The Derivative Function

1. Donagh feels bored with the toy rocket and decides to play with a remote control aircraft instead. He began playing around 11:30pm. The aircraft's height in meters is modeled by the function $f(t) = t^2$. Here $t = 0$ corresponds to 12pm, $t = 1$ corresponds to 12:01pm, etc.

   (a) Sketch a graph of f.

   (b) What is the meaning and unit of $f'(3)$ in the setting? How do you interpret $f'(3)$ in terms of graph?

   (c) Use the definition of derivative to calculate the values below

t_0	-3	-2	-1	0	1	2	3
$f'(t_0)$							

   (d) We can compute that $f'(4) = 8$. Which of the following statements are reasonable?

   A. At 12:04pm, the aircraft is 8 meters high.

   B. The aircraft goes up approximately 8 meters between 12:04pm and 12:05pm.

   C. The aircraft goes up approximately 8 meters between 12:03pm and 12:04pm.

   D. The aircraft's vertical velocity increases by 8 between $t = 0$ and $t = 4$.

   E. The aircraft's vertical velocity is exactly 8 m/min at 12:04pm.

(e) For which values of t_0 above is $f'(t_0)$ positive/negative? What does $f'(t_0) > 0$ mean in the setting? What does $f'(t_0) > 0$ mean in terms of graph?

(f) If $f'(t_0)$ is positive, what can we say about the original function $f(t)$ around $t = t_0$?

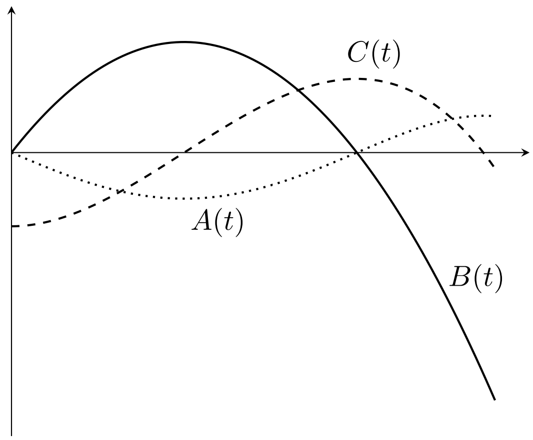
(g) By looking at the graph of f, sketch a graph of $f'(t)$.

(h) Write down the function $f'(t)$ explicitly using the definition of the derivative. Does your answer has similar graph as you predicted in (g)?

(i) By looking at the graph of f', sketch a graph of $f''(t)$.

(j) Write down the function $f''(t)$ explicitly using the definition of the derivative. What is the meaning and unit of $f''(t)$ in the setting?

2. A mouse is moving in a narrow lane. Three graphs are shown below: one shows the mouse's position at time t, another its velocity at time t, and the last its acceleration at time t. Which is which? How do you know?



3. Let $f(x) = \left|\dfrac{x^4 - 9x^2 + 20}{x^2 - 4}\right|$. Sketch the graph of f, and then sketch the graph of f'.

4. Let

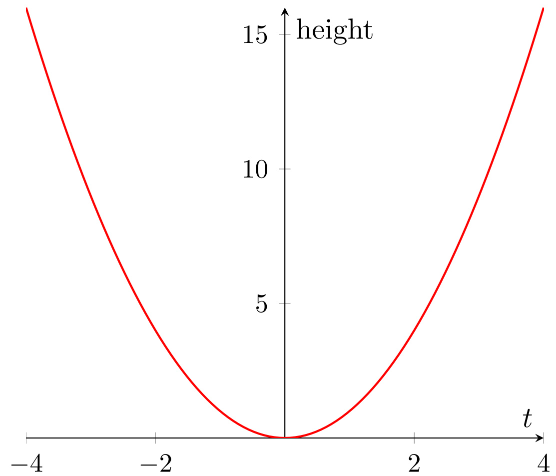
$$f(x) = \begin{cases} |x+2| & x \leq -1 \\ -x+1 & -1 < x \leq 0 \\ |x-1| & x > 0 \end{cases}$$

(a) Sketch the graph of f.

(b) Where does $f'(x)$ not exist?

The Derivative Function – Solutions

1. (a)



(b) $f'(3)$ is the aircraft's instantaneous vertical velocity at 12:03pm. The unit is meters per minute. Graphically, $f'(3)$ is the slope of the tangent line to f at $t = 3$.

(c)

t_0	-3	-2	-1	0	1	2	3
$f'(t_0)$	-6	-4	-2	0	2	4	6

For example

$$f'(-3) = \lim_{t\to-3} \frac{f(t) - f(-3)}{t - (-3)} = \lim_{t\to-3} \frac{t^2 - 9}{t + 3} = \lim_{t\to-3} \frac{(t + 3)(t - 3)}{t + 3} = \lim_{t\to-3}(t-3) = -6$$

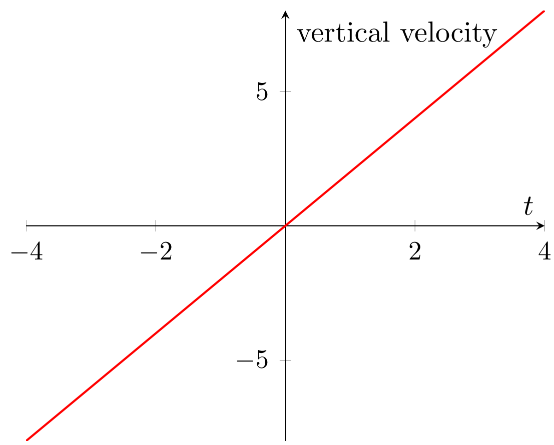
The other values $f'(t_0)$ can be computed in the same way.

