function $f\colon X \to Y$ where $f(x) = y$ if $\{y\} = \bigcap_n p_n(A_{x,n})$ where $A_{x,n}$ is the unique element of $\mathrm{dom}(p_n)$ such that $x \in A_n$. Conditions (a) and (b) ensure that $\bigcap_n p_n(A_{x,n})$ is a singleton for every $x$. We will construct $f$ in this way. Clearly a generic such sequence $p_n$ will have $\max\{\mathrm{diam}(U)\colon U \in \mathrm{ran}(p_n)\} \to 0$ and (1) in the statement of the Lemma will be true for a generic such $f$. We need to ensure (2) is true.

Since $R$, $S_n$ have countable vertical sections, by Lusin-Novikov uniformization [K95, 18.5], there are Borel functions $(g_i)_{i\in\omega}$ and $(h_{n,i})_{i,n\in\omega}$ where $g_i\colon X \to Z$ and $h_{n,i}\colon Y^n \to Z$ such that $x \mathrel{R} z$ iff $f_i(x) = z$ for some $i$, and $\vec{y} \mathrel{S_n} z$ iff $h_{n,i}(\vec{y}) = z$ for some $i$. By perhaps refining the sets $C_n$, we may assume that the functions $h_{n,i}$ are continuous on $C_n$, since any Borel function is continuous on a comeager set [K95, Theorem 8.38]. Fix a Borel action of a countable group $\Gamma$ generating $E$. Let $\mathcal{G}$ be the set of $z \in Z$ such that for some $n$, there is a nonmeager set of $\vec{y} \in Y^n$ such that $zS_n\vec{y}$. Note that $\mathcal{G}$ is countable.

Fix an approximation $p$, finitely many disjoint basic open sets $V_1, \ldots, V_n \subseteq X$ and group elements $\gamma_1, \ldots, \gamma_n \in \Gamma$, and $j, k \in \omega$. It suffices to show that we can refine $p$ to an approximation $p^*$ such that for all $x \in X$, if $\gamma_i \cdot x \in V_i$ for all $i$, either $h_j(x) \in \mathcal{G}$, or

$$(*) \qquad g_j(x) \notin (h_{n,k} \restriction C_n)(p(A'_1) \times \ldots \times p(A'_n))$$

where $A_i \in \mathrm{dom}(p')$ is such that $\gamma_i \cdot x \in A'_i$. That is, the value of $h_{n,k}(f(\gamma_1 \cdot x), \ldots, f(\gamma_n \cdot x))$ is "forced" by $p'$ to be different from $g_j(x)$.

Let $P = \mathrm{dom}(p)$. By refining $p$, we may assume every element of $P$ is either contained in or disjoint from $V_i$. For $x \in X$, let $[x]_P$ be the element of $P$ that contains $x$. If $A$ is a subset of $X$ contained in a single element of $P$, let $[A]_P$ be this element. Let $F$ be the equivalence relation on $\bigcup_i V_i$ where $x \mathrel{F} x'$ if $x \in V_i$, $x' \in V_j$, and

$$[\gamma_k\gamma_i^{-1} \cdot x]_P = [\gamma_k\gamma_j^{-1} \cdot x']_P \text{ for all } k.$$

Clearly $F$ has finitely many equivalence classes since there are finitely many $\gamma_i$ and element of $P$. We also have that for every equivalence class $A$ of $F$ and each $i$, $A \cap V_i$ is contained in a single element of $P$.

For each equivalence class $A$ of $F$, let $U_{i,A} = p([A \cap V_i]_P)$. Now let $U'_{i,A}, U''_{i,A} \subseteq U_{i,A}$ be basic open sets so that $(h_{n,k} \restriction C_n)((U'_{1,A}, \ldots, U'_{n,A})) \subseteq W'_A$ and $(h_{n,k} \restriction C_n)((U''_{1,A}, \ldots, U''_{n,A})) \subseteq W''_A$ for disjoint basic open sets $W'_A, W''_A \subseteq Z$. If such sets do not exist, then since $h_{n,k} \restriction C_n$ is continuous, then $(h_{n,k} \restriction C_n)((U_{1,A}, \ldots, U_{n,A}))$ must be a singleton, which must therefore be in $\mathcal{G}$. For $A \in F$, let $\phi(A)$ be the set of $x \in A$ such $g_j(\gamma_i^{-1} \cdot x) \in W'_A$, where $i$ is such that $x \in V_i$. Let $Q$ be the set of equivalence classes of $F$.

