Definition of the Derivative

1. Donagh sets off a toy rocket straight up into the air. The function $f(t) = t^2$ describes the toy rocket's height in meters t seconds after liftoff.

   (a) Fill in the table below.

t	average speed between 3 and t seconds after liftoff
5	
4	
3.5	
3.1	

   (b) Fill in the table below.

t	average speed between 3 and t seconds after liftoff
1	
2	
2.5	
2.9	

   (c) What do you think the (instantaneous) speed of the toy rocket 3 seconds after liftoff should be?

   (d) Sketch a graph describing the toy rocket's height t seconds after liftoff.

   (e) Interpret what you did in (a)–(c) in terms of the graph.

Definition

Suppose $f(x)$ is a function, a is a number in the domain of f. The derivative of f at a, written as $f'(a)$, and read as "f prime of a", is

$f'(a) =$ [ ] or [ ]

if it exists.

- $f'(a)$ is the instantaneous rate of change of f at $x = a$.
- Graphically, $f'(a)$ is the slope of the tangent line to the graph $y = f(x)$ at $x = a$.

2. Let $f(x) = \frac{1}{x}$. **If you have more time, do the same problems again for $g(x) = \frac{1}{4x}$.**

   (a) Calculate $f'(1)$ using the definition of the derivative f at 1.

   (b) Calculate the average rate of change of f between

   - $x = 1$ and $x = 1.5$
   - $x = 0.5$ and $x = 1$
   - $x = 0.75, 1.25$

   Do the answers justify your answer in (a)?

   (c) Sketch the graph of f and the tangent line to the graph of $y = f(x)$ at $x = 1$. Is it consistent with your answer in (a)?

(d) Find the equation of the tangent line to the graph of $y = f(x)$ at $x = 1$.

(e) Use the tangent line to approximate $\frac{1}{1.1}$. Is it an overestimate or an underestimate?

(f) Use the tangent line to approximate $\frac{1}{0.9}$. Is it an overestimate or an underestimate?

3. (a) Let $A(r)$ be the area of a circle of radius r cm. Find the derivative $A'(5)$. What is the unit of $A'(5)$? Explain the meaning of $A'(5)$ in words.

(b) Let $B(x)$ be the area of a square with side length x cm. Find the derivative $B'(5)$. What is the unit of $B'(5)$? Explain the meaning of $B'(5)$ in words.

Explanation in Pictures:

Observation

- For a concave up graph, a tangent line is __________ the graph, while a secant line is __________ the graph between the endpoints of the secant line, and __________ the graph outside the endpoints.
- For a concave down graph, a tangent line is __________ the graph, while a secant line is __________ the graph between the endpoints of the secant line, and __________ the graph outside the endpoints.

4. Peter is roasting a 14 lb turkey as a test run for Thanksgiving dinner. He begins at noon. At 1:00pm, he checks on the temperature and discovers that it has an internal reading of 35.6°C,[1] and it's rising at an instantaneous rate of 0.25°C per minute.

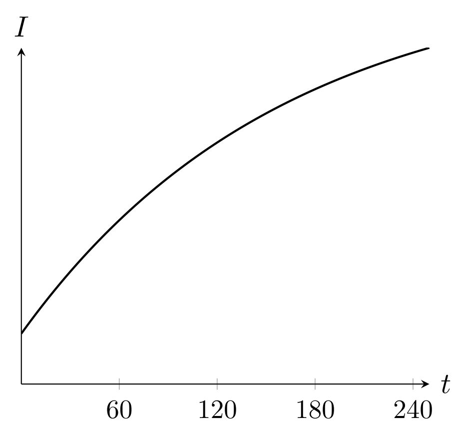
(a) *Approximate the temperature of the turkey at 1:06pm.*

Solution. The temperature of the turkey at 1:00 pm is 35.6°C. If we assume that the temperature continues rises at 0.25°C per minute for the next 6 minutes, then the turkey's temperature will be $35.6 + 6 \cdot 0.25 = 37.1$°C at 1:06 pm.

(b) *Let $I(t)$ be the turkey's internal temperature (in °C) t minutes after noon. Use functional notation to express what you were told about the turkey.*

Solution. We were told that $I(60) = 35.6$ and $I'(60) = 0.25$.

