SHIFTED WITT GROUPS OF SEMI-LOCAL RINGS

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Abstract. We show that the odd-indexed derived Witt groups of a semi-local ring with trivial involution vanish. We show that this is wrong when the involution is not trivial and we provide examples.

Introduction

Let us denote by $\mathrm{W}^{+}_{\mathrm{us}}$ and $\mathrm{W}^{-}_{\mathrm{us}}$ the usual Witt groups of symmetric and skew-symmetric spaces respectively, which classify such spaces up to isometry and modulo metabolic ones. Similarly, there exist Witt groups W^i for triangulated categories with duality, see [B1], which are 4-periodic $\mathrm{W}^i \cong \mathrm{W}^{i+4}$ in the index $i \in \mathbb{Z}$. These groups form a cohomology theory, in the sense that suitable localization long exact sequences can be produced. These so-called *triangular Witt groups* W^i constitute a very flexible and powerful device for studying quadratic forms, especially in algebraic geometry, see for instance [B3] and [BW].

In the special case where the considered triangulated category is the derived category $\mathrm{D^b}(\mathcal{E})$ of an exact category $\mathcal{E}$ with duality, like $\mathcal{E}$ the category of vector bundles over a scheme, or $\mathcal{E}$ the category of projective modules over a ring with involution, the Witt groups $\mathrm{W}^*(\mathrm{D^b}(\mathcal{E}))$ are called the *derived Witt groups* of $\mathcal{E}$. The even-indexed groups $\mathrm{W}^0(\mathrm{D^b}(\mathcal{E}))$ and $\mathrm{W}^2(\mathrm{D^b}(\mathcal{E}))$ are the usual Witt groups $\mathrm{W}^{+}_{\mathrm{us}}(\mathcal{E})$ and $\mathrm{W}^{-}_{\mathrm{us}}(\mathcal{E})$ of $\mathcal{E}$ respectively, see [B2, Thm. 4.3], at least when we assume that 2 is invertible, as we shall always do below. The odd-indexed groups, also known as *the shifted Witt groups*, can be given by generators and relations as well, see Walter [W, § 8]. When moreover $\mathcal{E}$ is split exact, like the category of projective modules, these $\mathrm{W}^{2k+1}(\mathrm{D^b}(\mathcal{E}))$ are nothing but the Wall–Mischenko–Ranicki odd-indexed L-groups of $\mathcal{E}$, which are groups of formations.

In the present paper, we are interested in these odd-indexed groups W^1 and W^3 when $\mathcal{E} = R-\mathrm{proj}$ is the category of finitely generated projective left R-modules over a *semi-local ring* R with an involution σ. We denote these groups by $\mathrm{W}^*(R_\sigma-\mathrm{proj})$ or simply by $\mathrm{W}^*(R)$ when no confusion can arise. Let us stress that the category $R-\mathrm{proj}$ is split exact and the presence of $\frac{1}{2} \in R$, so, these groups coincide with the odd-indexed L-groups $L^p_{2i+1}(R)$ of R, in the notation of [DR]. We are interested in possibly non-commutative semi-local rings, but we shall see that the real distinction arises from the involution being trivial or non-trivial.

In [B2, Thm. 5.6], it is proven that $\mathrm{W}^{2k+1}(R) = 0$ when R is commutative local with trivial involution $\sigma = \mathrm{id}$. Our first goal is to generalize this result to the semi-local situation (see Theorem 2.3 below):

Theorem. *Let R be a semi-local commutative ring containing $\frac{1}{2}$ equipped with the trivial involution $\sigma = \mathrm{id}$. Then* $\mathrm{W}^{2k+1}(R) = 0$.

In fact, we give here a *very elementary proof*, which does not rely on the connections between derived Witt groups and L-theory and which does not use much

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of the triangular theory either, except for the definition and for two easy lemmas of [B2, B3] recalled in Section 1. This settles the case of a trivial involution.

When the involution $\sigma : R \to R$ is non-trivial, the odd-indexed Witt groups do not vanish. In fact, they do not vanish *even if R is commutative*. So, to produce examples, we can and shall concentrate on commutative rings. (Readers looking for non-commutative examples can replace our rings by matrix rings over them.)

As usual with semi-local rings, our ideal 3-step choreography would be:

(1) Go modulo the radical $R \longrightarrow \bar{R} := R/\mathrm{Rad}(R)$.
(2) Prove something for semi-simple rings, like $\bar{R}$.
(3) Lift the information by Nakayamian arguments from $\bar{R}$ to R.

At first sight, this looks hopeless because of the following result of Ranicki: Denote by $\bar{\sigma}$ the involution on $\bar{R}$ induced by σ, then $\mathrm{W}^{2k+1}(\bar{R}_{\bar{\sigma}}-\mathrm{proj}) = 0$, see [R]. On the other hand, we can as well use the split exact category $\mathcal{E} = R-\mathrm{free}$ of finitely generated free R-modules, instead of $R-\mathrm{proj}$. In this case, the odd-indexed Witt groups are the $L^h_{2i+1}(R)$ in the notation of [DR]. The good news is that the lift of information is now possible, by a result of Davis-Ranicki, see [DR, Cor. 5.2 (ii)], which says that in the following diagram, the map on the left is a monomorphism:

$$\begin{array}{ccc} \mathrm{W}^{2k+1}(R_\sigma-\mathrm{free}) & \longrightarrow & \mathrm{W}^{2k+1}(R_\sigma-\mathrm{proj}) \\ \big\downarrow & & \big\downarrow \\ \mathrm{W}^{2k+1}(\bar{R}_{\bar{\sigma}}-\mathrm{free}) & \longrightarrow & 0\,. \end{array}$$

The problem becomes to understand the kernel and cokernel of the horizontal map. This involves the so-called *Rothenberg sequence* in L-theory, which now has a triangular formulation due to Hornbostel and Schlichting, recalled in Section 3. Then, we need to compute $\mathrm{W}^{2k+1}(\bar{R}_{\bar{\sigma}}-\mathrm{free})$. Indeed in Sections 4 and 5 we prove:

