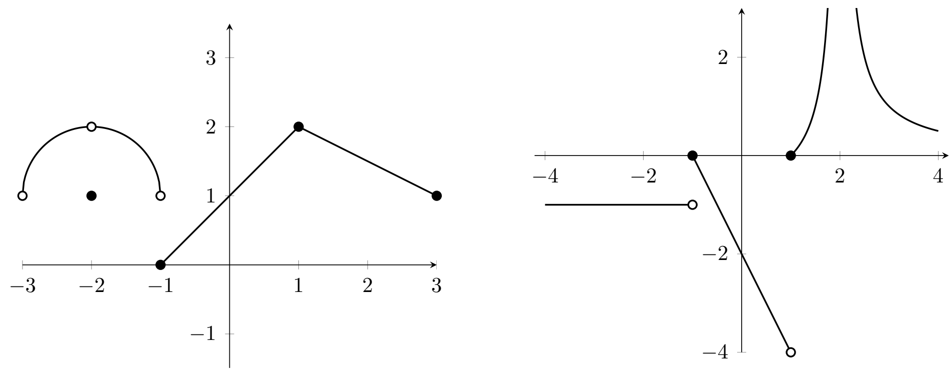
Midterm Review

1. Below is the graphs of two functions. The left graph is f and the right one is g. Find the following values.



(Limits)

(a) $\lim_{x \to -2} f(x)$

(b) $f(-2)$

(c) $\lim_{x \to -1} f(x)$

(d) $\lim_{x \to 2} g(x)$

(e) $\lim_{x \to -1} [f(x) + g(x)]$

(f) $\lim_{x \to 0} \frac{f(x) - 1}{g(x) + 2}$

(g) $\lim_{x \to -2} \frac{f(x) - f(-2)}{x + 2}$

(h) $\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$

(Continuity)

(i) Find all the discontinuities of f and use the limit definition to specify their types.

(j) Find all the points at which f is continuous but not differentiable. And find all the points at which f is differentiable but not continuous.

(Differential Rules)

(k) Find the value of $\frac{d}{dx}[f(x) \cdot g(x)]$ at $x = 0$.

(l) Compute the derivative of $h(x) = \dfrac{3x^2 + 2}{x + 1}$.

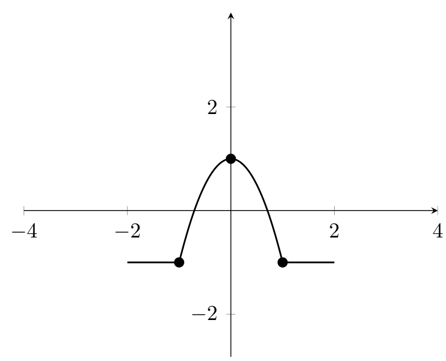
2. **(Continuity and Differentiation)** Let f be the function defined by

$$f(x) = \begin{cases} \frac{1}{x} & \text{for } x > 1 \\ ax + b & \text{for } x \leq 1 \end{cases}$$

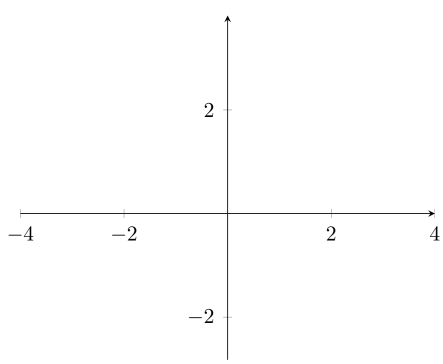
(a) For what values of a and b will f be continuous at $x = 1$?

(b) For what values of a and b will f be differentiable at $x = 1$?

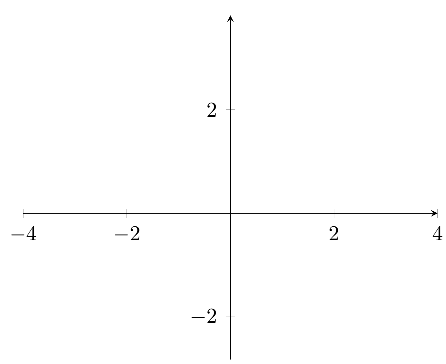
3. **(Graph transformation)** Below is the graph of $f(x)$. Sketch the following functions.



