Midterm 1
Math 31B, Winter 2021

Name:	**UID:**

Honor Statement
I assert, on my honor, that I have not received assistance of any kind from any other people, including posting exam questions on online forums while working on this Final Exam. I have only used non-human resources, for example Internet, calculators, textbook, notes, and lecture videos, during the period of this evaluation.

Signature: ________________________

Directions—Please read carefully!

- You have a 24-hour window

  Feb 3 (Wed) 8am – Feb 4 (Thurs) 8am PST

  to complete the exam, but the exam is designed to be able to be finished in an hour.
- You are allowed to use any non-human resources including internet, calculators, textbook, notes, lecture videos, etc. **You are NOT allowed to seek help from other people, including posting exam questions on online forums.**
- In order to receive full credit, you must **show your work or explain your reasoning**; your final answer is less important than the reasoning you used to reach it. Correct answers without work will receive little or no credit. Please write neatly. **Illegible answers will be assumed to be incorrect.** Circle or box your final answer when relevant.
- On Feb 3 (Wed) 10-11am and 3-4pm PST, I will be on Zoom for any questions about statements of exam problems. I will also be monitoring emails and Piazza notifications more closely, during normal awake time at PST.
- The exam is on Gradescope. Please either
  - Write your answers on the pdf file of the exam, then submit onto Gradescope,
  - Print the exam and write your answers on the exam paper, then scan and submit onto Gradescope, or
  - Use blank sheets of paper, **copy the honor statement and sign.** Then write your answers on them, scan and submit onto Gradescope.

Good luck!

Integral Formulas you can directly use without derivation (and supposedly you have memorized them):

$$\begin{aligned}
\int \frac{dx}{\sqrt{1-x^2}} &= \sin^{-1} x + C \\
\int \frac{dx}{1+x^2} &= \tan^{-1} x + C \\
\int \frac{dx}{\sqrt{x^2+1}} &= \sinh^{-1} x + C \\
\int \frac{dx}{\sqrt{x^2-1}} &= \cosh^{-1} x + C \\
\int \sec x \, dx &= \ln|\tan x + \sec x| + C
\end{aligned}$$

1. You do NOT need to provide explanation for the following questions.

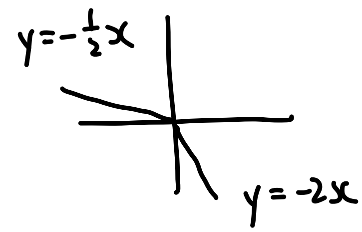
(3) (a) Suppose the population in an area grows exponentially. The population had 1600 people in 1990 and 2000 people in 2000. What should be the population in 2020?

A. 2800 people
✓ B. 3125 people
C. 3200 people
D. 3275 people
E. 4000 people

1990 : 1600 people $\cdot \frac{5}{4}$
2000 : 2000 people $\cdot \left(\frac{5}{4}\right)^2$
2020 : $2000 \cdot \left(\frac{5}{4}\right)^2 = 3125$

(3) (b) Which of the following function $f(x)$ satisfy $f^{-1}(x) = f(x)$? Choose ALL which are correct.

A. $f(x) = \frac{1}{x^2}$ $\quad y = \frac{1}{x^2}, \ x = \pm\frac{1}{\sqrt{y}}$
B. $f(x) = 3x$ $\quad y = 3x, \ x = \frac{1}{3}y$
✓ C. $f(x) = 2 - x$ $\quad y = 2-x, \ x = 2-y$
✓ D. $f(x) = -x$ $\quad y = -x, \ x = -y$
✓ E. $f(x) = \begin{cases} -\frac{1}{2}x & x \leq 0 \\ -2x & x \geq 0 \end{cases}$



$\cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

(3) (c) True or False: $\cos(\sin^{-1}(-\frac{1}{\sqrt{2}})) = -\frac{1}{\sqrt{2}}$.

(3) (d) True or False: The reduction formula

$$\int \sin^n x \, dx = -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

was proved by using Integration by Parts.

(3) (e) What substitution can be used to evaluate the integral $\int \frac{dx}{\sqrt{9x^2 - 1}}$? Choose ALL which are possible.