Theorem. *Let (S,σ) be a semi-simple ring with involution such that $\frac{1}{2} \in S$. Let $\mathbf{so}(S)$ and let $\mathbf{ss}(S)$ be the numbers of simple factors of S on which the involution is split-orthogonal, respectively split-symplectic, see Def. 5.2. Then the odd-indexed Witt groups $\mathrm{W}^{2k+1}(S_\sigma-\mathrm{free})$ are finitely generated $\mathbb{Z}/2$-modules with*

$$\dim_{\mathbb{Z}/2} \mathrm{W}^1(S_\sigma-\mathrm{free}) \;=\; \mathbf{ss}(S)$$

whereas

$$\mathbf{so}(S) - 1 \;\le\; \dim_{\mathbb{Z}/2} \mathrm{W}^3(S_\sigma-\mathrm{free}) \;\le\; \mathbf{so}(S)\,.$$

This is Theorem 5.6, where the reader can also find a more precise description of the rank of $\mathrm{W}^3(S_\sigma-\mathrm{free})$ in terms of the simple factors of S with split-orthogonal involution.

In Section 6 we assume the ring R to be *commutative* and semi-local. Then, the situation is the following. First, we have $\mathrm{W}^1(R_\sigma-\mathrm{proj}) = 0$ for any involution. Secondly, $\mathrm{W}^3(R_\sigma-\mathrm{proj})$ is a finite $\mathbb{Z}/2$-vector space whose dimension is strictly less than $\mathbf{so}(\bar{R})$, the number of simple factors of $\bar{R}$ on which the involution $\bar{\sigma}$ is trivial. Simple factors of $\bar{R}$ with non-trivial involution and pairs of simple factors switched by the involution do not play any role. See Theorem 6.2. Let us stress the peculiar situation: to produce a (big) non-zero $\mathrm{W}^3(R)$, we need the ring R itself to carry a *non-trivial* involution, but modulo the radical, we need as many factors as possible with *trivial* involution.

In Section 6, we also explain the significance of the dimension of $\mathrm{W}^3(R)$ and show that maximal ideals of R can in fact be "detected" quite explicitly by $\mathrm{W}^3(R)$ if some simple properties are observable in the ring R, see Definition 6.4. Using this method, we can produce semi-local domains R with arbitrary big W^3. More precisely, these domains can even be chosen to be semi-localizations of rings of integers in suitable number fields. See Theorem 6.10, which says in particular:

Theorem. *Let $n \in \mathbb{N}$ be an integer. Then there exists a semi-local Dedekind domain with non-trivial involution (R, σ) such that $\mathrm{W}^3(R) \simeq (\mathbb{Z}/2)^n$.*

One importance of our vanishing result comes from the Gersten Conjecture for Witt groups, see [B3, BGPW], which, combined with the Balmer-Walter spectral sequence [BW], requires in particular $\mathrm{W}^{2k+1}(R) = 0$ for all semi-local *regular* rings with trivial involution. Conversely, the vanishing of $\mathrm{W}^{2k+1}(R)$ for R local has been used in the proof of the conjecture; namely, this vanishing facilitates the reformulation of the conjecture in terms of Zariski cohomology with coefficients in the Witt sheaf, see [BGPW]. In this logic, the vanishing result of the present article allows us to extend the Gersten-Witt Conjecture to semi-local equicharacteristic regular rings, as confirmed by Mitchell in [M].

Another consequence of our vanishing result is that the Gersten Conjecture for Witt groups holds up to dimension 4 without assuming the existence of a ground field, that is, [BW, Cor. 10.4] now holds for semi-local rings as well.

With these facts in mind, it is surprising, though not contradictory, that this vanishing $\mathrm{W}^{2k+1}(R) = 0$ holds for all semi-local rings with trivial involution, but fails when the involution is non-trivial, even for R commutative, *regular* and of dimension 1.

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Contents

1. Explicit classes in odd-indexed Witt groups

In this paper, R denotes a ring with unit, *such that* 2 *is invertible in* R. Let σ be an involution on R, that is, an anti-isomorphism $\sigma : R \xrightarrow{\sim} R$, $\sigma(r_1\, r_2) = \sigma(r_2)\, \sigma(r_1)$, such that $\sigma^2 = \mathrm{id}_R$. When $A = \big(a_{ij}\big) \in \mathrm{M}_{m,n}(R)$ is an $m \times n$ matrix, we shall denote by $A^* = \sigma(A^t)$ the $n \times m$ matrix $(A^*)_{ij} = \sigma(a_{ji})$.

The involution σ allows us to turn a right R-module N into a left R-module via $r \cdot n := n \cdot \sigma(r)$. This applies in particular to the dual $N = \mathrm{Hom}_R(M, R)$ of a left R-module M and gives: $(r \cdot f)(m) = f(m) \cdot \sigma(r)$.

Without mention, R-modules are left R-modules. In the sequel, we shall consider the category $R{-}\mathrm{proj}$ of finitely generated projective R-modules and its subcategory $R{-}\mathrm{free}$ of free ones. Both are endowed with the duality $(-)^* := \mathrm{Hom}_R(-, R)$ using the involution as usual to identify an object and its double dual: $\pi : \mathrm{Id} \xrightarrow{\sim} * \circ *$. Explicitly, the isomorphism $\pi_M : M \to M^{**}$ is defined on an element $m \in M$ to be $\pi_M(m) : f \mapsto \sigma(f(m))$ for all $f \in M^*$.

