Midterm 1
Math 132, Spring 2021

Name:	UID:

Honor Statement
I assert, on my honor, that I have not received assistance of any kind from any other people, including posting exam questions on online forums while working on this Final Exam. I have only used non-human resources, for example Internet, calculators, textbook, notes, and lecture videos, during the period of this evaluation.

Signature: ________________________

Directions—Please read carefully!

- You have a 24-hour window

  April 23 (Fri) 8am PDT – April 24 (Sat) 8am PDT

  to complete the exam, but the exam is designed to be able to be finished in 1 hour.

- On April 23 (Fri) 11–11:50am PDT, I will be on Zoom for any questions about statements of exam problems. You can also use Piazza to ask, but please use private message to avoid leaking your solution to others.

- You are allowed to use any non-human resources including internet, calculators, textbook, notes, lecture videos, etc. **You are NOT allowed to seek help from other people, including posting exam questions on online forums.**

- In order to receive full credit, you must **show your work or explain your reasoning**; your final answer is less important than the reasoning you used to reach it. Correct answers without work will receive little or no credit. Please write neatly. **Illegible answers will be assumed to be incorrect.** Circle or box your final answer when relevant.

- The exam is on Gradescope. Please either
  - Write your answers on the pdf file of the exam, then submit onto Gradescope,
  - Print the exam and write your answers on the exam paper, then scan and submit onto Gradescope, or
  - Use blank sheets of paper, **copy the honor statement and sign.** Then write your answers on them, scan and submit onto Gradescope.

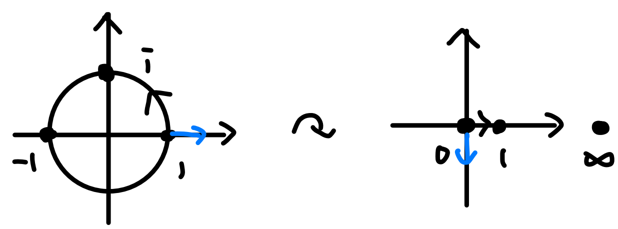
Good luck!

1. You do NOT need to provide explanation to the following questions.

(8) (a) Under the stereographic projection, what is the image of the following curves on sphere to the complex plane. Select THE BEST description.

B Latitude circles in upper hemisphere
E Longitude circles
C Circles through south pole but not north pole
F Circles through north pole but not south pole

A. Circles centered at origin with radius ≤ 1.
B. Circles centered at origin with radius ≥ 1.
C. Circles through origin.
D. Circles not through origin.
E. Lines through origin.
F. Lines not passing through origin.



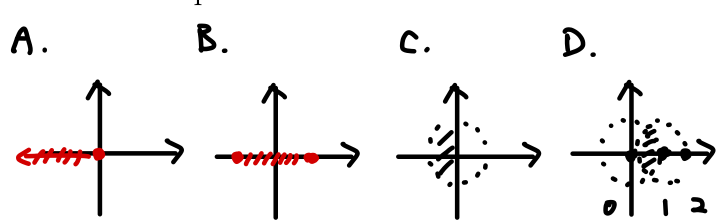
(4) (b) Let $f(z)$ be the fractional linear transformation which maps $(1, i, -1)$ to $(0, 1, \infty)$. What is the image of the following curves under $f(z)$?

A Unit circle, positively oriented (counterclockwise)
D Real axis, positively oriented (from $-\infty$ to ∞)

A. Real axis, positively oriented (from $-\infty$ to ∞)
B. Real axis, negatively oriented (from ∞ to $-\infty$)
C. Imaginary axis, positvely oriented (from $-i\infty$ to $i\infty$)
D. Imaginary axis, negatively oriented (from $i\infty$ to $-i\infty$)
E. Unit circle, positively oriented (counterclockwise)
F. Unit circle, negatively oriented (clockwise)

(4) (c) For the following subsets of $\mathbb{C}$, select ALL which are open.