(a) $f(x + 2) + 1$



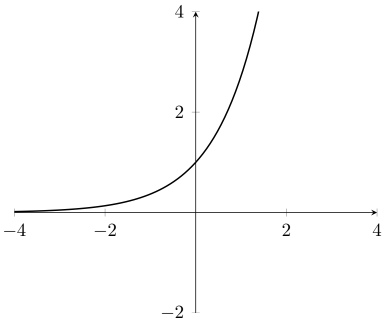
(b) $2f(0.5x)$



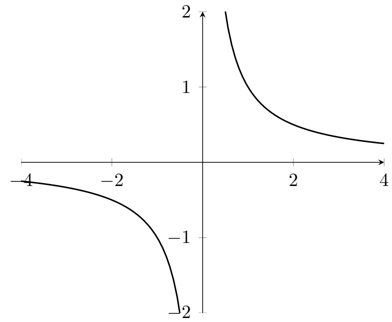
(Think of a way to check whether your answer is correct or not.)

4. **(Graph and derivative)** For the following graphs, first sketch the graph of its derivative and then sketch the graph of its anti-derivative.

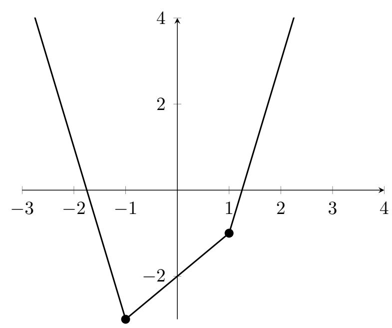
(a)



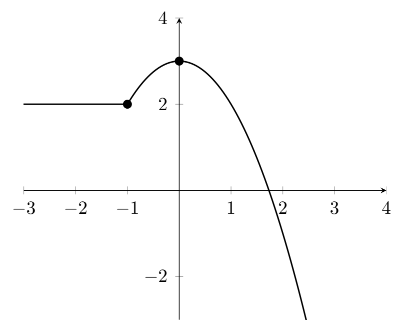
(b)



(c)



(d)



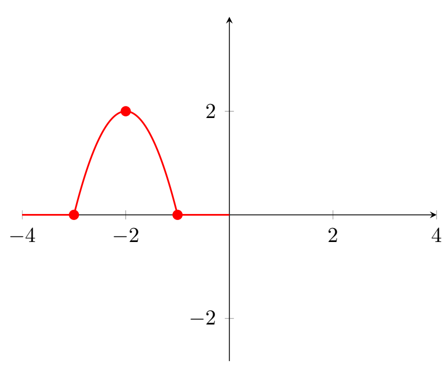
5. **(Modeling)** cf. Fall 2011 #1
6. **(Linear Approximation)** cf. Fall 2011 #2
7. **(Piecewise linear functions)** ...
8. **(Even and odd functions)** ...

Midterm Review – Solutions

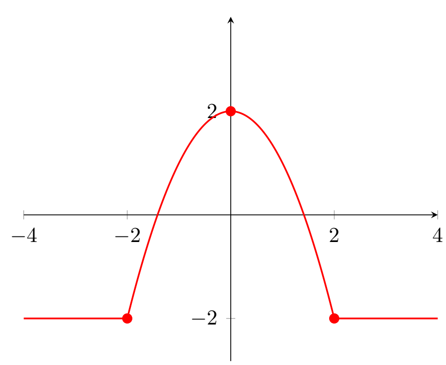
1. (a) 2
   (b) 1
   (c) Does not exist.
   (d) Does not exist. (∞)
   (e) 0
   (f) $-\frac{1}{2}$
   (g) Does not exist.
   (h) $-\frac{1}{2}$
   (i) $x = -2$ is a removable discontinuity as $\lim_{x\to-2} f(x) = 2$ but $f(-2) = 1$. $x = -1$ is a jump discontinuity as $\lim_{x\to-1^+} f(x) = 0 \neq 1 = \lim_{x\to-1^-} f(x)$.
   (j) At $x = 1$, $f(x)$ has a corner, so it is continuous but not differentiable. There is no point where f is differentiable but no continuous since differentiability implies continuity.
   (k) We have the product rule,
   $$[f(x) \cdot g(x)]' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$$
   Hence
   $$(fg)'(0) = f'(0)g(0) + f(0)g'(0) = 1 \cdot (-2) + 1 \cdot (-2) = -4$$
   (l) We use the quotient rule to compute $h'(x)$:
   $$h'(x) = \frac{(6x) \cdot (x+1) - (3x^2+2) \cdot 1}{(x+1)^2} = \frac{6x^2 + 6x - 3x^2 - 2}{(x+1)^2} = \frac{3x^2 + 6x - 2}{(x+1)^2}$$