We study the derived Witt groups of these categories, that is, the W^* of their derived categories. In both cases, the derived categories are nothing but the homotopy categories $\mathrm{K^b}(R\text{−proj})$ and $\mathrm{K^b}(R\text{−free})$, *i.e.* the categories of bounded complexes with morphisms up to homotopy. We denote by $(-)^{\#}$ the duality induced on them by $(-)^*$ and which turns them into triangulated categories with duality in the sense of Balmer [B1]. Their Witt groups W^k are defined using k-shifted dualities, *i.e.* the functors $(-)^{\#}[k]$. In addition to [B1, § 2], the reader can find in [B2, § 5] a specific discussion of odd-indexed Witt groups. These references apply in particular to the sign conventions, which are so that for the 1-shifted duality $(-)^{\#}[1]$, the identification ϖ between an object and its double dual is $-\pi$ in each degree, whereas it is π in each degree for the 3-shifted duality $(-)^{\#}[3]$.

We want to construct some explicit classes in W^1 and W^3. For this, we use *short complexes*, in the terminology of [W], namely complexes of length one.

Definition 1.1. Let $n \geq 0$ be an integer and let $A, B \in \mathrm{M}_n(R)$ be two $n \times n$ matrices such that the following

$$0 \longrightarrow R^n \xrightarrow{\begin{pmatrix} A \\ B \end{pmatrix}} R^n \oplus R^n \xrightarrow{\begin{pmatrix} -B^* & A^* \end{pmatrix}} R^n \longrightarrow 0 \qquad (1)$$

is an exact sequence. (In particular, $A^* \cdot B - B^* \cdot A = 0$.) Equivalently, one can say that the following is a symmetric 1-space $(P_\bullet, \varphi)$ in $\mathrm{K^b}(R\text{−free})$, that is, a symmetric space for the 1-shifted duality:

$$\begin{array}{ccccccccccc}
P_\bullet := & & \cdots 0 & \longrightarrow & 0 & \longrightarrow & R^n & \xrightarrow{A} & R^n & \longrightarrow & 0 \longrightarrow 0 \cdots \\
\downarrow{\scriptstyle \varphi :=} & & & & \downarrow & & \downarrow{\scriptstyle B} & & \downarrow{\scriptstyle -B^*} & & \downarrow \\
(P_\bullet)^{\#}[1] = & & \cdots 0 & \longrightarrow & 0 & \longrightarrow & R^n & \xrightarrow{-A^*} & R^n & \longrightarrow & 0 \longrightarrow 0 \cdots
\end{array}$$

Here, R^* is canonically identified with R as usual. The complex $P_\bullet$ has non-zero modules in homological degree 1 and 0. The vertical map φ is an isomorphism in $\mathrm{K^b}(R\text{−free})$; indeed φ is a homotopy equivalence by condition (1). The Witt class of this symmetric 1-space will be denoted by

$$[\, n\,,\, A\,,\, B\,]^{(1)} \in \mathrm{W}^1(R_\sigma\text{−free}).$$

Similarly, we define explicit classes in W^3 as follows.

Definition 1.2. Let $n \geq 0$ and $A, B \in \mathrm{M}_n(R)$ such that

$$0 \longrightarrow R^n \xrightarrow{\begin{pmatrix} A \\ B \end{pmatrix}} R^n \oplus R^n \xrightarrow{\begin{pmatrix} B^* & A^* \end{pmatrix}} R^n \longrightarrow 0 \qquad (2)$$

is an exact sequence. Then we have the following symmetric 3-space $(Q_\bullet, \psi)$:

$$\begin{array}{ccccccccccc}
Q_\bullet := & & \cdots 0 & \longrightarrow & 0 & \longrightarrow & R^n & \xrightarrow{A} & R^n & \longrightarrow & 0 \longrightarrow 0 \cdots \\
\downarrow{\scriptstyle \psi :=} & & & & \downarrow & & \downarrow{\scriptstyle B} & & \downarrow{\scriptstyle B^*} & & \downarrow \\
(Q_\bullet)^{\#}[3] = & & \cdots 0 & \longrightarrow & 0 & \longrightarrow & R^n & \xrightarrow{-A^*} & R^n & \longrightarrow & 0 \longrightarrow 0 \cdots
\end{array}$$

where the complex $Q_\bullet$ has non-zero modules in homological degree 2 and 1. The Witt class of this 3-space will be denoted by

$$[\, n\,,\, A\,,\, B\,]^{(3)} \in \mathrm{W}^3(R_\sigma\text{−free}).$$

Remark 1.3. For $k = 1$ or 3, we might again denote by

$$[\, n\,,\, A\,,\, B\,]^{(k)} \in \mathrm{W}^k(R_\sigma\text{−proj}),$$

the image of the class $[\,n\,,\,A\,,\,B\,]^{(k)}$ defined above, under the natural homomorphism $\mathrm{W}^k(R_\sigma\text{−free}) \to \mathrm{W}^k(R_\sigma\text{−proj})$, although the latter needs not be injective in general.

We now collect some results which are used in the proof of the main theorem. The following proposition is easily deduced from TWG, namely from [B2].

Proposition 1.4. *Suppose that $R^m \simeq R^n$ implies $m = n$ (for instance, R commutative or R semi-local). Let $k = 1$ or 3. Then, any class in $\mathrm{W}^k(R_\sigma\text{−free})$ is represented by an element of the form $[\,n\,,\,A\,,\,B\,]^{(k)}$ as in Definitions 1.1 and 1.2.*

Proof. From [B2, Prop. 5.2], we know that any symmetric k-space is Witt-equivalent to a symmetric k-space $(P_\bullet, \varphi)$ where $P_\bullet$ is a short complex, *i.e.* vanishes except for two consecutive indices, as above. *A priori*, the complex $P_\bullet$ is of the form

$$P_\bullet = \qquad \cdots 0 \to 0 \to R^m \to R^n \to 0 \to 0 \to \cdots$$

and we have to prove that $m = n$. The symmetric form $\varphi : P_\bullet \to (P_\bullet)^\#[k]$ being a homotopy equivalence means that its cone is a split exact complex of the form

$$0 \to R^m \to R^n \oplus R^n \to R^m \to 0\,.$$

It implies in particular that $R^{2m} \simeq R^{2n}$ and in turn $m = n$. □

Remark 1.5. Walter has given a general description of the odd-indexed triangular Witt groups $\mathrm{W}^1(\mathcal{E})$ and $\mathrm{W}^3(\mathcal{E})$ by generators and relations, for any exact category with duality $\mathcal{E}$, see [W, § 8].