(c) *Here is a graph of $I(t)$. Use a sketch to explain the approximation you made in (a).*



Solution. In 0a, we assumed that the turkey's temperature would continue to

[1] According to the USDA, a turkey should be roasted to an internal temperature of 73°C.

rise at the rate of 0.25°C for the next 6 minutes; that is, we assumed that the turkey's temperature would change linearly.

(d) *Based on your sketch, was your approximation too high or too low?*

(e) *Would you be comfortable using the same method to predict the turkey's temperature at 3 pm? Explain. Could we use the* ***definition of the derivative*** *to get the exact answer in this case?*

Definition of the Derivative – Solutions

1. (a)

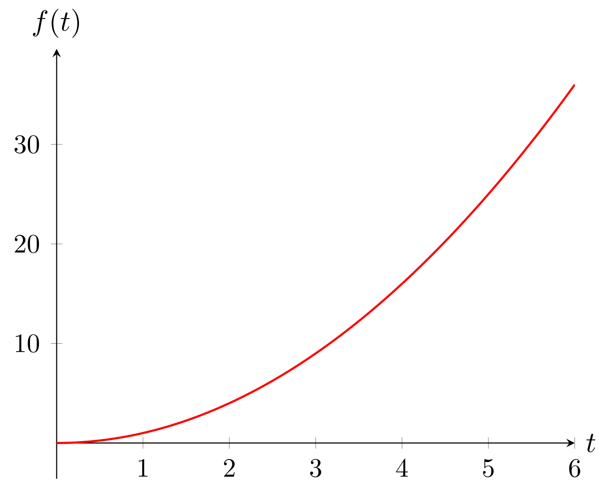
t	average speed between 3 and t seconds after liftoff
5	8
4	7
3.5	6.5
3.1	6.1

(b)

t	average speed between t and 3 seconds after liftoff
1	4
2	5
2.5	5.5
2.9	5.9

(c) From the two tables above, it seems that the average speed between t seconds and 3 seconds gets closer and closer to 6 as t gets closer to 3. Hence it would be fair to guess that the instantaneous speed at 3 is 6.

(d)



(e) What we were doing in (a)(b) was computing the slope of the secant line passing through t and 3, and let the point with coordinate t goes closer and closer to the point with coordinate 3. Meanwhile the secant line through the two points will become the tangent line at $t = 3$.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

2. (a)

$$f'(a) = \lim_{x \to 1} \frac{\frac{1}{x} - 1}{x - 1} = \lim_{x \to 1} \frac{1 - x}{x(x-1)} = \lim_{x \to 1} \frac{-1}{x} = -1$$

or

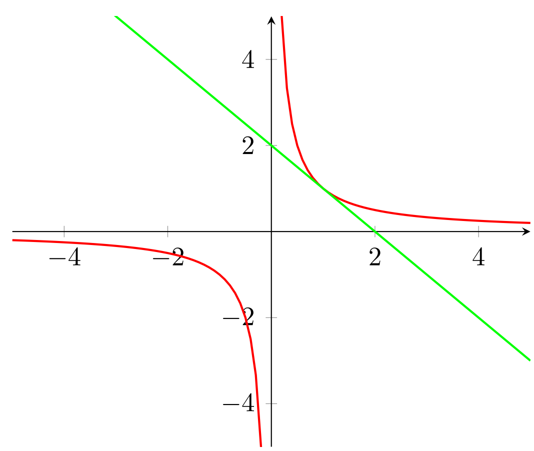
$$f'(a) = \lim_{h \to 0} \frac{\frac{1}{1+h} - 1}{h} = \lim_{h \to 0} \frac{1 - (1+h)}{h(1+h)} = \lim_{h \to 0} \frac{-h}{h(1+h)} = \lim_{h \to 0} \frac{-1}{1+h} = -1$$

(b)

- The average rate of change of f in $[1, 1.5]$ is $\frac{\frac{1}{1.5} - 1}{1.5 - 1} = \frac{-\frac{1}{3}}{0.5} = -\frac{2}{3}$.
- The average rate of change of f in $[0.5, 1]$ is $\frac{1 - \frac{1}{0.5}}{1 - 0.5} = \frac{-1}{0.5} = -2$
- The average rate of change of f in $[0.75, 1.25]$ is $\frac{\frac{1}{1.25} - \frac{1}{0.75}}{1.25 - 0.75} = \frac{\frac{5}{6} - \frac{4}{3}}{0.5} = -1$

All of the above are not too far from the instantaneous rate of change -1 we get in (a).

