Quadratic Functions

1. On the earth, acceleration due to gravity is approximately 9.8 m/sec^2 or 32 ft/sec^2. One day Johnny leans out his dorm window (a height of 40 ft above the ground) and hurls an apple straight up in the air with initial velocity 48 ft/sec.

   (a) Write down the function $h(t)$ describing the height of the apple in feet t seconds after Johnny tosses it.

   (b) What is the greatest height that the apple reaches?

2. Mikaela is selling tickets for the Pitches' monthly a capella event. From pervious months, when the price was \$10, 45 tickets were sold, and when the price was \$7, 72 tickets were sold. Suppose that she wants to use a linear function $T(x)$ to model how many tickets will sell when the price is x dollars. Based on the model $T(x)$, $A(x)$ is the revenue from ticket sales when the price is x dollars.

   (a) Write down the formula for $A(x)$. Sketch its graph and label the intercepts.

   (b) What is the price that maximizes revenue?

   (c) Find $A'(x)$. What is the meaning of it?

3. Sketch the following parabolas, and write down the coordinate of the vertex.

   (a) $y = -2x^2 + 4x - 7$ (b) $y = 2(x-1)(x-5)$ (c) $y = -2(x-1)^2 + 18$

4. For each of the following, are there any parabola or quadratic function satisfying the description? If so, write down the equation for it. And specify if there are more than one such parabola or quadratic functions.

   (a) A quadratic function whose derivative is $2x + 3$.

   (b) A parabola with x-intercepts at $x = 1$ and $x = 4$ and y-intercept at $y = -1$.

   (c) A quadratic function with zeros at $x = 2$ and $x = 4$ and has maximum at $x = 1$.

   (d) A parabola with vertex at $(2, -4)$ and an x-intercept at $x = 5$.

   (e) A quadratic function assuming its minimum -3 at $x = 1$.

   (f) A parabola passing through the points $(0, 3)$, $(-1, 6)$, and $(2, 9)$.

Quadratic Functions – Solutions

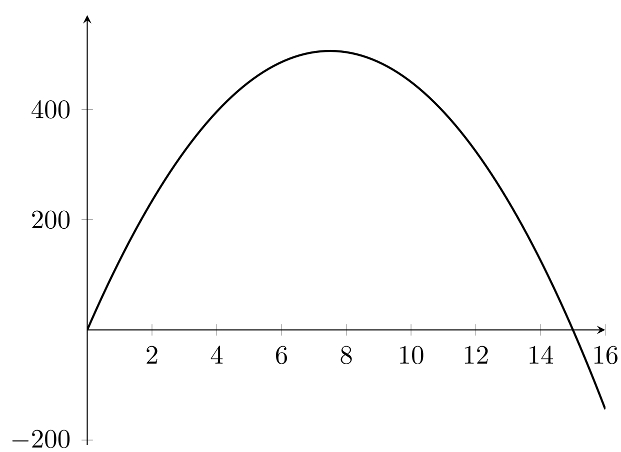
1. (a) We know the acceleration function $a(t)$ is a constant -32. The velocity function $v(t)$, as an antiderivative of $a(t)$, is $-32t + c$ where c is a constant. Since the initial velocity $v(0)$ is 48, we have $c = 48$, namely $v(t) = -32t + 48$ Now we also know the height function $h(t)$ is an antiderivative of $v(t)$ with $h(0) = 40$, so $h(t) = -16t^2 + 48t + 40$.

   (b) We can find the maximum of $h(t)$ in two ways. First of all we can do completing square to $h(t)$.

$$h(t) = -16t^2 + 48t + 40 = -16(t^2 - 3t + \frac{9}{4}) + 76 = -16(t - \frac{3}{2})^2 + 76$$

   Hence the greatest height happens at $t = \frac{2}{3}$ and the maximal height is 76 feets. We may also use the derivative to find the maximum. $h'(t) = v(t) = -32t + 48$, which is zero when $t = \frac{48}{32} = \frac{3}{2}$. So the maximum height is $h(\frac{3}{2}) = -16 \cdot \frac{9}{4} + 48 \cdot \frac{3}{2} + 40 = -36 + 72 + 40 = 76$.