(d) Statement B, C, E are correct.

(e) $f'(t_0)$ is positive for $t_0 = 1, 2, 3$. This means the aircraft's vertical velocity is positive at $t = t_0$, or that the aircraft is rising. Graphically, this means the slope of the tangent lines at $t = t_0$ is positive.

(f) If $f'(t_0)$ is positive, we can say that $f(t)$ is increasing around t_0.

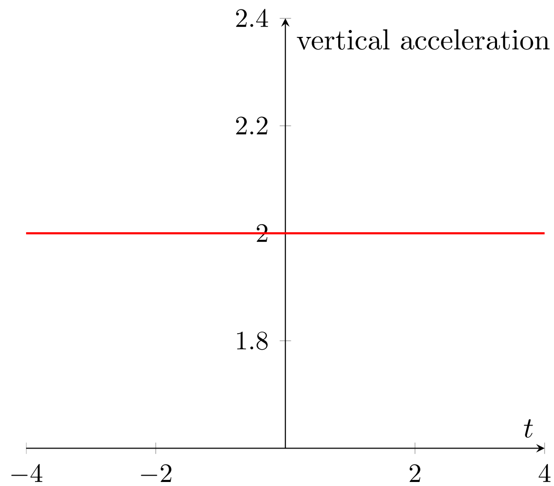
(g)



(h)

$$f'(t) = \lim_{s\to t} \frac{f(s) - f(t)}{s - t} = \lim_{s\to t} \frac{s^2 - t^2}{s - t} = \lim_{s\to t} \frac{(s+t)(s-t)}{s - t} = \lim_{s\to t}(s + t) = 2t$$

(i)

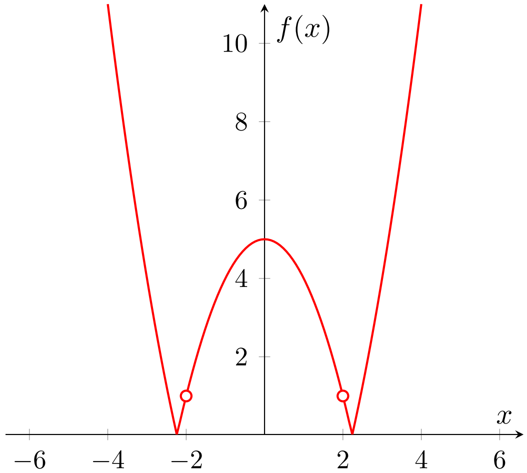


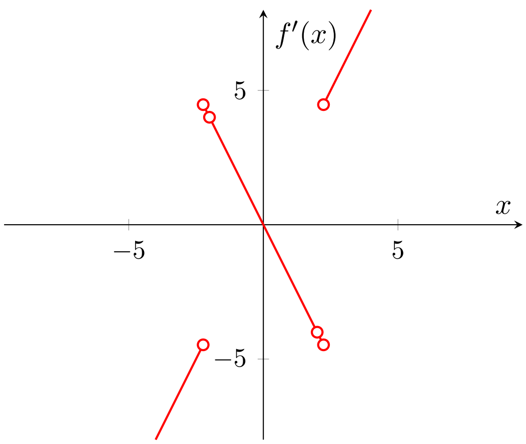
(j)

$$f''(t) = \lim_{s\to t} \frac{f'(s) - f'(t)}{s - t} = \lim_{s\to t} \frac{2s - 2t}{s - t} = \lim_{s\to t} \frac{2(s - t)}{s - t} = \lim_{s\to t} 2 = 2$$

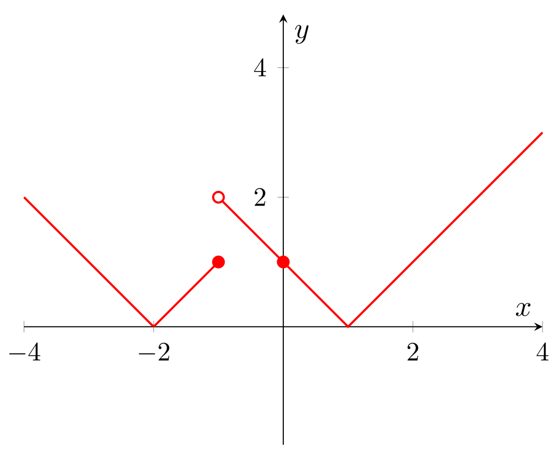
$f''(t)$ is the aircraft's vertical acceleration at time t.

2. A is the position function, C is the velocity function and B is the acceleration function.

3. 



4. (a)



(b) $f'(x)$ does not exist when the graph of f is not continuous at x or when it has a corner. Hence $f'(x)$ does not exist for $x = -2, -1, 1$.