Optimization

1. An aluminum soft drink can has a volume of 128π cubic centimeters. In order to conserve resources, a soda company wants to minimize the amount of aluminum needed for a single can. What dimension should they make their cans? **Solution.**

   1. **Name (and understand) some variables! (If applicable, draw a picture to help with this.)** We are asked for dimensions of the can, so let's give names to these. Let's call the radius of the can r and the height of the can h.

   2. **Figure out what to optimize.** We are asked to "minimize the amount of aluminum needed for a single can," so let's try to write an expression for this amount of aluminum. We know we need aluminum for the sides of the can as well as the top and bottom. The top and bottom are circles of radius r, so they each have area πr^2. The sides have area $2\pi rh$. So, the total amount of aluminum (in square inches) is $2\pi r^2 + 2\pi rh$.

   3. **Express the thing you're optimizing as a function of *one* variable.** Right now, we've expressed the total amount of aluminum as a function of both r and h; we need to get it down to just one variable. To do this, we need to **relate the variables r and h.** In this case, we can use the fact that the can has volume 128π cubic centimeters, which tells us that $\pi r^2 h = 128\pi$.
   We can either write r in terms of h or h in terms of r. It really makes no difference, but it looks like it's very slightly easier to solve for h in terms of r, so let's do that. Dividing the equation $\pi r^2 h = 128\pi$ by πr^2 gives $h = \frac{108}{r^2}$.
   Therefore, the total amount of aluminum, as a function of just r, is $2\pi r^2 + 2\pi r\left(\frac{128}{r^2}\right) = 2\pi r^2 + \frac{256\pi}{r}$.

   4. **Find the domain of the function we're maximizing.** This is really important because the method we use for finding the absolute maximum or minimum of a function depends on whether the domain is a closed interval. The problem statement is specific about the lower bound for r

   $$1 \leq r$$

   The upper bound for r happens when h is as small as possible.

   $$128\pi = \pi r^2 2$$

   $$r^2 = 64$$

   $$r = 8$$

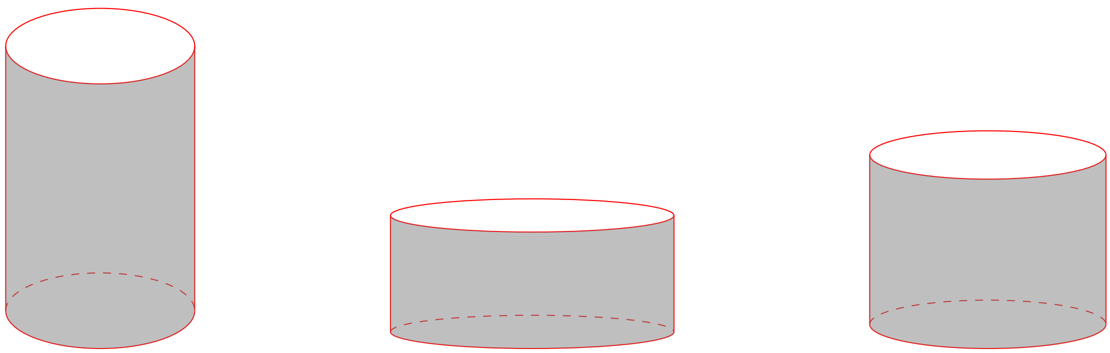
   It is now exactly the problem solved above.

2. The soda company realizes that they need to use stronger aluminum for the tops and bottoms of the cans, and this stronger aluminum costs 3 times as much as the aluminum used for the sides. If the company wants to minimize the cost of each can, what dimensions should the can be?

Solution.

(a) Goal: Minimize cost.

(b) Understand the problem: There are many different cylinders with the same volume. We want to find the dimensions that minimize the cost. Below are several different cylinders with the same volume:



Note that the cost is different than the surface area since the material for the tops and the bottoms cost more per square inch than the material for the sides. Let r be the radius of the cylinder and h be the height.

(c) Our strategy moving forward should be to find a single variable equation for the cost and then use the Closed Interval Test.

