Math Ma

Exponential Wrap up

Fall 2016

1. Johnnie decides to open a bank account with an opening deposit of \$1000.

   (a) Suppose that the account earns a *nominal annual interest* rate of 6%, compounded annually. How much money does the account have t years after Johnnie opens it?

   (b) Suppose that Johnnie had instead deposited his money in a bank that offered quarterly compounding. That is, the bank offers a 6% nominal annual interest rate with 4 compounding periods a year. How much money would the account have after t years?

2. Find an exponential function through the points $(4, 3)$ and $(6, 48)$.

3. Growing bacteria is in a fashion now! Suppose at $t = 0$ (t is measured in hours) there are 8000 bacteria, and they are growing at a rate of 400 bacteria per hour. Assume that the bacteria are growing exponentially.

   (a) How quickly will the bacteria population be growing when there are 10000 bacteria?

      i. 400 bacteria per hour
      ii. 500 bacteria per hour
      iii. 1000 bacteria per hour
      iv. There is not enough information to determine the growing rate. We need to know exactly when the population reaches 10000 bacteria.

   (b) Write down the formula for $P(t)$, the number of bacteria at time t.

4. Find a function $f(x)$ such that $f(0) = 2$ and $f'(x) = 7f(x)$ for all x.

5. The function $f(x) = \dfrac{3x}{e^x}$ is an example of a surge function. Sketch the graph of $f(x)$.
What properties do I want to know about the graph?

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Exponential Wrap up – Solutions

1. (a) $1000 \cdot 1.06^t$
   (b) $1000 \cdot 1.015^{4t}$

2. Let the exponential function be $f(x) = B \cdot A^x$.

$$48 = B \cdot A^6 \quad (1)$$
$$3 = B \cdot A^4 \quad (2)$$

   Dividing (1) by (2), we have
$$16 = A^2.$$
   Thus $A = 4$. Now solving $3 = B \cdot 4^4$, we have $B = \frac{3}{256}$. So $f(x) = \frac{3}{256}4^x$ passes through $(4, 3)$ and $(6, 48)$.

> Problem 1 and 2 are trying to tell you that **there is exactly one exponential function through two points**, just as there is exactly one line through two points.

3. (a) ii. is true.
   (b) $P(t) = 8000e^{\frac{t}{20}}$

4. $f(x)$ is an exponential function. $f(x) = 2e^{7x}$.

> Problem 3 and 4 are trying to tell you that **an exponential function is proportional to its derivative**. If $f(x)$ is an exponential function with constant of proportionality $m_A = f'(x)/f(x)$, then $f(x)$ has base e^{m_A}. Another way to say this is $A = e^{m_A}$.

5. To sketch the graph of $f(x)$, we want to know
   - when is f positive/negative: We have $f(x) = \frac{3x}{e^x}$. Since e^x is always positive, we see that $f(x)$ is positive when $x > 0$ and negative when $x < 0$.
   - when is f increasing/decreasing: We have to determine when f' is positive/negative. First compute
$$f'(x) = 3e^{-x} + 3x(e^{-x}) = -3(x-1)e^{-x}.$$
     So $f'(x)$ is positive when $x < 1$ and negative when $x > 1$. Namely $f(x)$ is increasing when $x < 1$ and decreasing when $x > 1$.
   - when is f concave up/down: We have to determine when f'' is positive/negative. Compute
$$f''(x) = -3e^{-x} + (-3)(x-1)(-e^{-x}) = (3x-6)e^{-x}.$$
     So $f'(x)$ is positive when $x > 2$ and negative when $x < 2$. Namely $f(x)$ is increasing when $x > 2$ and decreasing when $x < 2$.

- what happens to f when x goes to $\pm\infty$: We compute the limits

$$\lim_{x\to\infty}\frac{3x}{e^x}=0,\ \lim_{x\to-\infty}\frac{3x}{e^x}=-\infty$$

Finally we can sketch the graph of f