Modeling with Functions

1. Chi-Yun wants to plant some roses in front of her house. She has 28 feet of wire fencing to fence off a rectangular garden. Does it affect the area of the garden if she builds up the fence differently? If so, what is the maximum possible area of the rose garden?

   Your solution:

   Chi-Yun's lengthy explanation:

   - **Step 1:** ____________________
   - **Step 2:** ____________________
   - **Step 3:** ____________________
   - **Step 4:** ____________________

2. Tegan is eating an chocolate ice cream cone. The cone has 2-inch radius and 10-inch height. There are only some melting ice cream left in the cone.

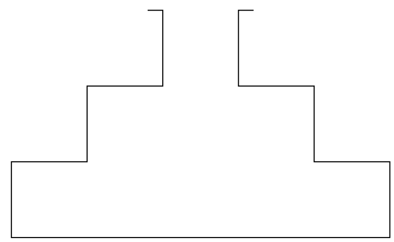
   (a) If the cone is filled half-way, what is the volume of the ice cream? (The volume of a cone with radius R and height H is $\frac{1}{3}\pi R^2 H$.)

   (b) Suppose now the radius of the surface area of the ice cream is r, what is the volume of the ice cream? What are the domain and range?

   (c) Express the volume of the ice cream as a function of the height of it. What are the domain and range?

   (d) Express the height of the ice cream as a function of its volume. What are the domain and range?

3. There is a bottle in a strange shape as below, which Johnny wants to use to do calibration. The bottom width is 5 inches, the middle width is 3 inches and the top width is 1 inch. Also the height of the bottle is 3 inches.



   (a) Draw the bottle calibration graph.

   (b) Write down the bottle calibration function $f(v)$, expressing the height of water as a function of it volume. What is the domain and range of f? Is f continuous?

4. Hunter is sending a birthday card to his friend. According to USPS [1] ,the cost of sending a first-class stamped letter is as follows: it costs \$0.47 for letters not over 1 ounce, and \$0.21 for each additional 1 ounce. However, the weight cannot exceed 3.54 ounce.

   (a) Draw the cost versus weight graph.

   (b) Write down the function $f(w)$, expressing the cost as a function of weight. What is the domain and range of f? Is f continuous?

[1] http://pe.usps.com/text/dmm300/Notice123.htm

5. Let $d(x)$ be the distance between x and 5 on the number line.

   (a) What is the domain and range of $d(x)$?

   (b) For what value of x is $d(x) = 0$? How about $d(x) = 1$? How about $d(x) > 1$.

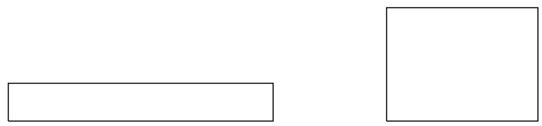
   (c) Draw the graph of $d(x)$.

   (d) Write down the formula of $d(x)$.

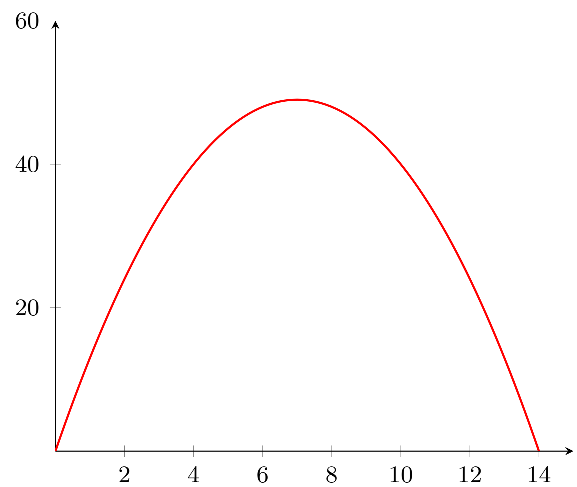
Modeling with Functions – Solutions

1.

- **Step 1: Understand the problem**
  Our goal is to find the maximum area of the garden. Draw several pictures of the garden, as below.