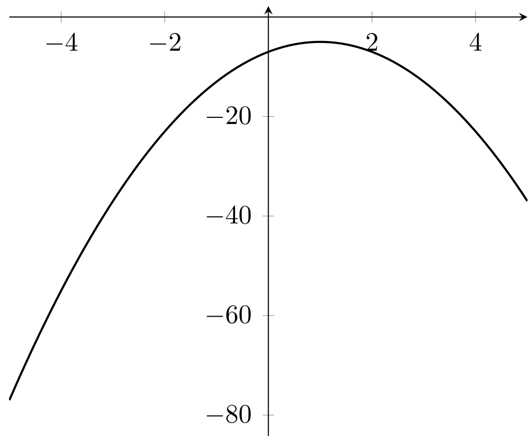
2. (a) We first find $T(x)$. The slope of the linear function is $\frac{72-45}{7-10} = -9$. Knowing $T(10) = 45$ we may find $T(x) = -9x+135$. Then the revenue function $A(x)$ would be ticket price times the number of tickets sold, which is $x \cdot T(x) = x(-9x + 135)$. The graph of $A(x)$ is as follows.



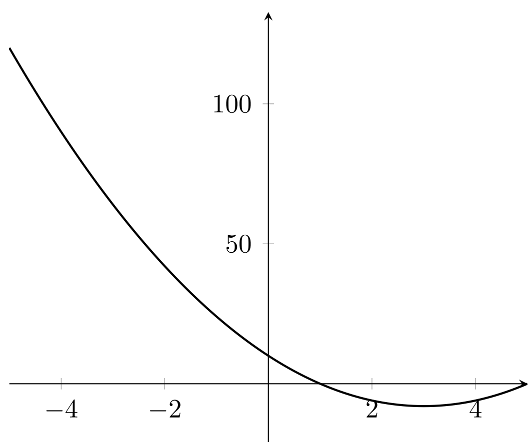
   (b) As in problem 1, there are two ways to find the maximum revenue: either do completing square or observing that the derivative at the maximum is zero. However there is a third way (which is faster) to find the maximum here. We observe that the two x-intercepts are 0 and $\frac{135}{9} = 15$. Then the maximum must be half way between the two x-intercepts, which is 7.5. Hence the maximum revenue is $A(7.5) = 7.5 \cdot (-9 \cdot 7.5 + 135) = 7.5 \cdot 67.5 = 506.25$.

   (c) We have $A(x) = x(-9x+135) = -9x^2+135x$, so $A'(x) = -18x+135$. Be caution that $A'(x) \neq T(x)$. The meaning of $A'(x)$ is the rate of change of the revenue when the ticket price is x, and $T(x)$ is the number of tickets sold when the ticket price is x.

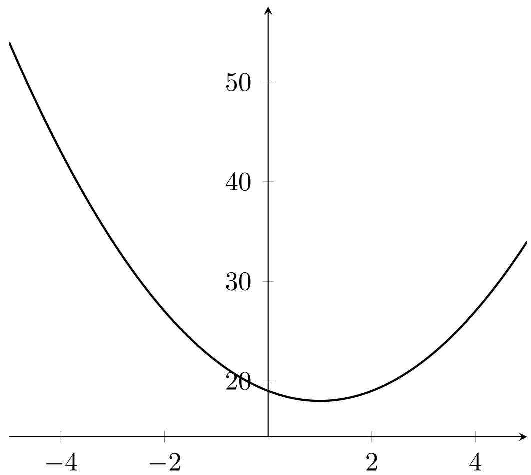
3. (a) The derivative is $-4x + 4$. Setting $-4x + 4 = 0$, we get the x-coordinate of the vertex is 1. Plugging back into the quadratic function, the coordinate of the vertex is $(1, -5)$.



(b) The x-intercepts are $x = 1$ and $x = 5$. The vertex should be half way between the x-intercepts and hence has x-coordinate 3. Plugging back into the quadratic function, the coordinate of the vertex is $(1, 5)$.



(c) The coordinate of the vertex can be immediately read off from the expression: $(1, 18)$.



4. (a) The quadratic functino can be $x^2 + 3x + c$, where c is any number.

   (b) We let the equation of the parabola be $y = a(x-1)(x-4)$. Knowing the y-intercept is -1, we have $-1 = a \cdot (-1) \cdot (-4)$, and thus $a = -\frac{1}{4}$.

   (c) Such quadratic function cannot exist because the maximum should appear exactly in the middle of the zeros, namely at $x = 3$.

   (d) We let the equation of the parabola be $a(x-2)^2 - 4$. Knowing an x-intercept is 5, we have $0 = a \cdot (5-2)^2 - 4$, and thus $a = \frac{4}{9}$.

   (e) The quadratic function can be $y = a(x-1)^2 - 3$ with $a > 0$.

   (f) We let the equation of the parabola be $y = ax^2 + bx + c$. The fact that it passes through the points $(0, 3)$, $(-1, 6)$ and $(2, 9)$ gives us a system of equation

$$\begin{aligned} 3 &= c \\ 6 &= a - b + c \\ 9 &= 4a + 2b + c. \end{aligned}$$

   Solving it we have $a = 2, b = -1, c = 3$.