(d) Implementing our strategy: Since the tops and the bottoms of the cylinder are made of material that is three times as expensive the cost can be modeled as

$$C = 3 \cdot 2 \cdot \pi r^2 + 2\pi r h$$

$$C = 6\pi r^2 + 2\pi r h$$

We currently have C as a function of both h and r. Our strategy depends on getting C in terms of one variable. Looking for a relationship between h and r we note that the volume is constant

$$128\pi = \pi r^2 h$$

We have a choice here about which variable to isolate. It is more efficient to solve for h since solving for r would require taking a square root.

$$h = \frac{128\pi}{\pi r^2} = \frac{128}{r^2}$$

Now we can find C as a function of r

$$C = 6\pi r^2 + 2\pi r \frac{128}{r^2}$$

$$C = 6\pi r^2 + \frac{256\pi}{r}$$

Now we find the critical points

$$C' = 12\pi r - \frac{256\pi}{r^2} = 0$$

$$12r = \frac{256}{r^2}$$

$$r^3 = \frac{256}{12}$$

$$r = \sqrt[3]{\frac{256}{12}}$$

The domain of our function is $(0, \infty)$. Note that since we do not have a closed interval, we cannot make use of the Closed Interval Test. Instead, we can use the second derivative to classify the one critical point we do have.

$$C'' = 12 + \frac{512}{r^3} > 0$$

This means that that at the critical point $r = \sqrt[3]{\frac{256}{12}}$ the function is concave up and there there is a local minimum at $\sqrt[3]{\frac{256}{12}} \approx 2.77$. Since this is the only critical point on the domain it is the absolute minimum.
We should now find the corresponding height.

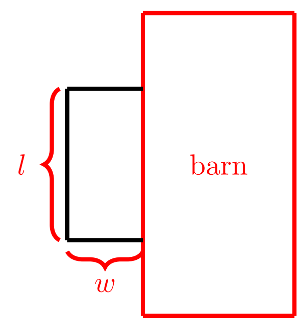
$$h = \frac{128}{\left(\sqrt[3]{\frac{256}{12}}\right)^2} \approx 16.688$$

3. A farmer has 40 feet of fencing, and he wants to fence off a rectangular pen next to his barn. The barn will be one side of the pen, so that side needs no fencing. In order for the cow to be able to turn around in the pen, the pen needs to be at least 5 feet long and 5 feet wide. What is the largest area the pen could have?

Solution.

(a) Goal: We want to maximize area. It is important to articulate the goal of the problem. It sets the tone for everything to follow.

(b) Understanding the problem. Here is a great place to draw a picture and label variables that might become important.



(c) Once we have a goal is clearly articulated and we have a understanding of what the problem is asking, we can formulate a strategy. Because we want to optimize area we should try and find a single variable function for the area of the pen. Once we have a single variable function we can use the Closed Interval Test.

(d) Carrying out the strategy is now straightforward.

$$A = wl$$

Note that we start off with area the function of two variables, but in order to make use of the calculus we have learned we need a function of one variable. We have 40 feet of fencing available so there is a relationship between the variables

$$40 = 2w + l$$

$$40 - 2w = l$$

So we can find area as a function of w,

$$A(w) = w(40 - 2w)$$

What is the domain of this function? From the problem statement we know that $5 \leq w$. Since there is only so much fencing, there is an upper bound to w too.

That upper bound happens when l is at its lower bound:

$$40 = 2w + 5$$

$$w = 17.5$$

The critical points are the endpoints $w = 17.5$ and $w = 5$ and where $A'(w) = 40 - 4w = 0$ or $w = 10$. To determine the maximum area we can now use the Closed Interval Test.

w	5	10	17.5
$A(w)$	150	200	87.5

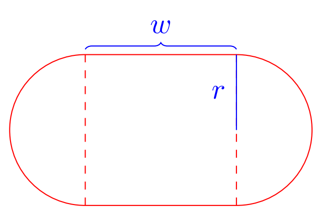
This shows that $w = 10$ is an absolute maximum.

4. The University is building a new running track. It is to be the perimeter of a region obtained by putting two semicircles on opposite ends of a rectangle. The administration has decided to grow roses in the rectangular portion of the area surrounded by the track. If the track is to be 440 yards long, what dimensions would maximize the area for growing roses?

Solution. We start by identifying our goal.

(a) Goal: Maximize the area of the rectangular portion.

