Differentiation Rules

1. **(Power Rule)**

(a) We have computed the derivative of the following functions in the past. Fill in the table below.

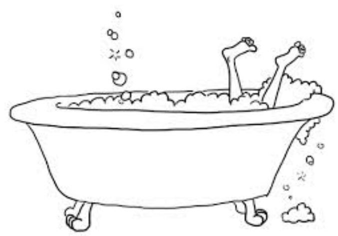
$f(x)$	1	x	x^2	$x^{-1} = \frac{1}{x}$	$x^{-2} = \frac{1}{x^2}$
$f'(x)$					

(b) Let $f(x) = x^n$. Based on the table above, what would you guess $f'(x)$ should be?

(c) We may write $(x+h)^n = \_\_\_\_\_ x^n + \_\_\_\_\_ x^{n-1}h + \_\_\_\_\_ x^{n-2}h^2 + \cdots$. What are the first two coefficients?

(d) Use the definition of derivative to find $\frac{d}{dx}(x^n)$.

Chandler is preparing to take a bath. Suppose at time t (measured in minutes), $f(t)$ gallons of hot water has been poured into the tub, and $g(t)$ gallons of cold water has been poured into the tub.



2. **(Sum Rule)**

   (a) What function describes the amount of water (in gallons) in the tub at time t? What is the instantaneous rate of change (gallons/min) of the amount of water at time t?

   (b) Use the definition of derivative to find $\frac{d}{dx}\left[f(x)+g(x)\right]$.

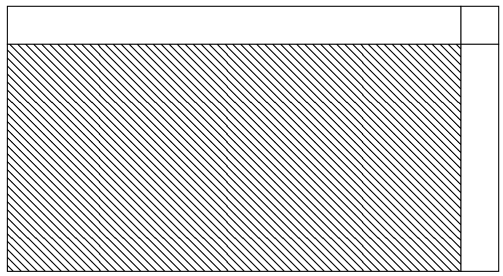
3. **(Constant Multiple Rule)**

   (a) What function describes the amount of hot water (in liters) in the tub at time t? What is the instantaneous rate of change (liter/min) of the amount of hot water at time t? (1 gallon $\approx$ 3.78 liters)

   (b) Use the definition of derivative to find $\frac{d}{dx}\left[c \cdot f(x)\right]$, where c is a constant.

4. **(Product Rule)**

Olivia is doing an experiment on bacterial culture. The magical bacteria always grow in the shape of rectangle. Suppose $f(t)$ is the length in cm of the rectangular colony at time t, and $g(t)$ is the width in cm.



(a) Suppose $f'(3) = 2$. Roughly how much longer does the colony grows between $t = 3$ and $t = 3.1$?

(b) More generally, if h is a very small number, write down an approximation of the length of the colony at time $t + h$, in terms of the length and growing rate of length of the colony at time t.

(c) What is the growing rate of the area of the colony at time t?

(d) Use the definition of derivative to find $\frac{d}{dx}[f(x) \cdot g(x)]$.

Differentiation Rules

- Power Rule: $\frac{d}{dx}(x^n) =$
- Sum Rule: $\frac{d}{dx}[f(x) + g(x)] =$
- Constant Multiple Rule: For a constant c, $\frac{d}{dx}[c \cdot f(x)] =$
- Product Rule: $\frac{d}{dx}[f(x) \cdot g(x)] =$
- Quotient Rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$
  (You will derive this rule in your homework!)

5. Find the derivative of the following functions.

   (a) $f(x) = 2x^5 + 3x^2 + 5x + 4$

   (b) $f(x) = x + \frac{1}{x} + 1$

   (c) $f(x) = \pi^5$

   (d) $f(x) = (3x^2 + 1)(x + \frac{1}{x})$

Differentiation Rules – Solutions

1.

(a)

$f(x)$	1	x	x^2	$x^{-1} = \frac{1}{x}$	$x^{-2} = \frac{1}{x^2}$
$f'(x)$	0	1	$2x$	$-x^{-2}$	$-2x^{-3}$

(b) $f'(x) = nx^{n-1}$.

(c) $(x+h)^n = x^n + nx^{n-1}h + \cdots$, which we can get by drawing the Pascal's triangle.

(d)

$$\begin{aligned}\frac{d}{dx}(x^n) &= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = \lim_{h\to 0}\frac{(x^n + nx^{n-1}h + \_\_x^{n-2}h^2 + \cdots + h^n) - x^n}{h}\\ &= \lim_{h\to 0}\frac{nx^{n-1}h + \_\_x^{n-2}h^2 + \cdots + h^n}{h} = \lim_{h\to 0}\left(nx^{n-1} + \_\_x^{n-2}h + \cdots + h^{n-1}\right)\\ &= nx^{n-1}\end{aligned}$$

2. (a) The function $f(t) + g(t)$ describes the amount of water in the tub at time t? The instantaneous rate of change of the amount of water at time t is $f'(t) + g'(t) = (f(t) + g(t))'$.

(b)

$$\begin{aligned}\frac{d}{dx}[f(x) + g(x)] &= \lim_{h\to 0}\frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h}\\ &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} + \lim_{h\to 0}\frac{g(x+h) - g(x)}{h}\\ &\textbf{(when both limits exist)}\\ &= f'(x) + g'(x)\end{aligned}$$

3. (a) The function $3.78f(t)$ describes the amount of water in the tub at time t? The instantaneous rate of change of the amount of water at time t is $(3.78f(t))' = 3.78f'(t)$.

(b)

$$\begin{aligned}\frac{d}{dx}[c \cdot f(x)] &= \lim_{h\to 0}\frac{c \cdot f(x+h) - c \cdot f(x)}{h}\\ &= c \cdot \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}\\ &= c \cdot f'(x)\end{aligned}$$

4. (a) 0.2
   (b) $f(t+h) \approx f(t) + hf'(t)$
   (c) The white area is the difference $A(t+h)$ and $A(t)$, which is approximately $hf'(t) \cdot g(t) + f(t) \cdot hg'(t) + hf'(t) \cdot hg'(t)$. Divided by h, which is the time passed between t and $t+h$, the growing rate of the colony would roughly be $f'(t) \cdot g(t) + f(t) \cdot g'(t)$. The last term was omitted as h is very small.
   (d)

$$
\begin{aligned}
\frac{d}{dx}\left[f(x) \cdot g(x)\right] &= \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h} \\
&= \lim_{h \to 0} \frac{(f(x+h) \cdot g(x+h) - f(x) \cdot g(x+h)) - (f(x) \cdot g(x+h) - f(x) \cdot g(x))}{h} \\
&\textbf{(we add the second term and subtract it at the third term)} \\
&= \lim_{h \to 0} \frac{[f(x+h) - f(x)] \cdot g(x+h) - f(x) \cdot [g(x+h) - g(x)]}{h} \\
&= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \cdot g(x+h) + f(x) \cdot \frac{g(x+h) - g(x)}{h} \right) \\
&= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot \lim_{h \to 0} g(x+h) + f(x) \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&\textbf{(when all three limits exist)} \\
&= f'(x) \cdot g(x) + f(x) \cdot g'(x)
\end{aligned}
$$

5. (a) $f'(x) = 10x^4 + 6x + 5$
   (b) $f'(x) = 1 - \frac{1}{x^2}$
   (c) $f'(x) = 0$
   (d) $f'(x) = 6x \cdot (x + \frac{1}{x}) + (3x^2 + 1) \cdot (1 - \frac{1}{x^2})$