✓ A. $x = \frac{1}{3} \cosh u$
B. $x = 3 \cos u$
C. $x = \frac{1}{3} \sin u$
✓ D. $x = \frac{1}{3} \sec u$
E. $x = 3 \tan u$

2. Find the following limits.

(5) (a) $\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{x} = \frac{\cos \frac{\pi}{2}}{\frac{\pi}{2}} = 0$

(10) (b) $\lim_{x \to \infty} (\frac{x+1}{x})^x$

$= \lim_{x \to \infty} e^{x \ln \frac{x+1}{x}}$

$= e^{\lim_{x \to \infty} \frac{\ln \frac{x+1}{x}}{\frac{1}{x}}}$

$\underset{L'H}{=} e^{\lim_{x \to \infty} \frac{\frac{x}{x+1} \cdot (-\cancel{\frac{1}{x^2}})}{-\cancel{\frac{1}{x^2}}}}$

$= e^{\lim_{x \to \infty} \frac{x}{x+1}}$

$= e^1$

$= e$

3. Evaluate the following integrals.

(15) (a) $\int x \tan^{-1} x \, dx$

$$= \frac{1}{2}x^2 \cdot \tan^{-1}x - \int \frac{1}{2}x^2 \cdot \frac{1}{1+x^2}\, dx$$

IBP

$$= \frac{1}{2}x^2 \tan^{-1}x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right) dx$$

$$= \frac{1}{2}x^2 \tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + C$$

(15) (b) Evaluate $\int (\cos^4 x - \sin^4 x)\, dx$ using the double angle formula. You are NOT allowed to use the reduction formula for $\int \cos^n x \; dx$ or $\int \sin^n x \; dx$, but any algebraic manipulations are allowed.

$$\int (\cos^4 x - \sin^4 x)\, dx$$

$$= \int \left[\frac{1}{2}(1+\cos 2x)\right]^2 - \left[\frac{1}{2}(1-\cos 2x)\right]^2 \, dx$$

$$= \frac{1}{4}\int (\cancel{1} + 2\cos 2x + \cancel{\cos^2 2x}) - (\cancel{1} - 2\cos 2x + \cancel{\cos^2 2x}) \, dx$$

$$= \int \cos 2x \, dx$$

$$= \frac{1}{2}\sin 2x + C$$

(10) 4. Compute the arc length of $y = \ln(\cos x)$ over the inverval $[0, \frac{\pi}{3}]$.

$$\text{arc length} = \int_0^{\frac{\pi}{3}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

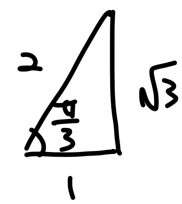
$$= \int_0^{\frac{\pi}{3}} \sqrt{1 + \left(\frac{-\sin x}{\cos x}\right)^2}\, dx$$

$$= \int_0^{\frac{\pi}{3}} \sec x \, dx$$

$$= \ln|\sec x + \tan x| \Big|_0^{\frac{\pi}{3}}$$

$$= \ln(2+\sqrt{3}) - \ln(1+0)$$

$$= \ln(2+\sqrt{3})$$



(15) 5. (a) Write down the partial fraction decomposition of $\dfrac{3x^2+2x+5}{(x-1)(x^2+2x+2)}$.

$$\frac{3x^2+2x+5}{(x-1)(x^2+2x+2)} = \frac{a}{x-1} + \frac{bx+c}{x^2+2x+2}$$

$$\rightsquigarrow 3x^2+2x+5 = a(x^2+2x+2) + (x-1)(bx+c)$$

$x=1$: $10 = 5a \Rightarrow a = 2$

$x=0$: $5 = 2a - c \Rightarrow c = -1$

$x=-1$: $6 = a + (-2)(-b+c) \Rightarrow b = 1$

$$\Rightarrow \frac{3x^2+2x+5}{(x-1)(x^2+2x+5)} = \frac{2}{x-1} + \frac{x-1}{x^2+2x+2}$$

(15) (b) Evaluate the indefinite integral

$$\int \frac{3x^2+2x+5}{(x-1)(x^2+2x+2)}\, dx.$$

$$= \int \left(\frac{2}{x-1} + \frac{x-1}{x^2+2x+2} \right) dx$$

$$= 2\ln|x-1| + \int \frac{x-1}{(x+1)^2+1}\, dx$$

$$\underset{u=x+1}{=} 2\ln|x-1| + \int \frac{u-2}{u^2+1}\, du$$

$$= 2\ln|x-1| + \frac{1}{2}\ln|u^2+1| - 2\tan^{-1}u + C$$

$$= 2\ln|x-1| + \frac{1}{2}\ln|x^2+2x+2| - 2\tan^{-1}(x+1) + C$$