We're ready to define $p^*$. Let $\mathrm{dom}(p^*)$ be the partition

$$\begin{aligned}\mathrm{dom}(p^*) = \{B \in P\colon \forall i \in nB \not\subseteq V_i\} \cup \{\phi(A) \cap V_i\colon i \in n \wedge A \in Q\} \\ \cup \{(A \setminus \phi(A)) \cap V_i\colon i \in n \wedge A \in Q\}\end{aligned}$$

For $B \in P$ such that $B \not\subseteq V_i$ for all $i$, let $p^*(B) = p(B)$. For $A \in Q$, let $p^*(\phi(A) \cap V_i) = U''_{i,A}$, and let $p^*((A \setminus \phi(A)) \cap V_i) = U'_{i,A}$. Then $p^*$ satisfies (*) by definition. □

We remark that there are interesting open problems about the extent to which the Kuratowski-Mycielski theorem can be generalized. For example,

**Question 4.4.** *Does there exist a Borel function $g\colon 2^\omega \to 2^\omega$, so that for all distinct $x, y$ with $x \leq_T y$, $g(x)$ and $g(y)$ are mutually $x$-generic?*

## 5. A nonuniform construction

In this section, we prove that a positive answer to Question 3.4 implies the existence of various non-uniform invariant functions, using the following theorem:

**Theorem 5.1.** *Suppose $E$ is a hyper-Borel-finite Borel equivalence relation on $2^\omega$. Then there exists an injective Borel function $f\colon 2^\omega \to 2^\omega$ such that for all $x_0, x_1 \in 2^\omega$*

1. *If $x_0 \mathrel{E} x_1$, then $f(x_0) \equiv_1 f(x_1)$*
2. *If $x_0 \not\mathrel{E} x_1$, then $f(x_0) \not\equiv_m f(x_1)$.*
3. *For every noncomputable $x$, $f(x)$ is $\leq_m$-incomparable with both $x$ and $\overline{x}$.*
4. *For all $x \in 2^\omega$, there does not exist an infinite sequence $(x_i)_{i\in\omega}$ of distinct elements of $[x]_E$ such that $f(x) \geq_m \bigoplus_i f(x_i)$.*

Hence, we have the following corollaries

**Corollary 5.2.** *Suppose $\equiv_T$ is hyper-recursively-finite. Then*

1. *There is a Borel homomorphism from $\equiv_T$ to $\equiv_m$ which is not uniform on any pointed perfect set. Hence, Conjecture 2.3 is false.*
2. *$\equiv_m$ and $\equiv_1$ on $2^\omega$ are universal countable Borel equivalence relations. Hence, there is a universal countable Borel equivalence relation which is not uniformly universal. So* [M, Conjecture 1.1] *is false.*

*Proof.* To prove (1), let $f$ be as in Theorem 5.1 for the equivalence relation $E = \equiv_T$. Let $\Phi_e\colon 2^\omega \to 2^\omega$ be a total Turing functional with inverse $\Phi_d\colon 2^\omega \to 2^\omega$ such that $x, \Phi_e(x), \Phi_e(x)^2, \ldots$ are all distinct and have the same Turing degree. Then if $f$ was uniformly $(\equiv_T, \equiv_m)$-invariant it would contradict condition (4) of Theorem 5.1.

To prove (2), note first that if every countable Borel equivalence relation $E$ is hyper-Borel-finite, the function $f$ given in Theorem 5.1 is a Borel reduction from $E$ to many-one equivalence $\equiv_m$ on $2^\omega$. However, $\equiv_m$ is not uniformly universal by [M, Theorem 1.5.(5)]. □

It is open whether there is a counterexample to Martin's conjecture or Steel's conjecture assuming $\equiv_T$ is hyper-recursively-finite.

**Question 5.3.** *Assume $\equiv_T$ is hyper-recursively-finite. Is Martin's conjecture false? Is Steel's conjecture false?*

We are now ready to prove Theorem 5.1.

*Proof of Theorem 5.1.* A very rough idea of the proof is as follows. Let $g\colon 2^\omega \to 2^\omega$ be the Borel function from Lemma 4.2 where $\mathrm{ran}(g)$ is a set of mutual 1-generics and $y$ and $g(x)$ form a minimal pair for all $y \mathrel{E} x$. We will build a Borel function $f$ so that $f(x)(n) = g(x)(n)$ for infinitely many $n$, and the remaining bits of $f(x)$ will "code" the values of $f(y)$ for $y$ such that $y \mathrel{E} x$, so $f$ satisfies (1). Precisely, we will fix a witness $E_0 \subseteq E_1 \subseteq \ldots$ to the hyper-Borel-finiteness of $E$, and at stage $s$ of the construction, we will code $f(y)$ into $f(x)$ for all $y$ such that $y \mathrel{E_s} x$. We will show that we cannot find infinitely many places in $f(x)$ where we have coded values of $f(y)$ in a computable way; this would contradict our choice of the equivalence relations $E_s$. Then mutual 1-genericity of $g$ ensures that from finitely many $g(x_1), \dots, g(x_n)$ we cannot compute $g(y)$ for any other $y$. This will imply (2). The fact that $x$ and $g(x)$ form a minimal pair will imply (3).