(b) Understanding the problem: There are many different plans for a track of this shape. We draw an examples below:



(c) Our strategy is to find a single varaible function for the area of the rectangle and used the Closed Interval Test.

(d) The track is comprised of a circle and a rectangle so the area is

$$A = 2rw$$

We want to express A as a function of just one variable. To do this, we must relate r and w somehow, by using something else we're told in the problem. We're told that the track must be 440 yards long. In terms of r and w, the length of the track is $2\pi r + 2w$, so

$$\begin{aligned} 2\pi r + 2w &= 440 \\ 2w &= 440 - 2\pi r \\ w &= 220 - \pi r \end{aligned}$$

Plugging this into our formula for A, we get A in terms of just r:

$$A(r) = 2r\,(220 - \pi r)\,.$$

Now, maximize! Note that since the area is a quadratic function with leading coeffeient negative, we know that the unique critical point will be a maximum. $A'(r) = 440 - 4\pi r$. So, the track should have $\boxed{\text{semicircles of radius } \frac{110}{\pi} \text{ yards}}$ and $\boxed{\text{straight sides of length 110 yards}}$.

5. Chi-Yun is planning to buy a custom-made jewelry box for her mother's next birthday. The box will have a square base. The sides and bottom will be made out of mahogany, which costs 30 cents per square inch. The top will be made out of maple, which costs 50 cents per square inch. Chi-Yun has $60 to spend on the present and wants to get a box with the largest volume possible. What dimensions should the box be?

   Solution. We start by identifying our goal.

   (a) Goal: Maximize volume of the jewelry box.

   (b) Understanding the problem. There are many different possible boxes that Chi-Yun could construct. The limiting factor is the cost of the box. Let the length of the square base be denoted by s and the height of the box by h.

   (c) Strategy We will find a single variable function to describe the volume of the box and then use the Close Interval Test.

   (d) Implementing the strategy: Formula for thing we're trying to maximize

$$V = s^2 h.$$

(This is the area of the base times the height.) Next, we want to express V as a function of just one variable. To do this, we need to relate s and h using something we know. We are told that the cost of the box is $60. Let's work in cents:

$$\begin{aligned} 6000 &= \text{cost of box (in cents)} \\ &= \text{cost of top} + \text{cost of bottom} + 4(\text{cost of each side}) \end{aligned}$$

The top costs 50 cents per square inch, while all other sides cost 30 cents per square inch:

$$\begin{aligned} &= (50)(\text{area of top}) + (30)(\text{area of bottom}) + 4(30)(\text{area of each side}) \\ &= 50s^2 + 30s^2 + 4(30)sh \\ &= 80s^2 + 120sh \end{aligned}$$

Now, we can solve this equation for h:

$$6000 = 80s^2 + 120sh$$

Divide both sides by 40 to make the numbers simpler:

$$\begin{aligned} 150 &= 2s^2 + 3sh \\ 150 - 2s^2 &= 3sh \\ \frac{150 - 2s^2}{3s} &= h \\ \frac{50}{s} - \frac{2s}{3} &= h \end{aligned}$$

Plugging this into our volume formula gives

$$\boxed{V = s^2\left(\frac{50}{s} - \frac{2s}{3}\right) = 50s - \frac{2s^3}{3}}$$

We want the critical points, so we should first find the derivative of $V(s)$, which is $V'(s) = 50 - 2s^2$. So, the critical points are where

$$\begin{aligned} 50 - 2s^2 &= 0 \text{ } (V'(s) \text{ is never undefined}) \\ 50 &= 2s^2 \\ 25 &= s^2 \\ 5 &= s^1 \end{aligned}$$

But wait, we're not done yet! We've found a critical point, but we still need to check that it's the absolute maximum.
Let's use the Second Derivative Test:

$$V''(s) = -4s$$

$$V''(5) = -20 < 0$$

. Therefore, 5 is a local maximum. Since it's the only critical point in our domain, it must be an absolute maximum as well. So, $s = 5$ is an absolute maximum. Since we found that $h = \frac{50}{s} - \frac{2s}{3}$, the height when $s = 5$ is $\frac{50}{5} - \frac{2 \cdot 5}{3} = \frac{20}{3}$. So, the dimensions for the box should be $\boxed{5 \text{ inches} \times 5 \text{ inches} \times \frac{20}{3} \text{ inches}}$.