✓ A. The slit plane $\mathbb{C} \setminus (-\infty, 0]$
✓ B. The slit plane $\mathbb{C} \setminus [-1, 1]$
✓ C. $\{z \in \mathbb{C} \colon |z| < 1, \operatorname{Re} z < 0\}$
✓ D. $\{|z| < 1\} \cap \{|z - 1| < 1\}$

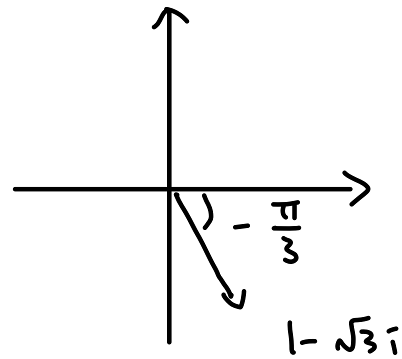


2. Write the following expressions in polar coordinates $z = re^{i\theta}$. (Caution: The expressions are not always single numbers.)

(8) (a) $(1-\sqrt{3}i)^3$

$$= \left(2e^{-\frac{\pi}{3}i}\right)^3$$

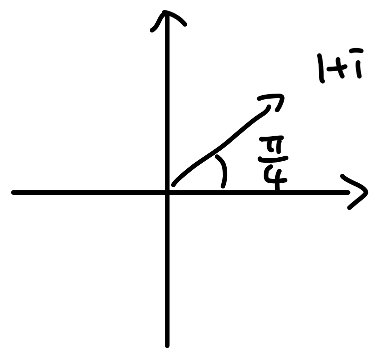
$$= 8e^{-\pi i}$$



(8) (b) $\sqrt[4]{1+i}$

$$= \left(\sqrt{2}e^{(\frac{\pi}{4}+2m\pi)i}\right)^{\frac{1}{4}}$$

$$= \sqrt[8]{2}\; e^{(\frac{1}{16}+\frac{1}{2}m)\pi i}, \quad m=0,1,2,3$$



(10) (c) $(-1)^{1+i}$

$$= e^{\log(-1)\cdot(1+i)}$$

$$= e^{(-\pi+2m\pi)i\cdot(1+i)}$$

$$= e^{(-\pi+2m\pi)(-1+i)}$$

$$= e^{\pi-2m\pi}\cdot e^{-\pi i}, \quad m\in\mathbb{Z}$$

3. Let $u(x, y) = \dfrac{y}{x^2 + y^2}$.

(13) (a) Show that $u(x, y)$ is a harmonic function.

$$\frac{\partial u}{\partial x} = \frac{-y \cdot 2x}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-2y(x^2+y^2)^2 - (-2xy)\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4} = \frac{-2y(x^2+y^2) + 2xy \cdot 4x}{(x^2+y^2)^3}$$

$$= \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$$

$$\frac{\partial u}{\partial y} = \frac{1\cdot(x^2+y^2) - y\cdot 2y}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-2y(x^2+y^2)^2 - (x^2-y^2)\cdot 2(x^2+y^2)\cdot 2y}{(x^2+y^2)^4}$$

$$= \frac{-2y(x^2+y^2) - (x^2-y^2)\cdot 4y}{(x^2+y^2)^3}$$

$$= \frac{-6x^2y + 2y^3}{(x^2+y^2)^3}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(13) (b) Find the harmonic conjugate $v(x,y)$ of $u(x,y)$.

$$\begin{cases} \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} = -\dfrac{x^2-y^2}{(x^2+y^2)^2} & ① \\ \dfrac{\partial v}{\partial y} = \dfrac{\partial u}{\partial x} = \dfrac{-2xy}{(x^2+y^2)^2} & ② \end{cases}$$

$$② \xRightarrow{\int dy} v(x,y) = \frac{x}{x^2+y^2} + v_0(x)$$

for some function $v_0(x)$

$$\text{So } \frac{\partial v}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2} + v_0'(x) \overset{①}{=} -\frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\Rightarrow v_0'(x) = 0$$

$$\Rightarrow v_0(x) = C \quad \text{constant}$$

$$\Rightarrow v(x,y) = \frac{x}{x^2+y^2} + C$$

(12) 4. Let $f(z)$ be an analytic function on $\mathbb{C}$. Write $f(z) = u(r,\theta) + iv(r,\theta)$, where $z = re^{i\theta}$. Derive the polar form of the Cauchy–Riemann equation WITHOUT using the Cartesian form of the Cauchy–Riemann equation.