Let $\rho_0, \rho_1, \dots$ be an enumeration of all total computable functions from $\omega$ to $\omega$. That is, all many-one reductions. To begin, we define infinite computable sets $(D_i)_{i\in\omega}$ where $D_i \subseteq \omega$ and computable injective functions $c_s \colon \{(n,i,m) \in \omega^3 \colon i,n < s\} \to \omega$. We will use the sets $D_i$ for diagonalization; for every $x$, $f(x) \upharpoonright D_i$ will be generic. We will use the functions $c_s$ for coding $f(y)$ into $f(x)$ for every $y$ such that $y\ E\ x$. Precisely, we'll have that if $x\ E\ y$, then there is some fixed $n, i, s$ such that for all $m$, $f(y)(m) = f(x)(c_s(n,i,m))$.

Let $D_0$ be any infinite coinfinite computable set such that $0 \in D_0$, and let $c_0$ be the empty function. We will define $D_{s+1}$ and $c_{s+1}$ simultaneously by induction. They will have the properties that $D_s \subseteq D_{s+1}$ and $c_s \subseteq c_{s+1}$. We will also ensure that:

(1) For every $s$, $D_s$ and $\operatorname{ran}(c_s)$ are disjoint and their union is coinfinite.
(2) For every $i, n < s$, the function $m \mapsto c_s(n,i,m)$ is strictly increasing. In particular, this implies $\operatorname{ran}(c_s)$ is computable, and $c_s$ has a computable inverse.

We will define a function $c'_s$ on sequences which iterates applying the coding $c_s$. Let $c'_s$ be defined on sequences $(n_0, i_0, \dots, n_k, i_k, m)$ where $i_j, n_j < s$ by

$$c'_s(n_0, i_0, \dots, n_k, i_k, m) = c_s(n_0, i_0, c_s(n_1, i_1, \dots c_s(n_k, i_k, m)))).$$

Since $c_s$ is injective, if $c'_s(n_0, i_0, \dots, n_k, i_k, m) = c'_s(n'_0, i'_0, \dots, n'_{k'}, i'_{k'}, m')$, then either $(n_0, i_0, \dots, n_k, i_k)$ is an initial segment of $(n'_0, i'_0, \dots, n'_{k'}, i'_{k'})$, or vice versa. Since $m \mapsto c_s(n,i,m)$ is strictly increasing, for each $m$, there is some sequence $(n_0, i_0, \dots, n_k, i_k, m')$ of maximal length such that $c'_s(n_0, i_0, \dots, n_k, i_k, m') = m$. Define the function $d'_s$ by $d'_s(m) = (n_0, i_0, \dots, n_k, i_k, m')$ where $(n_0, i_0, \dots, n_k, i_k, m')$ is the maximal length sequence such that $c'_s(n_0, i_0, \dots, n_k, i_k, m') = m$. We think of $d'_s$ as "decoding" $m$ according to $c'_s$. Note that $d'_s$ is computable, since $c_s$ has a computable inverse by (2).

Next we define $D_{s+1}$. First, let

$$\pi((n_0, i_0, \dots, n_k, i_k, m)) = m$$

be the projection of a sequence onto its last coordinate. Let $A_s = \pi(d'_s(\operatorname{ran}(\rho_s)))$, so $A_s$ is the bits that elements in $\operatorname{ran}(\rho_s)$ are eventually "decoded" to. Note that $A_s \cap \operatorname{ran}(c_s) = \emptyset$ by definition of $d'_s$. The set $A_s$ is c.e. since $d'_s$ is computable. If $A_s$ is finite, then let $D_{s+1} = D_s$. If $A_s$ is infinite, then let $A_s^*$ be an infinite computable subset of $A_s$ such that $A_s \setminus A_s^*$ is infinite. Furthermore, let $A_s^*$ be so that if $(C_n)_{n\in\omega}$ is the $e$th uniformly c.e. sequence of subsets of $\omega$ where $e \leq s$, and $C_n$ is infinite for every $n$ and $C_n \subseteq A_s$ for every $n$, then $C_n \cap A_n^*$ is infinite for every $n$. (Such an $A^*$ exists by a straightforward argument). Let $D_{s+1} = D_s \cup A_s^*$.