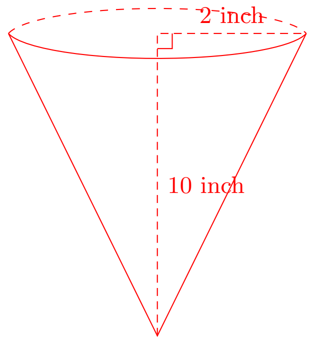


- **Step 2: Make a plan**
  Introduce variable ℓ: length of the garden, w: width of the garden. From the problem we know $2\ell + 2w = 28$. And the goal is to find the maximum possible $A = \ell w$. Our strategy is to use the first equality to express w in terms of ℓ, write A as a function of ℓ only, and find its maximum. Since $A(\ell)$ will be a quadratic polynomial, we can either draw the graph or complete the square to find its maximum.
- **Step 3: Realize the plan**
  We find that $w = 14 - \ell$ and $A = \ell w = \ell(14 - \ell)$.



  Looking at the graph, when $\ell = 7$ is exactly between 0 and 7, the area is maximum. Alternatively, $-\ell(14 - \ell) = -(\ell^2 - 7)^2 + 49$, so the maximum area is 49, happening at $\ell = 7$.
- **Step 4: Look back**
  The answer we obtained says that the area of the garden is maximal when it is in the shape of a square.

2.



(a) When the cone is filled half-way, the height of ice cream is 5 inches, and the radius of the surface area is 1 inch. By the formula of the volume of a right circular cone, the volume is

$$\frac{1}{3} \cdot \pi \cdot 1^2 \cdot 5 = \frac{5\pi}{3}$$

(b) The ratio of radius and height is the same no matter how high the ice cream is. From the dimension of the cone, we know this ratio is $\frac{1}{5}$. If the radius of the surface area of the ice cream is r, then the height is $5r$. Again by the formula, the volume is

$$\frac{1}{3} \cdot \pi \cdot r^2 \cdot 5r = \frac{5\pi r^3}{3}$$

The domain of this function consists of all the possible radii, and thus it is $[0, 2]$. The range consists of all possible volumes, and the largest of which is $\frac{40\pi}{3}$ when $r = 2$. Hence the range is $[0, \frac{40\pi}{3}]$.

(c) If the height of the ice cream is h, then the radius of the surface area is $\frac{h}{5}$. Hence the volume is

$$\frac{1}{3} \cdot \pi \cdot \left(\frac{h}{5}\right)^2 \cdot h = \frac{\pi h^3}{75}$$

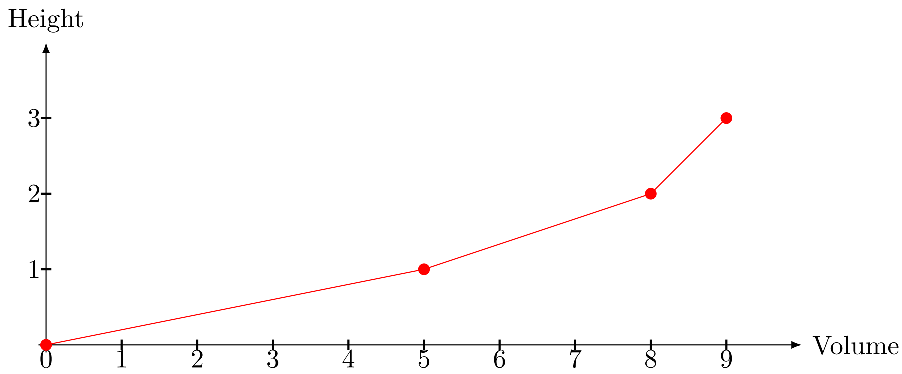
The domain is $[0, 10]$ and the range is $[0, \frac{40\pi}{3}]$.

(d) From the previous question we know the volume $V = \frac{\pi h^3}{75}$. Solving for h,

$$h = \sqrt[3]{\frac{75V}{\pi}}$$

The domain is $[0, \frac{40\pi}{3}]$ and the range is $[0, 10]$.

