Linear Approximation

1. (a) How would you use what we have learned so far to approximate $\sqrt{3}$? Is your approximation an overestimate or underestimate?

   (b) Now try to approximate $\sqrt[3]{9}$ using both methods. Are they overestimates or underestimates?

2. (a) Review how we use the limit definition to obtain the derivative of $\sqrt{x}$.

(b) Can you think of a way to use the product rule $(fg)' = f'g + fg'$ instead to derive $(x^{\frac{1}{2}})' = \frac{1}{2}x^{-\frac{1}{2}}$? (This proves the power rule for $(x^n)' = nx^{n-1}$ for $n = \frac{1}{2}$.)

(c) Use the limit definition to find the derivative of $\sqrt[3]{x}$.

(d) Use the product rule instead to find the derivative of $\sqrt[3]{x}$.

Linear Approximation– Solutions

1. (a) **Tangent line method:**
We choose to make use of the tangent line at $(4, 2)$ to approximate, as we know the square root of 4 and it is near 3. By power rule, if $f(x) = \sqrt{x}$, then $f'(x) = \frac{1}{2\sqrt{x}}$, and thus $f'(4) = \frac{1}{4}$. Using point-slope formula, we find the equation of the tangent line to f at $(4, 2)$:
$$y - 2 = \frac{1}{4}(x - 4)$$
The point on the tangent line with $x = 3$ has $y = \frac{7}{4}$, which is our approximation for $\sqrt{3}$. This is an an overestimate since $\sqrt{x}$ is a concave down function.
Secant line method:
We can also use the secant line through the points $(1, 1)$ and $(4, 2)$ to do approximation. The equation of the secant line is
$$y - 1 = \frac{1}{3}(x - 1)$$
Hence the approximated value of $\sqrt{3}$ is $\frac{5}{3}$. This is a underestimate since $\sqrt{x}$ is a concave down function and $x = 3$ is between the two endpoints $x = 1$ and $x = 4$ of the secant line.

(b) **Tangent line method:**
We will use the tangent line at $(8, 2)$. Let $f(x) = \sqrt[3]{x}$. Compute $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$ and thus $f'(8) = \frac{1}{12}$. So the equation of the tangent line is
$$y - 2 = \frac{1}{12}(x - 8)$$
and the approximated value of $\sqrt[3]{9}$ us $\frac{25}{12}$. This is an overestimate.
Secant line method:
We use the secant line through $(1, 1)$ and $(8, 2)$. The equation of the secant line is
$$y - 1 = \frac{1}{7}(x - 1)$$
so the approximated value of $\sqrt[3]{9}$ is $\frac{15}{7}$. This is an overestimate, too, since $\sqrt[3]{x}$ is concave up and $x = 9$ is outside of the two endpoints $x = 1$ and $x = 8$ of the secant line.

2. (a)

$$
\begin{aligned}
(\sqrt{x})' &= \lim_{y\to x} \frac{\sqrt{y}-\sqrt{x}}{y-x} = \lim_{y\to x} \frac{(\sqrt{y}-\sqrt{x})(\sqrt{y}+\sqrt{x})}{(y-x)(\sqrt{y}+\sqrt{x})} = \lim_{y\to x} \frac{y-x}{(y-x)(\sqrt{y}+\sqrt{x})} \\
&= \lim_{y\to x} \frac{1}{\sqrt{y}+\sqrt{x}} = \frac{1}{2\sqrt{x}}
\end{aligned}
$$

(b) Let $f(x) = \sqrt{x}$. Then $f(x)^2 = x$. Take the derivative of both sides. The derivative of the right hand side is 1. The derivative of the left hand side can be found by using the product rule:

$$[f(x)^2]' = [f(x) \cdot f(x)]' = f'(x) \cdot f(x) + f(x) \cdot f'(x) = 2f(x)f'(x)$$

We have $2f'(x)f(x) = 1$ and thus $f'(x) = \frac{1}{2f(x)} = \frac{1}{2\sqrt{x}}$.

(c)

$$
\begin{aligned}
(\sqrt[3]{x})' &= \lim_{y\to x} \frac{\sqrt[3]{y}-\sqrt[3]{x}}{y-x} = \lim_{y\to x} \frac{(\sqrt[3]{y}-\sqrt[3]{x})(\sqrt[3]{y}^2+\sqrt[3]{y}\sqrt[3]{x}+\sqrt[3]{x}^2)}{(y-x)(\sqrt[3]{y}^2+\sqrt[3]{y}\sqrt[3]{x}+\sqrt[3]{x}^2)} \\
&= \lim_{y\to x} \frac{y-x}{(y-x)(\sqrt[3]{y}^2+\sqrt[3]{y}\sqrt[3]{x}+\sqrt[3]{x}^2)} = \lim_{y\to x} \frac{1}{(\sqrt[3]{y}^2+\sqrt[3]{y}\sqrt[3]{x}+\sqrt[3]{x}^2)} = \frac{1}{3\sqrt[3]{x^2}}
\end{aligned}
$$

(d) Let $f(x) = \sqrt[3]{x}$. Then $f(x) \cdot f(x) \cdot f(x) = x$. Take the derivative of both sides. The derivative of the right hand side is 1. The derivative of the left hand side can be found by using the product rule:

$$
\begin{aligned}
f(x)^3]' &= [f(x)^2 \cdot f(x)]' \\
&= [f(x)^2]' \cdot f(x) + f(x)^2 \cdot [f(x)]' \\
&= [2f(x)f'(x)] \cdot f(x) + f(x)^2 \cdot f'(x) \\
&= 3f(x)^2 f'(x)
\end{aligned}
$$

We have $3f'(x)f(x) = 1$ and thus $f'(x) = \frac{1}{3f(x)} = \frac{1}{3\sqrt[3]{x}}$.