Finally, let $c_{s+1}$ be a computable injection extending $c_s$ satisfying (1) and (2), and such that $\operatorname{ran}(c_{s+1})$ contains the least element of $\omega \setminus (\operatorname{ran}(c_s) \cup D_{s+1})$. Let $c = \bigcup_s c_s$, $c' = \bigcup_s c'_s$, and $D = \bigcup_s D_s$. Since $\operatorname{ran}(c_{s+1})$ contains the least element of $\omega \setminus (\operatorname{ran}(c_s) \cup D_{s+1})$ we have that $D \cup \operatorname{ran}(c) = \omega$.

Now that we have defined $D_s$ and $c_s$, we will define $f$ in terms of them, and then show it has the required properties.

Fix a Borel action of a countable group $\Gamma$ generating $E$, and an enumeration $\{\gamma_0, \gamma_1, \ldots\}$ of $\Gamma$. Let $d\colon 2^\omega \to (2^\omega)^{(\omega^{<\omega})}$ be the function where $d(x)$ is the function $(i_0, \ldots, i_k) \mapsto \gamma_{i_k} \cdots \gamma_{i_0} \cdot x$, which codes the "diagram" of the action of $\Gamma$ on $x$. (The reason we do not use the simpler function $i \mapsto \gamma_i \cdot x$ is that we're not making any assumptions on how complicated group multiplication is to compute.)

Fix an increasing sequence of Borel equivalence relations $E_0 \subseteq E_1 \subseteq \ldots$ witnessing that $E$ is hyper-$(f_i)$-finite with respect to the functions $f_i$ where $f_i(x) \in (2^\omega)^\omega$ is the $i$th sequence arithmetically definable from $d(x)$. That is, $d(x) \geq_A f_i(x)$ via the $i$th arithmetic reduction.

Let $\ell\colon \{(x,y)\colon x\ E\ y\} \to \omega$ be the function where $\ell(x,y) = n$ if $n$ is the least number such that $x\ E_n\ y$. Let $g\colon 2^\omega \to 2^\omega$ be the function $f$ in Lemma 4.2, letting the relation $E$ be our given equivalence relation $E$.

We are now ready to define our function $f\colon 2^\omega \to 2^\omega$. Define

$$f(x)(n) = \begin{cases} f(\gamma_i \cdot x)(m) & \text{if } n \in \operatorname{ran}(c) \text{ and } n = c(\ell(x, \gamma_i \cdot x), i, m) \\ g(x)(n) \text{ otherwise.} & \end{cases}$$

This definition is self-referential, but it is not circular. If $f(x)(n) = f(\gamma_i \cdot x)(m)$ after applying the definition, then $m < n$ by condition (2) above. Hence, given any $n$, by applying the above definition finitely many times we must obtain some sequence of $i_0, \ldots, i_k$ and an $m$ such that $f(x)(n) = g(\gamma_{i_k} \cdots \gamma_{i_0} \cdot x)(m)$.

Let $G_x$ be the set of $n$ such that $n \notin \operatorname{ran}(c)$ or $n = c(k,i,m)$ and $k \neq \ell(\gamma_i \cdot x, x)$. Hence, $f(x) \upharpoonright G_x = g(x) \upharpoonright G_x$. We think of $G_x$ as being the "generic bits" of $f(x)$.

Our central claim is the following:

**Claim.** *Fix $x \in 2^\omega$, and suppose $z \leq_m f(x)$. Then either $z$ is computable, or there is an infinite computable set $B \subseteq \omega$ and some $y\ E\ x$ so that $z \upharpoonright B \leq_m g(y)$.*

*Proof.* Let $\rho_s$ be a many-one reduction witnessing $z \leq_m f(x)$. Recall our decoding function $d'_s$. If $d'_s(m) = (n_0, i_0, \ldots, n_k, i_k, m')$, then $f(x)(m) = g(\gamma_{i_k} \cdots \gamma_{i_0} \cdot x)(m')$ provided that for all $j \leq k$, $n_j = \ell(\gamma_{i_j} \cdots \gamma_{i_0} \cdot x, \gamma_{i_{j-1}} \cdots \gamma_{i_0} \cdot x)$. We now define a version $d_s$ of this decoding which depends on both our fixed $x \in 2^\omega$ and $Y \subseteq [x]_E$, where $Y$ is a finite subset of the equivalence class of $x$ corresponding to places where we are permitted to decode. Precisely, let

$$d_s(m, Y) = (n_0, i_0, \ldots, n_k, i_k, m')$$

if $c'_s(n_0, i_0, \ldots, n_k, i_k, m') = m$ and $(n_0, i_0, \ldots, n_k, i_k, m')$ is the longest such sequence so that for every $j \leq k$, we have $\gamma_{i_j} \cdots \gamma_{i_0} \cdot x \in Y$.