Lemma 1.6. *The following hold true :*

(a) *If $[\,n\,,\,A\,,\,B\,]^{(1)} \in \mathrm{W}^1(R_\sigma\text{−free})$ and if $F \in \mathrm{M}_n(R)$ is such that $F = F^*$ then $[\,n\,,\,A\,,\,B + FA\,]^{(1)}$ is defined and we have*

$$[\,n\,,\,A\,,\,B\,]^{(1)} = [\,n\,,\,A\,,\,B + FA\,]^{(1)}$$

in $\mathrm{W}^1(R_\sigma\text{−free})$.

(b) *If $[\,n\,,\,A\,,\,B\,]^{(3)} \in \mathrm{W}^3(R_\sigma\text{−free})$ and if $F \in \mathrm{M}_n(R)$ is such that $F = -F^*$ then $[\,n\,,\,A\,,\,B + FA\,]^{(3)}$ is defined and we have*

$$[\,n\,,\,A\,,\,B\,]^{(3)} = [\,n\,,\,A\,,\,B + FA\,]^{(3)}$$

in $\mathrm{W}^3(R_\sigma\text{−free})$.

Proof. For (a), it is easy to check that the matrix F provides a homotopy between the two morphisms φ of Definition 1.1, once using B and once using $B + FA$. This means that they define the same morphism $P_\bullet \to (P_\bullet)^\#[1]$ in $\mathrm{K^b}(R\text{−free})$. The two Witt classes coincide because the two symmetric spaces defining them are equal. The proof for (b) is the same, *mutatis mutandis* □

Lemma 1.7. *Let $k = 1$ or 3. Let $U, V \in \mathrm{GL}_n(R)$ be two invertible matrices. For any class $[\,n\,,\,A\,,\,B\,]^{(k)} \in \mathrm{W}^k(R_\sigma\text{−free})$, the class $[\,n\,,\,V^{-1}AU\,,\,V^*BU\,]^{(k)}$ is defined and we have*

$$[\,n\,,\,A\,,\,B\,]^{(k)} = [\,n\,,\,V^{-1}AU\,,\,V^*BU\,]^{(k)}$$

in $\mathrm{W}^k(R_\sigma\text{−free})$.

Proof. The classes coincide because the spaces are isometric via (U, V). □

More generally, we have :

Lemma 1.8. *Let $k = 1$ or 3 and let $[\,n\,,\,A\,,\,B\,]^{(k)}$ be a class in $\mathrm{W}^k(R_\sigma\text{–free})$. Let $m \geq 0$ and let $A' \in \mathrm{M}_m(R)$ and $U, V \in \mathrm{M}_{n,m}(R)$ be matrices such that the following*

$$0 \longrightarrow R^m \xrightarrow{\begin{pmatrix} A' \\ U \end{pmatrix}} R^m \oplus R^n \xrightarrow{\begin{pmatrix} V & -A \end{pmatrix}} R^n \longrightarrow 0$$

is an exact sequence. Then the class $[\,m\,,\,A'\,,\,V^*BU\,]^{(k)}$ is defined and we have

$$[\,n\,,\,A\,,\,B\,]^{(k)} = [\,m\,,\,A'\,,\,V^*BU\,]^{(k)}$$

in $\mathrm{W}^k(R_\sigma\text{–free})$.

Proof. The classes coincide because the spaces are isometric in $\mathrm{K^b}(R\text{–free})$ via

$$\begin{array}{ccccccccccc}
\cdots 0 & \longrightarrow & 0 & \longrightarrow & R^m & \xrightarrow{A'} & R^m & \longrightarrow & 0 & \longrightarrow & 0 \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle U} & & \downarrow{\scriptstyle V} & & \downarrow & & \\
\cdots 0 & \longrightarrow & 0 & \longrightarrow & R^n & \xrightarrow{A} & R^n & \longrightarrow & 0 & \longrightarrow & 0 \cdots
\end{array}$$

which is a homotopy equivalence by assumption. □

The following Interchange Lemma is proven in TWG. We restate it here for the convenience of the reader. We stress that Proposition 1.4 and Lemma 1.9 are essentially the only facts on Triangular Witt groups that are used in the proof of the vanishing result, Theorem 2.3.

Lemma 1.9. *Let $k = 1$ or 3. In $\mathrm{W}^k(R_\sigma\text{–free})$, we have*

$$[\,n\,,\,A\,,\,B\,]^{(k)} = -[\,n\,,\,B\,,\,A\,]^{(k)}\,.$$

Proof. This follows from [B3, Lemma 3.2 (2)] and also appears in [W, 7.2 (c)]. □

Lemma 1.10. *The following hold true:*

(a) *If $[\,n\,,\,A\,,\,B\,]^{(1)} \in \mathrm{W}^1(R_\sigma\text{–free})$ and if $E \in \mathrm{M}_n(R)$ is such that $E = E^*$ then $[\,n\,,\,A + EB\,,\,B\,]^{(1)}$ is defined and we have*

$$[\,n\,,\,A\,,\,B\,]^{(1)} = [\,n\,,\,A + EB\,,\,B\,]^{(1)}$$

in $\mathrm{W}^1(R_\sigma\text{–free})$.

(b) *If $[\,n\,,\,A\,,\,B\,]^{(3)} \in \mathrm{W}^3(R_\sigma\text{–free})$ and if $E \in \mathrm{M}_n(R)$ is such that $E = -E^*$ then $[\,n\,,\,A + EB\,,\,B\,]^{(3)}$ is defined and we have*

$$[\,n\,,\,A\,,\,B\,]^{(3)} = [\,n\,,\,A + EB\,,\,B\,]^{(3)}$$

in $\mathrm{W}^3(R_\sigma\text{–free})$.