(c)



(d) The tangent line of $y = f(x)$ at $x = 1$ has slope -1, as shown in (a). Using point-slope formula, the equation of the tangent line is $y - 1 = -(x - 1)$, or $y = -x + 2$.

(e) $\frac{1}{1.1} \approx -1.1 + 2 = 0.9$. As $\frac{1}{1.1} = \frac{10}{11} > \frac{9}{10} = 0.9$, this is an underestimate. That this is an underestimate can also be seen from the graph, as $y = \frac{1}{x}$ is concave up around $x = 1$.

(f) $\frac{1}{0.9} \approx -0.9 + 2 = 1.1$. As $\frac{1}{0.9} = \frac{10}{9} > \frac{11}{10} = 1.1$, this is an underestimate. That this is an underestimate can also be seen from the graph, as $y = \frac{1}{x}$ is concave up around $x = 1$.

3. (a) The area of a circle with radius r is $A(r) = \pi r^2$. Then by definition of derivative of A at 5

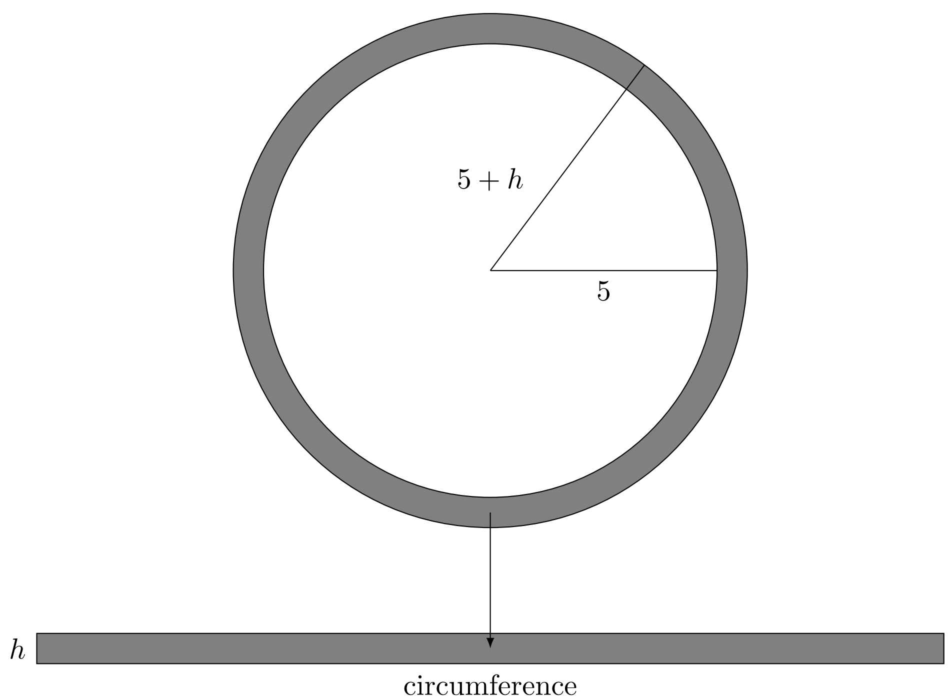
$$A'(5) = \lim_{r\to 5} \frac{\pi r^2 - 25\pi}{r-5} = \lim_{r\to 5} \frac{\pi(r+5)(r-5)}{r-5} = \lim_{r\to 5} \pi(r+5) = 10\pi$$

The unit of $A'(5)$ is cm^2/cm. This means that when the circle has radius 5, if we increase the radius by 1 cm, the area increases roughly by 10π cm^2.

(b) The area of a square with side length x is $B(x) = x^2$. Then by definition of derivative of B at 5

$$B'(5) = \lim_{x\to 5} \frac{x^2 - 25}{x-5} = \lim_{x\to 5} \frac{(x+5)(x-5)}{x-5} = \lim_{x\to 5}(x+5) = 10$$

The unit of $B'(5)$ is cm^2/cm. This means that when the square has side length 5, if we increase the side length by 1 cm, the area increases roughly by 10 cm^2.



We observe that $A'(5) = 10\pi$ is exactly the circumference of the circle of radius 5.