Define an increasing sequence $Y_0 \subseteq Y_1 \subseteq \ldots$ of finite subsets of $[x]_E$ as follows: $Y_0 = \{x\}$ and

$$Y_{j+1} = Y_j \cup \{\gamma_{i_j} \cdots \gamma_{i_0} \cdot x \colon (\forall k \leq j) i_k < s\}.$$

So $Y_j$ is the set of elements of $[x]_E$ that can be reached from $x$ by multiplying by at most $j+1$ many group elements from $\{\gamma_0, \ldots, \gamma_{s-1}\}$.

To each $y \in Y_j$, associate the set

$$N_{j,y} = \{m \in \operatorname{ran}(\rho_s) \colon d_s(m, Y_j) = (n_0, i_0, \ldots, n_k, i_k, m_k) \wedge \gamma_{i_k} \cdots \gamma_{i_0} \cdot x = y\}.$$

Intuitively, this is the set of $m \in \operatorname{ran}(\rho_s)$ which are decoded to eventually correspond to some bit of $f(y)$, provided the sequence $n_0, n_1, \ldots n_k$ corresponds to the appropriate values of $\ell$. Note that $N_{j,y}$ is c.e. This is because $c_s$ has a computable inverse, and we just need to use the finite information of how multiplication by

$\gamma_0, \ldots, \gamma_{s-1}$ works inside the finite set $Y_j$. By definition, the sets $N_{j,y}$ partition $\mathrm{ran}(\rho_s)$. We now break into cases.

*Case 1:* There exists $j$ and $y \in Y_j$ such that there are infinitely many $m \in N_{j,y}$ such that $d_s(m, Y_j) = (n_0, i_0, \ldots, n_k, i_k, m')$, and for some $k' \leq k$, we have $n_{k'} \neq \ell(\gamma_{i_{k'}} \cdots \gamma_{i_0} \cdot x, \gamma_{i_{k'-1}} \cdots \gamma_{i_0} \cdot x)$.

Define the function $r$ on such tuples $(n_0, i_0, \ldots, n_k, i_k, m)$ by:

$$r((n_0, i_0, \ldots, n_k, i_k, m')) = (n_0, i_0, \ldots, n_{k'}, i_{k'}, m'')$$

where $k' \leq k$ is least such that $n'_k \neq \ell(\gamma_{i_{k'}} \cdots \gamma_{i_0} \cdot x, \gamma_{i_{k'-1}} \cdots \gamma_{i_0})$, and $m''$ is the unique value such that $c'_s(n_0, i_0, \ldots, n_k, i_k, m') = c'_s(n_0, i_0, \ldots, n_{k'-1}, i_{k'-1}, m'')$. In this case, since $n'_k$ is not the "correct" value of $\ell$, we have $m'' \in G_{\left(\gamma_{i_{k'-1}} \cdots \gamma_{i_0} \cdot x\right)}$. So by the definition of $f$,

$$m \in f(x) \iff \pi(r(d_s(m, Y_j))) \in g(\gamma_{i_{k'-1}} \cdots \gamma_{i_0} \cdot x).$$

By the pigeonhole principle, there must be some $y' \in Y_j$ so that there are infinitely many $m \in N_{j,y}$ such that if $r(d_s(m, Y_j)) = (n_0, i_0, \ldots, n_{k'-1}, i_{k'-1}, m'')$, we have $y' = \gamma_{i_{k'-1}} \cdots \gamma_{i_0} \cdot x$. Fixing an enumeration $y_0, \ldots, y_l$ of $Y_j$, the function $(n_0, i_0, \ldots, n_k, i_k, m') \mapsto r((n_0, i_0, \ldots, n_k, i_k, m'))$ is computable from the finitely many values of $\ell$ on elements of $Y_j$, and so there is an infinite computable set $B$ so that $z \restriction B \leq_m g(y')$ via the function $n \mapsto \pi(r(d_s(\rho_s(n), Y_j)))$. This proves the claim in this case.