2. (a) We see that $\lim_{x\to1^+} f(x) = 1$ and $\lim_{x\to1^-} f(x) = a + b$. Thus $f(x)$ is continuous at $x = 1$ whenever $a + b = 1$.
   (b) For $f(x)$ to be differentiable at $x = 1$, it has to be continuous first of all, so we have $a + b = 1$. Then we need the limit $\lim_{x\to1} \frac{f(x)-f(1)}{x-1}$ to exist. For the right limit, $\lim_{x\to1^+} \frac{f(x)-f(1)}{x-1} = \underline{\text{derivative of } \frac{1}{x} \text{ at } x = 1}$. Since $\left(\frac{1}{x}\right)' = -\frac{1}{x^2}$, the right limit is -1. For the left limit, $\lim_{x\to1^-} \frac{f(x)-f(1)}{x-1} = a$. Thus we have $a = -1$ and $b = 1 - a = 2$.

3. (a)

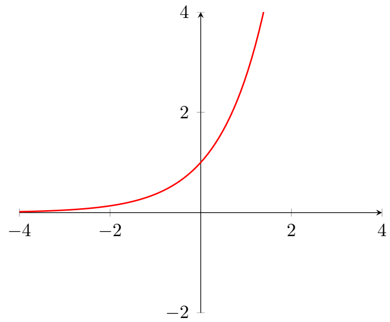


(b)

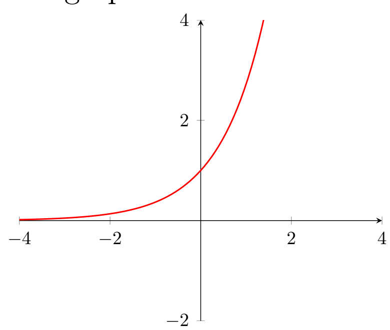


4. (a)

The graph of the derivative

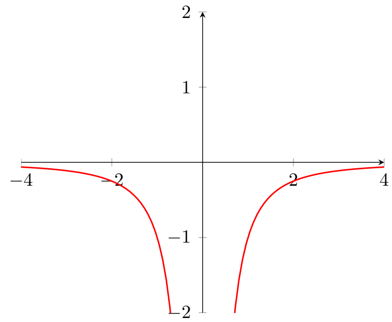


The graph of the anti-derivative

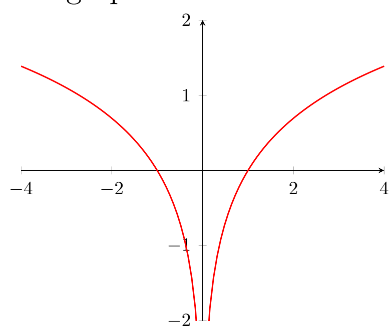


(b)

The graph of the derivative

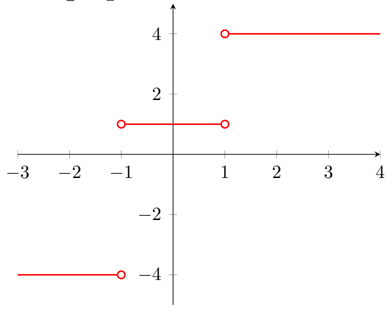


The graph of the anti-derivative

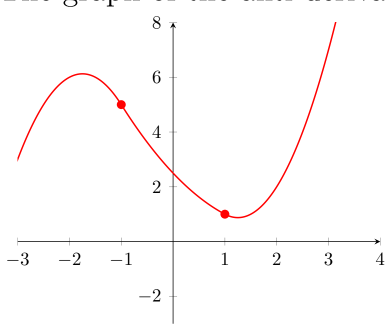


(c)

The graph of the derivative

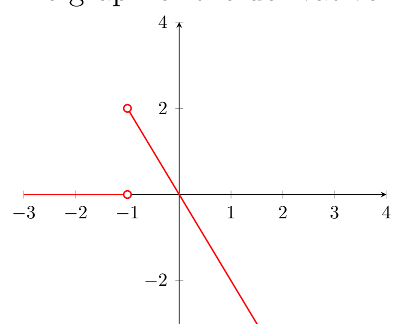


The graph of the anti-derivative



(d)

The graph of the derivative



The graph of the anti-derivative