Proof. This is Lemma 1.6 conjugated with the Interchange Lemma 1.9. □

Lemma 1.11. *Let $k = 1$ or 3 and a class $[\,n\,,\,A\,,\,B\,]^{(k)} \in \mathrm{W}^k(R_\sigma\text{–free})$. If either of A or B is an invertible matrix then $[\,n\,,\,A\,,\,B\,]^{(k)} = 0$ in $\mathrm{W}^k(R_\sigma\text{–free})$.*

Proof. If A is an isomorphism, the complex supporting the form is acyclic, *i.e.* it is zero in $\mathrm{K^b}(R\text{–free})$. Hence the space is necessarily trivial. If B is an isomorphism, the result follows from the above case and the Interchange Lemma 1.9. □

Lemma 1.12. *Let $k = 1$ or 3 and a class $[\,n\,,\,A\,,\,B\,]^{(k)} \in \mathrm{W}^k(R_\sigma\text{–free})$ for $n \geq 2$. Assume that the $(1,1)$-entry of the matrix A is invertible, that is, $a_{11} \in R^\times$. Then there exist $(n-1) \times (n-1)$ matrices A' and B' such that $[\,n\,,\,A\,,\,B\,]^{(k)} = [\,n - 1\,,\,A'\,,\,B'\,]^{(k)}$ in $\mathrm{W}^k(R_\sigma\text{–free})$.*

Proof. By elementary operations on rows and columns, *i.e.* replacing A by $V^{-1}AU$ for suitable elementary matrices U, $V \in \mathrm{GL}_n(R)$ and using Lemma 1.7, we can assume that A is of the following type

$$\begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A' & \\ 0 & & & \end{pmatrix}.$$

where A' is an $(n-1)\times(n-1)$ matrix over R. Consider the $n \times (n-1)$ matrix T corresponding to the inclusion $R^{n-1} \hookrightarrow R^n$ in the last $n-1$ factors. Then the result follows from Lemma 1.8 applied to $m = n-1$ and $U = V = T$. □

Example 1.13. Let R be a ring. Consider the ring $\tilde{R} = (R \times R^{op})$ with the switch involution sw, *i.e.* $\mathrm{sw}(r, s^{op}) = (s, r^{op})$. We now show that $\mathrm{W}^k(\tilde{R}_{\mathrm{sw}}\text{−free}) = 0$ and $\mathrm{W}^k(\tilde{R}_{\mathrm{sw}}\text{−proj}) = 0$ for every integer k. This is a particular case of the following more general set-up.

Given an additive category $\mathcal{F}$, consider the additive category $\mathcal{E} = \mathcal{F} \times \mathcal{F}^{op}$. Let $\mathrm{sw} : \mathcal{E}^{op} \to \mathcal{E}$ be the obvious *switch* duality: $(A, B^{op})^{\mathrm{sw}} = (B, A^{op})$. We consider the additive category with duality, $(\mathcal{E}, \mathrm{sw}, \pi)$ where $\pi : \mathrm{id} \to \mathrm{sw} \circ \mathrm{sw}$ is the obvious identification. It is easy to see that any symmetric space over $(\mathcal{E}, \mathrm{sw}, \pi)$ is of the form (P, ϕ) where $P = (A, B^{op})$ and $\phi = (\phi_1, \phi_1^{op})$ for an isomorphism $\phi_1 : A \xrightarrow{\sim} B$ in $\mathcal{F}$. Using this isomorphism to replace B by A, we see that any symmetric space is isometric to a symmetric space of the form $\big((A,0) \oplus (A,0)^{\mathrm{sw}}\, ,\, H\big)$ where $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Hence every symmetric space (P, ϕ) over $(\mathcal{E}, \mathrm{sw}, \pi)$ is the *hyperbolic space*. A similar argument works for skew-symmetric forms.

Applying the above to Witt groups, and *a fortiori* when $\mathcal{F}$ is an exact category or a triangulated category, we see that the usual Witt groups and the shifted Witt groups all vanish for such categories with duality $(\mathcal{F} \times \mathcal{F}^{op}, \mathrm{sw}, \pi)$.

For the ring $\tilde{R} = (R \times R^{op})$ with the switch involution sw as above, both the derived categories $\mathrm{K^b}(R\text{−proj})$ and $\mathrm{K^b}(R\text{−free})$ are examples of such categories with duality and hence $\mathrm{W}^k(\tilde{R}_{\mathrm{sw}}\text{−free}) = 0$ and $\mathrm{W}^k(\tilde{R}_{\mathrm{sw}}\text{−proj}) = 0$ for every integer k.

2. Trivial involution : the vanishing theorem

Let R be a ring with unit and such that 2 is invertible in R. Recall that R is called *semi-local* if the ring $\bar{R} := R/\mathrm{Rad}(R)$ is *semi-simple*, that is, a product of simple artinian rings (matrix algebras over division rings), where $\mathrm{Rad}(R)$ is the Jacobson radical of R, that is, the intersection of all the maximal left ideals in R. See more in [L, §20]. We shall use the following result :

Theorem 2.1 (Bass [Bs, 6.4])**.** *Let R be a semi-local ring. If $b \in R$ and $\mathfrak{a}$ is a left ideal in R such that $R \cdot b + \mathfrak{a} = R$ then $b + \mathfrak{a}$ contains a unit.*

Remark 2.2. Let R be a *commutative* semi-local ring. Then, we have :

(a) Any finitely generated projective R-module of constant rank is free [L, §20].
(b) The Witt group of skew-symmetric forms over R with the trivial involution vanishes, $\mathrm{W}^2(R) = \mathrm{W}^-_{\mathrm{us}}(R) = 0$. This can be proven very easily (*i.e.* as over a field) using Theorem 2.1 and elementary row and column operations.

In this section we consider a *commutative* semi-local ring R with identity as the involution and study its k-indexed Witt groups with k odd.