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

(Hint: Compute $f'(z)$ in two ways, varying r and varying θ, and identify the two expressions.)

Write $f(z) = u(r,\theta) + iv(r,\theta)$ where $z = re^{i\theta}$

$$f'(z) = \lim_{\Delta r \to 0} \frac{f((r+\Delta r)e^{i\theta}) - f(re^{i\theta})}{\Delta r e^{i\theta}}$$

$$= \lim_{\Delta r \to 0} \frac{u(r+\Delta r,\theta) + iv(r+\Delta r,\theta) - u(r,\theta) - iv(r,\theta)}{\Delta r e^{i\theta}}$$

$$= \frac{1}{e^{i\theta}}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$$

$$f'(z) = \lim_{\Delta\theta \to 0} \frac{f(re^{i(\theta+\Delta\theta)}) - f(re^{i\theta})}{r(e^{i(\theta+\Delta\theta)} - e^{i\theta})}$$

$$= \lim_{\Delta\theta \to 0} \frac{u(r,\theta+\Delta\theta) + iv(r,\theta+\Delta\theta) - u(r,\theta) - iv(r,\theta)}{r(e^{i(\theta+\Delta\theta)} - e^{i\theta})}$$

$$= \frac{1}{r}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right) \cdot \lim_{\Delta\theta \to 0} \frac{\Delta\theta}{e^{i(\theta+\Delta\theta)} - e^{i\theta}}$$

$$= \frac{1}{r}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right) \cdot \frac{1}{\frac{d}{d\theta}(e^{i\theta})}$$

$$= \frac{1}{r} \cdot \frac{1}{ie^{i\theta}}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right).$$

$$\Rightarrow \left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{1}{ri}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)$$

$$\Rightarrow \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

(10) 5. (a) Write $\operatorname{Log} z = u(r,\theta) + iv(r,\theta)$, where $z = re^{i\theta}$. Find the functions $u(r,\theta)$ and $v(r,\theta)$.

$$\operatorname{Log} z \overset{\text{def}}{=} \log|z| + i\operatorname{Arg} z$$

$$= \log r + i\theta$$

$$\Rightarrow \quad u(r,\theta) = \log r, \quad v(r,\theta) = \theta$$

Remember when we used Cartesian coordinate,

$$u(x,y) = \frac{1}{2}\log(x^2+y^2), \quad v(x,y) = \tan^{-1}\frac{y}{x}$$

(10) (b) Verify that the functions $u(r,\theta)$, $v(r,\theta)$ you found satisfy the polar form of the Cauchy–Riemann equations:

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}, \quad \frac{\partial v}{\partial \theta} = 1 \quad \Rightarrow \quad \frac{\partial u}{\partial r} = \frac{1}{r}\cdot\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = 0, \quad \frac{\partial u}{\partial \theta} = 0 \quad \Rightarrow \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Remember when we used Cartesian coordinate,

$$\frac{\partial u}{\partial x} = \frac{1}{2}\,\frac{2x}{x^2+y^2}, \quad \frac{\partial v}{\partial y} = \frac{\frac{1}{x}}{1+\left(\frac{y}{x}\right)^2} = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}\,\frac{2y}{x^2+y^2}, \quad \frac{\partial v}{\partial x} = \frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2} = \frac{-y}{x^2+y^2}$$

It was a more complicated computation