3. (a)



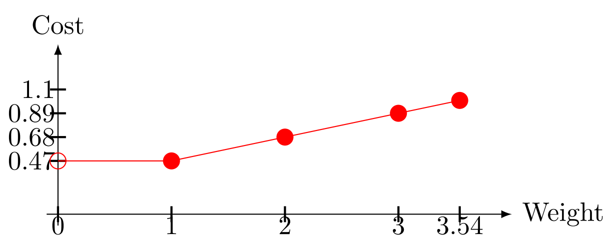
(b) The function $f(v)$ is

$$f(v) = \begin{cases} \frac{v}{5} & 0 \leq v \leq 5 \\ \frac{v-5}{3} + 1 & 5 \leq v \leq 8 \\ (v-8) + 2 & 8 \leq v \leq 9 \end{cases}$$

The domain is $[0, 9]$ and the range is $[0, 3]$, and f is continuous.

4. There are two ways to interpret the question. You can think of the cost as steadily increasing after the weight exceeds 1 ounce. Or the cost is added by \$0.21 once the weight exceeds by 1 ounce. In the first case,

(a)



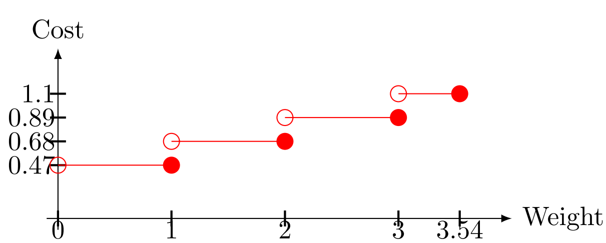
(b) The function $f(w)$ is

$$f(w) = \begin{cases} 0.47 & 0 < w \leq 1 \\ 0.21(w-1) + 0.47 & 1 < w \leq 3.54 \end{cases}$$

The domain is $(0, 3.54]$ and the range is $[0.47, 1.0034]$, and f is continuous.

In the second case,

(a)



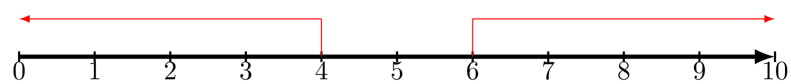
(b) The function $f(w)$ is

$$f(w) = \begin{cases} 0.47 & 0 < w \leq 1 \\ 0.68 & 1 < w \leq 2 \\ 0.89 & 2 < w \leq 3 \\ 1.1 & 3 < w \leq 3.54 \end{cases}$$

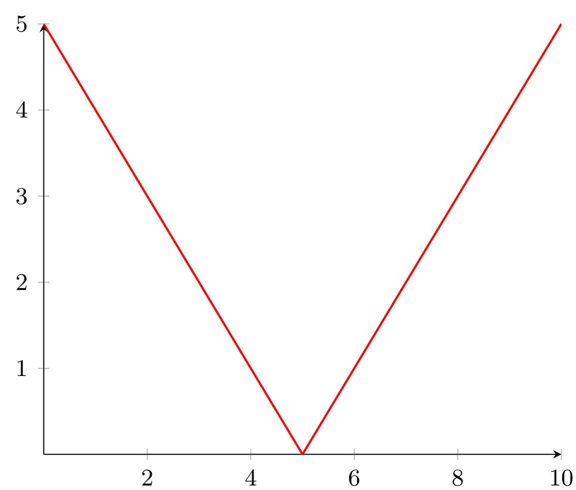
The domain is $(0, 3.54]$ and the range is $\{0.47, 0.68, 0.89, 1.1\}$, and f is not continuous.

5. (a) The domain is $(-\infty, \infty)$ and the range is $[0, \infty)$.

(b) $d(x) = 0$ when the distance between x and 5 is 0, which can only happen when $x = 0$. $d(x) = 1$ exactly when $x = 4, 6$. Finally $d(x) > 1$ when $x > 6$ or $x < 4$.



(c) Here is the graph of $d(x)$.



(d) In fact, $d(x) = |x - 5|$.