Theorem 2.3. *Let R be a commutative semi-local ring containing $\frac{1}{2}$, with the trivial involution. Then, its odd-indexed Witt groups vanish:* $\mathrm{W}^k(R_{\mathrm{id}}-\mathrm{proj}) = 0$ *for $k = 1$ or 3.*

Beginning of the proof. If $\mathrm{Spec}(R)$ is not connected then $R \cong R_1 \times R_2$ for commutative semi-local rings R_1 and R_2. We have $\mathrm{W}^*(R_{\mathrm{id}}-\mathrm{proj}) = \mathrm{W}^*((R_1)_{\mathrm{id}}-\mathrm{proj}) \oplus \mathrm{W}^*((R_2)_{\mathrm{id}}-\mathrm{proj})$. Hence, we can assume from now on that R is commutative semi-local and $\mathrm{Spec}(R)$ is connected. In this situation, any finitely generated projective module is free and we shall simply write

$$\mathrm{W}^*(R) := \mathrm{W}^*(R_{\mathrm{id}}-\mathrm{proj}) = \mathrm{W}^*(R_{\mathrm{id}}-\mathrm{free})$$

for the Witt groups of R. Recall from Proposition 1.4 that any class in $\mathrm{W}^k(R)$ is of the form $[\,n\,,\,A\,,\,B\,]^{(k)} \in \mathrm{W}^k(R)$, for $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ as in Definitions 1.1 or 1.2. Before distinguishing the case W^1 from the case W^3, we regroup the common parts of the two proofs in the following auxiliary lemma.

For elements $r_1, \cdots, r_n \in R$, we denote by $\langle\, r_1, \cdots, r_n \,\rangle$ the ideal they generate.

Lemma 2.4. *With the above hypotheses and notation, any class $[\,n\,,\,A\,,\,B\,]^{(k)}$ in $\mathrm{W}^k(R)$ is equal to a class of the form $[\,n\,,\,\tilde{A}\,,\,\tilde{B}\,]^{(k)}$, with same integer n and with matrices $\tilde{A} = (\tilde{a}_{ij})$ and $\tilde{B} = (\tilde{b}_{ij})$ such that $\langle\, \tilde{a}_{11}, \tilde{b}_{11} \,\rangle = R$.*

Proof. We treat simultaneously the cases $k = 1$ and $k = 3$. Set $\epsilon = 1$ for $k = 1$ and $\epsilon = -1$ for $k = 3$. By equations (1) or (2) of Section 1, we know that the first column of the matrix $\begin{pmatrix} A \\ B \end{pmatrix}$ is unimodular, *i.e.* the ideal it generates is the whole ring:

$$\langle\, a_{11}, a_{21}, \cdots, a_{n1}, b_{11}, b_{21}, \cdots, b_{n1} \,\rangle = R.$$

Hence for $n = 1$ the lemma is clear. Suppose now that $n \geq 2$. By Thm. 2.1, there exist $\alpha_i \in R$, for $2 \leq i \leq n$ and $\beta_j \in R$, for $1 \leq j \leq n$ such that $a_{11} + \sum_{i=2}^{n} \alpha_i a_{i1} + \sum_{j=1}^{n} \beta_j b_{j1}$ is a unit in R. Let us rewrite this fact:

$$(a_{11} + \sum_{j=2}^{n} \beta_j\, b_{j1}) + \sum_{i=2}^{n} \alpha_i\, a_{i1} + \beta_1 b_{11} \quad \text{is a unit in } R\,. \tag{3}$$

Consider the ϵ-symmetric $n \times n$ matrix

$$E := \begin{pmatrix} 0 & \beta_2 & \cdots & \beta_n \\ \epsilon\,\beta_2 & & & \\ \vdots & & 0 & \\ \epsilon\,\beta_n & & & \end{pmatrix} = \epsilon\, E^t\,.$$

Note that the $(1,1)$-entry of the matrix $A+EB$ is $a_{11}+\sum_{j=2}^{n} \beta_j\, b_{j1}$. By Lemma 1.10, we know that $[\,n\,,\,A\,,\,B\,]^{(k)} = [\,n\,,\,A+EB\,,\,B\,]^{(k)}$, so we can replace A by $A+EB$ and assume, because of (3) above, that:

$$\langle\, a_{11}, a_{21}, \ldots, a_{n1}, b_{11} \,\rangle = R\,.$$

We proceed dually to get rid of $a_{21}, \ldots, a_{n1}$. Namely, by Thm. 2.1, there exist elements $\gamma_1, \ldots, \gamma_n \in R$ such that $\sum_{i=1}^{n} \gamma_i\, a_{i1} + b_{11}$ is a unit in R, that is:

$$\gamma_1\, a_{11} + (b_{11} + \sum_{i=2}^{n} \gamma_i\, a_{i1}) \quad \text{is a unit in } R\,. \tag{4}$$

Consider the ϵ-symmetric $n \times n$ matrix

$$F := \begin{pmatrix} 0 & \gamma_2 & \cdots & \gamma_n \\ \epsilon\,\gamma_2 & & & \\ \vdots & & 0 & \\ \epsilon\,\gamma_n & & & \end{pmatrix} = \epsilon\, F^t\,.$$

Note that the $(1,1)$-entry of the matrix $B+FA$ is $b_{11}+\sum_{i=2}^n \gamma_i\, a_{i1}$. By Lemma 1.6, we know that $[\,n\,,\,A\,,\,B\,]^{(k)} = [\,n\,,\,A\,,\,B+FA\,]^{(k)}$, so we can replace B by $B+FA$ and assume, because of (4) above, that $\langle\, a_{11}, b_{11}\,\rangle = R$. This was precisely the claim of the Lemma. ☐

In the case of W^1, we have the following improvement:

Lemma 2.5. *With the above hypotheses and notation, any class* $[\,n\,,\,A\,,\,B\,]^{(1)}$ *in* $\mathrm{W}^1(R)$ *is equal to a class of the form* $[\,n\,,\,\tilde{A}\,,\,\tilde{B}\,]^{(1)}$*, with same integer* n*, and with matrices* $\tilde{A} = (\tilde{a}_{ij})$ *and* $\tilde{B} = (\tilde{b}_{ij})$ *such that* $\tilde{a}_{11}$ *is a unit in* R*.*