For the remaining cases, we may assume that for all $j$ and $y \in Y_j$, for all but finitely many $m \in N_{j,y}$, if $d_s(m, Y_j) = (n_0, i_0, \ldots, n_k, i_k, m')$, then $m \in f(x) \iff m' \in f(\gamma_{i_k} \cdots \gamma_{i_0} \cdot x)$.

Let

$$M_{j,y} = \pi(d_s(N_{j,y}, Y_j)),$$

so $M_{j,y}$ is c.e.

*Case 2:* Case 1 does not occur and there is some $j$ such that for every $y \in Y_j$, $M_{j,y}$ is finite.

That is, after decoding inside $Y_j$, the entire many-one reduction uses only finitely many bits from the finitely many $f(y)$ with $y \in Y_j$. In this case, $z$ is computable using the computability of $m \mapsto d_s(m, Y_j)$ (which uses the computability of the inverse of $c_s$, and using the finite amount of information of how group multiplication by $\gamma_0, \ldots, \gamma_{s-1}$ works on the set $Y_j$). This proves the claim in this case.

Let $M'_{j,y} \subseteq M_{j,y}$ be defined by

$$M'_{j,y} = \{m \in M_{j,y} \colon m \in \mathrm{ran}(\rho_s) \wedge m = c_s(n, i, m') \wedge \gamma_i \cdot y \notin Y_j\}.$$

Hence if $m \in M'_{j,y}$, we will be able to "decode" $m$ further inside $Y_{j+1}$. Note that if $m \in M_{j,y} \setminus M'_{j,y}$, then $m \in A_s$.

*Case 3:* Case 1 does not occur and there is some $j$ such that $M_{j,y}$ is infinite for some $y \in Y_j$, but $M'_{j,y}$ is finite for every $y \in Y_j$.

The sets $\{M_{j,y} \setminus M'_{j,y} \colon y \in Y_j\}$ cover $A_s$ modulo a finite set, and $A_s$ must be infinite. Hence, there must be some fixed $y' \in Y_j$ so that $M_{j,y'}$ contains infinitely many elements of $A^*_s$ by the pigeonhole principle. Since $A^*_s$ is computable and $M_{j,y'}$ is c.e., we can find an infinite computable subset $C \subseteq M_{j,y'}$ such that $\pi_\infty(d_s(C, Y_j))$ is an infinite computable subset of $A^*_s$, and $A^*_s \subseteq G_{y'}$. Let $B$ be an infinite computable set so that if $n \in B$, then $\rho_s(n) \in C$. Then $z \restriction B \leq_m g(y')$.

*Case 4:* Case 1 does not occur and for every $j$, there is some $y \in Y_j$ such that $M'_{j,y}$ is infinite.

Every $m \in M'_{j,y}$ has $m = c_s(n,i,m')$ for some $m'$, and $i,n < s$. So by the pigeonhole principle, there must be some $n,i$ such that there are infinitely many $m'$ such that $c_s(n,i,m') \in M'_{j,y}$. This implies that $\gamma_i \cdot y \notin Y_j$, and $\gamma_i \cdot y \; E_s \; y$ by induction and since Case 1 is not true. Hence, the set of $\gamma_i \cdot y$ such that there exists some $j$ and some $i,n < s$ such that there are infinitely many $m'$ with $c_s(n,i,m') \in M'_{j,y}$ is an infinite subset of $[x]_E$, which is arithmetically definable from $d(x)$. This contradicts our choice of $E_0 \subseteq E_1 \subseteq \ldots$. □ Claim.

We now check $f$ has all the desired properties.

(1) Since each function $m \mapsto c(n,i,m)$ is computable, it is clear that if $x \; E \; y$, then $f(x) \equiv_1 f(y)$.

(2) Suppose $f(y) \leq_m f(x)$ for some $x,y \in 2^\omega$. We wish to show that $y \; E \; x$. Then $f(y) \restriction D_0 = g(y) \restriction D_0$, so $g(y) \restriction D_0 \leq_m f(x)$. By our claim, there is some $y' \; E \; x$ so that $g(y) \restriction D_0 \leq_m g(y')$. We must have $y = y'$, otherwise a recursive subset of $g(y)$ is many-one reducible to $g(y')$ contradicting their mutual one-genericity.

