Application of Exponentials

1. **Sociology.** The population in a certain area of the country is increasing. In 1995 the population was 100,000, and by 2015 it was 200,000.

   (a) If the population has been increasing linearly, was the population in 2005 equal to 150,000, greater than 150,000, or less than 150,000?

   Solution. Equal to 150,000.

   (b) If the population has been increasing exponentially, was the population in 2005 equal to 150,000, greater than 150,000, or less than 150,000?

   Solution. Less than 150,000.

   (c) If the population has been increasing exponentially and continues to do so, what do you expect the population to be in 2020?

   Solution. Let $P(t)$ be the population t years after 2015. Since the population doubles every twenty years, $P(t) = 200,000 \cdot 2^{t/20} \implies P(5) = 200,000 \cdot \sqrt[4]{2}$

2. **Biology and Public Health.** *Escherichia coli*, better known as *E. Coli*, is a bacterium that can cause peritonitis, a potentially fatal disease. *E. Coli* is sometimes found in ground beef, so food safety inspectors would like to be able to test for it. The challenge is to be able to take a tiny sample of ground beef (so that the rest can be sold to customers) and check the sample for *E. Coli*, even if the sample has just a single bacterium. However, DNA tests are not sensitive enough to detect just a single bacterium, so the solution food inspectors have is to put the ground beef sample in "a broth infused with nutrients that *E. Coli* likes to eat, put in a warm place to rest for 10 hours"

   Under such conditions, it is estimated that a population of *E. coli* **doubles every 30 minutes**. Suppose that at time $t = 0$, t measured in minutes, the sample has just **two** *E. Coli* bacterium.

   (a) Make a table for the number of *E. coli* present t minutes later.

   Solution.

t minutes later	0	30	60	90	120	150
# of *E. coli*	2	4	8	16	32	64

   (b) Write a formula for the function $f(t)$ describing the number of *E. coli* present t minutes later.

   Solution. $f(t) = 2 \cdot 2^{t/30}$.

   (c) Write a formula for the function $g(h)$ describing the number of *E. coli* present h hours later.

   Solution. $g(h) = 2 \cdot 4^{h}$.

3. **Chemistry.** The half-life of the radioactive isotope radium-226 is approximately 1600 years.

   (a) Suppose there is currently a S_0 mg sample of radium-226. Write a formula for the amount of radium-226 that remains in the sample at time t, where t is measured in years and $t = 0$ means the current time.

   Solution. $S(t) = S_0 \left(\frac{1}{2}\right)^{t/1600}$.

   (b) By what percent does a sample of radium-226 decrease per year?

   Solution. $\dfrac{\text{OLD - NEW}}{\text{OLD}} = \dfrac{1}{2^{1/1600}} - 1.$

   (c) How many years would it take for a 100 mg sample of radium-226 to decay to 25 mg?

   Solution. This would take two half-lives, or 3200 years.

   (d) About how many years would it take for a 100 mg sample of radium-226 to decay to 5 mg?

   Solution. We are looking for the time t for which $100 \cdot \left(\frac{1}{2}\right)^{t/16} = 5$, or $\left(\frac{1}{2}\right)^{t/16} = \frac{1}{20}$. Unfortunately, we don't know how to solve equations like this (that is, we don't know how to solve for t if it is in the exponent of an equation, except by guess and check). What they can do is bound this - it will take 4 half lives to get down to 6.25 mg and 5 half lives to get to 3.125 mg - so it will have to take between 6400 and 8000 years. The actual answer is $\approx$ 6915 years.

4. **Finance.** Johnnie decides to open a bank account with an opening deposit of \$1000.

(a) Suppose that the account earns a *nominal annual interest* rate of 6%, compounded annually. How much money does the account have t years after Johnnie opens it?

Solution. At the end of each year, he earns 6% interest, which has the effect of multiplying his balance by $1 + 0.06$. Therefore, after t years, Johnnie has $1000(1 + 0.06)^t$.

(b) Suppose that Johnnie had instead deposited his money in a bank that offered quarterly compounding. That is, the bank offers a 6% nominal annual interest rate with 4 compounding periods a year. How much money would the account have after t years?

Solution. Quarterly compounding with a nominal annual interest rate of 6% means that, each quarter, the account earns $\frac{1}{4} \cdot 6\%$ interest. So, each quarter, the value of the account is multiplied by $1 + \frac{0.06}{4}$. After t years, $4t$ quarters have passed, so the account will have $\boxed{1000\left(1 + \frac{0.06}{4}\right)^{4t}}$ dollars.

(c) In this account, what is the percent increase in money per year? (This is called the *annual percentage yield* of the account.)

Solution. $\left(1 + \frac{0.06}{4}\right)^4 - 1$; using a calculator, this is 0.06136, or 6.136%.

(d) What if Johnnie had used a bank that offered n compounding periods a year? How much money would the account have after t years, and what would be the annual percentage yield?

Solution. After t years, he would have $\boxed{1000\left(1 + \frac{0.06}{n}\right)^{nt}}$ dollars. The annual percentage yield is then $\left(1 + \frac{0.06}{n}\right)^n - 1$.