Proof. We continue from Lemma 2.4, that is, we can consider a class $[\,n\,,\,A\,,\,B\,]^{(1)}$ for matrices $A = (a_{ij})$ and $B = (b_{ij})$ such that $\langle\, a_{11}, b_{11}\,\rangle = R$. By Thm. 2.1, there exists $r \in R$ such that $a_{11} + rb_{11}$ is a unit in R. Consider the symmetric $n \times n$ matrix

$$E := \begin{pmatrix} r & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{pmatrix} = E^t\,.$$

The matrix $A + EB$ has the (1, 1)-entry equal to $a_{11} + rb_{11}$ and so is a unit in R. We conclude via Lemma 1.10 (a). ☐

End of the proof of Theorem 2.3. The proof of $\mathrm{W}^1(R) = 0$ is now finished by induction on the integer n for the class $[\,n\,,\,\tilde{A}\,,\,\tilde{B}\,]^{(1)}$ as in Lemma 2.5. Indeed, for $n = 1$, such a matrix $\tilde{A}$ is invertible and Lemma 1.11 tells us that $[\,n\,,\,\tilde{A}\,,\,\tilde{B}\,]^{(1)} = 0$. For $n \geq 2$, we can reduce to $n-1$ by means of Lemma 1.12 and of Lemma 2.5 again. We now turn to the other case.

The case of W^3.

By Proposition 1.4, it is enough to establish the vanishing of any element of the form $[\,n\,,\,A\,,\,B\,]^{(3)} \in \mathrm{W}^3(R)$. We proceed by induction on n. By Lemma 2.4, we can assume further that $\langle\, a_{11}, b_{11}\,\rangle = R$.

Let $n = 1$. We have from Definition 1.2 that $a_{11} \cdot b_{11} = -b_{11} \cdot a_{11}$. Hence $a_{11} \cdot b_{11} = 0$ as R is commutative and contains $\frac{1}{2}$. For any $r \in R$, consider the principal open $D(r) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid r \notin \mathfrak{p}\}$ of $\mathrm{Spec}(R)$ defined by r. From $a_{11} \cdot b_{11} = 0$ we have $D(a_{11}) \cap D(b_{11}) = \varnothing$ and from $\langle\, a_{11}, b_{11}\,\rangle = R$ we have $D(a_{11}) \cup D(b_{11}) = \mathrm{Spec}(R)$. Since $\mathrm{Spec}(R)$ is connected, we deduce that $D(a_{11}) = \mathrm{Spec}(R)$ or $D(b_{11}) = \mathrm{Spec}(R)$, meaning that a_{11} or b_{11} is a unit in R. We conclude that $[\,1\,,\,a_{11}\,,\,b_{11}\,]^{(3)} = 0$ from Lemma 1.11.

Let now $n \geq 2$ and assume by induction that $[\,n-1\,,\,A'\,,\,B'\,]^{(3)} = 0$ for all $(n-1) \times (n-1)$ matrices A', B'. Consider a class $[\,n\,,\,A\,,\,B\,]^{(3)} \in \mathrm{W}^3(R)$ with $A = (a_{ij})$ and $B = (b_{ij})$ such that $\langle\, a_{11}, b_{11}\,\rangle = R$. By the Chinese Remainder Theorem, we can define an element of R by choosing its images under the various maps $R \to R/\mathfrak{m}$ for $\mathfrak{m} \in \mathrm{Max}(R)$. Let us choose an $r \in R$ such that

$$r \longmapsto \begin{cases} 0 \;\in R/\mathfrak{m} & \text{if } a_{11} \notin \mathfrak{m} \text{ or } a_{21} \notin \mathfrak{m} \\ 1 \;\in R/\mathfrak{m} & \text{if } a_{11} \in \mathfrak{m} \text{ and } a_{21} \in \mathfrak{m} \end{cases}$$

for every maximal ideal $\mathfrak{m} \in \operatorname{Max}(R)$. We claim that

$$\langle\, a_{11} - r \cdot b_{21} \;,\; a_{21} + r \cdot b_{11} \,\rangle \;=\; R \,. \tag{5}$$

To see this, one checks directly from the construction of r that there is no maximal ideal $\mathfrak{m}$ which contains this ideal $\langle\, a_{11} - r \cdot b_{21} \,,\, a_{21} + r \cdot b_{11} \,\rangle$. (Suppose the contrary and go modulo $\mathfrak{m}$ to find a contradiction, using that $1 \in \langle\, a_{11},\, b_{11} \,\rangle$.)

Using this element r, we define an $n \times n$ skew-symmetric matrix by

$$E := \begin{pmatrix} 0 & -r & 0 & \cdots & 0 \\ r & 0 & & & \\ 0 & & & & \\ \vdots & & & 0 & \\ 0 & & & & \end{pmatrix} = -E^t \,.$$

Observe that the $(1,1)$- and $(2,1)$-entries of $A + EB$ are respectively $a_{11} - r \cdot b_{21}$ and $a_{21} + r \cdot b_{11}$. Since by Lemma 1.10, $[\, n \,,\, A \,,\, B \,]^{(3)} = [\, n \,,\, A + EB \,,\, B \,]^{(3)}$ we can replace A by $A + EB$ and the choice of r, see (5) means that we can assume that

$$\langle\, a_{11}, a_{21} \,\rangle \;=\; R \,.$$

By Thm. 2.1, there exists $s \in R$ such that $a_{11} + s \cdot a_{21}$ is a unit in R. Consider the elementary matrix $C \in \mathrm{GL}_n(R)$ whose only non-zero entry outside the diagonal is s in the $(1,2)$-entry. Then the $(1,1)$-entry of $C \cdot A$ is our unit $a_{11} + s \cdot a_{21}$. Applying Lemma 1.7 to $U = \mathrm{id}$ and $V = C^{-1}$, we know that $[\, n \,,\, A \,,\, B \,]^{(3)} = [\, n \,,\, CA \,,\, V^t B \,]^{(3)}$ so we can assume that a_{11} is a unit. Using Lemma 1.12, we conclude by induction hypothesis that $[\, n \,,\, A \,,\, B \,]^{(3)} = [\, n - 1 \,,\, A' \,,\, B' \,]^{(3)} = 0$. □