(3) Since $G_x$ contains $D_0$ which is an infinite computable set, $x \not\geq_m f(x) \restriction D_0 = g(x) \restriction D_0$ by our choice of $g$. So $x \not\geq_m f(x)$. We see that $\overline{x} \not\geq_m f(x)$ similarly. Finally, suppose $x$ is incomputable. We claim $x \not\leq_m f(x)$ and $\overline{x} \not\leq_m f(x)$. To see this, let $z = \rho_s^{-1}(f(x))$. If $z = x$ or $z = \overline{x}$, then $z$ is not computable. Hence, by the claim, there is some computable set $B$ so that $z \restriction B \leq_m g(y)$ for some $y \; E \; x$ via a many-one reduction with infinite range. But this contradicts our choice of $g$.

(4) can be proved essentially by applying the claim uniformly in each column of the reduction. Let $\rho\colon \omega \to D_0$ be the unique increasing function from $\omega$ to $D_0$ (which is computable). For a contradiction, suppose $\bigoplus f(x_i) \leq_m f(x)$. Then $\bigoplus_i \rho^{-1}(f(x_i)) = \bigoplus_i \rho^{-1}(g(x_i)) \leq_m f(x)$. Fix the $\rho_s$ witnessing this many-one reduction. Let $d_s$ and $Y_j$ be defined as in the above claim. Define versions of $N_{j,y}$, $M_{j,y}$ and $M'_{j,y}$ which correspond to each column of the reduction $\rho_s$ as follows:

$$
\begin{aligned}
N_{n,j,y} &= \{m = \rho_s(\langle n,l\rangle)\colon d_s(m,Y_j) = (n_0,i_0,\ldots,n_k,i_k,m_k) \wedge \gamma_{i_k}\cdots\gamma_{i_0}\cdot x = y\}\\
M_{n,j,y} &= \pi(d_s(N_{n,j,y},Y_j))\\
M'_{n,j,y} &= \{m \in M_{n,j,y}\colon m \in \operatorname{ran}(c_s) \wedge m = c_s(n,i,m') \wedge \gamma_i \cdot y \notin Y_j\}.
\end{aligned}
$$

*Case 1*: There exists $j$ and $y \in Y_j$ such for infinitely many $n$ there exist infinitely many $m \in N_{n,j,y}$ such that $d_s(m,Y_j) = (n_0,i_0,\ldots,n_k,i_k,m')$, and for some $k' \leq k$, we have $n'_k \neq \ell(\gamma_{i'_k}\cdots\gamma_{i_0}\cdot x, \gamma_{i_{j-1}}\cdots\gamma_{i_0})$.

By the pigeonhole principle, it is clear that there is some $y' \in Y_j$ such that for infinitely many $n$ so there is some recursive infinite subset $B$ of $D_0$ such that $f(x_n) \restriction B = g(x_n) \restriction B$ is many-one reducible to $g(y')$. This contradicts the mutual genericity of $g$, proving (4) in this case.

*Case 2:* Case 1 does not occur and there exists $j$ so for infinitely many $n$, for all $y \in Y_j$, $M_{n,j,y}$ is finite.

Arguing as in Case 2 of the claim, this means for infinitely many $n$, $f(x_n) \restriction D_0 = g(x_n) \restriction D_0$ is computable, contradicting its genericity.

*Case 3:* Case 1 and 2 do not occur and there exists $j$ so that for all but finitely many $n$, $M'_{n,j,y}$ is finite for all $y \in Y_j$.

Fix such a $j$. Using the finite amount of information of how group multiplication by $\gamma_0,\ldots,\gamma_{s-1}$ works on the set $Y_j$, we see $d_s(m,Y_j)$ is computable, and hence the sets $C_n = \bigcup_{y\in Y_j} M_{n,j,y} \setminus \bigcup_{y\in Y_j} M'_{n,j,y}$ are uniformly c.e. sets. Choose $N$ sufficiently large so that for $n \geq N$, $C_n$ is infinite and is contained in $A_s$ modulo a finite set (since Case 2 does not occur). By the padding lemma we may assume $s$ is greater than the program computing the uniformly c.e. sequence $(C_n)_{n\geq N}$. By construction of $D_{s+1}$, for every $n \geq N$ there are infinitely many $m \in C_n$ such that $m \in D_{s+1}$.

Now by the pigeonhole principle, there is some $y' \in Y_j$, such that for infinitely many $n$ there is some recursive infinite subset $B$ of $D_0$ such that $f(x_n) \restriction B = g(x_n) \restriction B$ is many-one reducible to $g(y')$. This contradicts the mutual genericity of $g$.

*Case 4:* Case 1 does not occur and for every $j$, there are infinitely many $n$ so that $M'_{n,j,y}$ is infinite for some $y \in Y_j$.

This contradicts our choice of $E_0 \subseteq E_1 \subseteq \ldots$. Consider the set of $\gamma_i \cdot y$ such that there exists $j$ and $y \in Y_j$ such that for infinitely many $n$, there exist $n', i < s$ so that for infinitely many $m'$, we have $c_s(n', i, m') \in M'_{n,j,y}$ This is an infinite subset of $[x]_E$ that is arithmetically definable from $d(x)$. □

## 6. Open questions

We pose a conjecture which would give a negative answer to Question 3.4. It states in a strong way that Turing equivalence cannot be nontrivally written as an increasing union of Borel equivalence relations.

**Conjecture 6.1.** *Suppose we write Turing equivalence as an increasing union $(\equiv_T) = \bigcup_n E_n$ of Borel equivalence relations $E_n$ where $E_n \subseteq E_{n+1}$ for all $n$. Then there exists a pointed perfect set $P$ and some $i$ so that $E_i \restriction P = (\equiv_T \restriction P)$.*

In the context of probability measure preserving equivalence relations, an analogous phenomenon of non-approximability has been proved by Gaboriau and Tucker-Drob [GTD], e.g. for pmp actions of property (T) groups.

We know that Conjecture 6.1 implies some consequences of Martin's conjecture. In particular, Conjecture 6.1 implies that Martin measure is $E_0$-ergodic in the sense of [T].

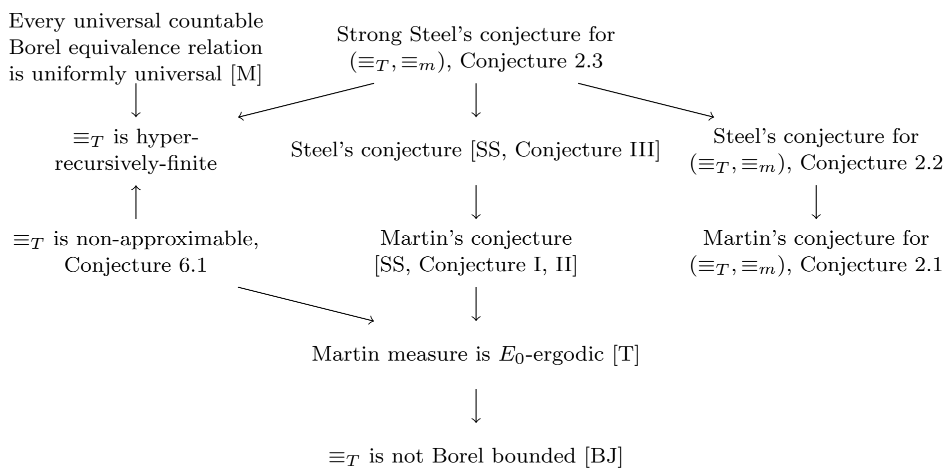
**Proposition 6.2.** *Suppose Conjecture 6.1 is true. Then if $f\colon 2^\omega \to 2^\omega$ is a Borel homomorphism from Turing equivalence to $E_0$, i.e. $x \equiv_T y \implies f(x) E_0 f(y)$, then the $E_0$-class of $f(x)$ is constant on a Turing cone.*

*Proof.* Let $E_n$ be the subequivalence relation of $\equiv_T$ defined by $x\ E_n\ y$ if $x \equiv_T y$ and $\forall k \geq n (f(x)(k) = f(y)(k))$. That is the $f(x)$ and $f(y)$ are equal past the first $n$ bits. By Conjecture 6.1, there is some $i$ and some pointed perfect set $P$ such that $E_i \restriction P = (\equiv_T \restriction P)$. Then by [MSS][Lemma 3.5] there is some pointed perfect set $P' \subseteq P$ such for $x, y \in P'$, if $x \equiv_T y$, then $f(x) = f(y)$. Define $f'(x) = f(y)$ if there is $y \in P'$ such that $x \equiv_T y$, and $f'(x) = \emptyset$ otherwise. Thus, $f'\colon 2^\omega \to 2^\omega$ is such that if $x \equiv_T y$, then $f'(x) = f'(y)$. Now any homomorphism from $\equiv_T$ to equality must be constant on a Turing cone, so $f'$ is constant on a Turing cone. This implies the $E_0$-class of $f$ is constant on a cone. □

It is open if Conjecture 6.1 implies Martin's conjecture.

**Question 6.3.** *Assume Conjecture 6.1 is true. Does this imply Martin's conjecture for Borel functions?*

The following is a diagram of some open questions surrounding Martin's conjecture. All relationships between these open problems which are not indicated by arrows are open.

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