3. Recalling cofinality and Tate cohomology

In this section we recall a 12-term periodic exact sequence due to Hornbostel and Schlichting relating the shifted Witt groups and some associated Tate cohomology groups. Let $(\mathcal{B}, \#, \varpi, \delta)$ be a triangulated $\mathbb{Z}[\frac{1}{2}]$-category with δ-duality. Let $\mathcal{A} \subset \mathcal{B}$ be a full triangulated subcategory invariant under the duality functor. Recall that $\mathcal{A}$ is said to be *cofinal* in $\mathcal{B}$ if every object of $\mathcal{B}$ is a direct summand of an object of $\mathcal{A}$. Let $K_0(\mathcal{A})$ and $K_0(\mathcal{B})$ be the 0-th K-theory groups. The duality $\#$ induces a $\mathbb{Z}/2$-action τ on $K_0(\mathcal{B})$ which coincides with the similar $\mathbb{Z}/2$-action on $K_0(\mathcal{A})$. Hence τ induces a $\mathbb{Z}/2$-action on $K_0(\mathcal{B})/K_0(\mathcal{A})$ which we denote by $\tilde{\tau}$.

Theorem 3.1 (Hornbostel-Schlichting [HS, App. A])**.** *Let $\mathcal{B}$ be a triangulated $\mathbb{Z}[\frac{1}{2}]$-category with δ-duality and $\mathcal{A}$ a full triangulated subcategory of $\mathcal{B}$ invariant under the duality and cofinal in $\mathcal{B}$. Then, there is a natural long exact sequence*

$$\cdots \to \mathrm{W}^n(\mathcal{A}) \to \mathrm{W}^n(\mathcal{B}) \to \hat{H}^n\big(\mathbb{Z}/2, K_0(\mathcal{B})/K_0(\mathcal{A})\big) \to \mathrm{W}^{n+1}(\mathcal{A}) \to \cdots$$

Let R be a ring (not necessarily commutative) with an involution σ and such that $\frac{1}{2} \in R$. We apply the above result with $\mathrm{K^b}(R\text{−proj})$ as $\mathcal{B}$ and $\mathrm{K^b}(R\text{−free})$ as $\mathcal{A}$. We have $K_0(\mathcal{B}) = K_0(R\text{−proj}) = K_0(R)$ and $K_0(\mathcal{A}) = K_0(R\text{−free}) = \mathbb{Z}$. We denote by $\widetilde{K_0}(R)$ the group $K_0(\mathcal{B})/K_0(\mathcal{A})$ which is the usual $\widetilde{K_0}(R) = \operatorname{coker}(\,\mathbb{Z} \to K_0(R))$. With this notation, the above theorem gives us the following 12-term periodic exact sequence :

$$\begin{array}{c} \mathrm{W}^0(R_\sigma\text{–free}) \longrightarrow \mathrm{W}^0(R_\sigma\text{–proj}) \longrightarrow \hat{H}^0(\mathbb{Z}/2,\widetilde{K_0}(R)) \longrightarrow \mathrm{W}^1(R_\sigma\text{–free}) \longrightarrow \mathrm{W}^1(R_\sigma\text{–proj}) \longrightarrow \hat{H}^1(\mathbb{Z}/2,\widetilde{K_0}(R)) \\ \longrightarrow \mathrm{W}^2(R_\sigma\text{–free}) \longrightarrow \mathrm{W}^2(R_\sigma\text{–proj}) \longrightarrow \hat{H}^0(\mathbb{Z}/2,\widetilde{K_0}(R)) \longrightarrow \mathrm{W}^3(R_\sigma\text{–free}) \longrightarrow \mathrm{W}^3(R_\sigma\text{–proj}) \longrightarrow \hat{H}^1(\mathbb{Z}/2,\widetilde{K_0}(R)) \longrightarrow \mathrm{W}^0(R_\sigma\text{–free}) \end{array} \tag{6}$$

Consider the natural exact sequence of complexes

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & K_0(R) & \xrightarrow{1-\tau} & K_0(R) & \xrightarrow{1+\tau} & K_0(R) & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \widetilde{K_0}(R) & \xrightarrow{1-\tilde{\tau}} & \widetilde{K_0}(R) & \xrightarrow{1+\tilde{\tau}} & \widetilde{K_0}(R) & \longrightarrow & \cdots \end{array} \tag{7}$$

The homology of the second and third rows is Tate cohomology, see [S, § VIII.4]. Hence (7) induces the following 5-term exact sequence :

$$\begin{array}{c} 0 \longrightarrow \hat{H}^1(\mathbb{Z}/2,K_0(R)) \longrightarrow \hat{H}^1(\mathbb{Z}/2,\widetilde{K_0}(R)) \\ \longrightarrow \mathbb{Z}/2 \longrightarrow \hat{H}^0(\mathbb{Z}/2,K_0(R)) \longrightarrow \hat{H}^0(\mathbb{Z}/2,\widetilde{K_0}(R)) \longrightarrow 0\,. \end{array} \tag{8}$$

Remark 3.2. We recall from [HS, App. A] the definition of the map

$$\alpha : \mathrm{W}^{2k}(R_\sigma\text{–proj}) \longrightarrow \hat{H}^0(\mathbb{Z}/2,\widetilde{K_0}(R))$$

which appears in the above sequence (6) for $k=0$ or 1. Consider a Witt class $[P,\phi] \in \mathrm{W}^{2k}(R_\sigma\text{–proj})$. By definition, we have an isomorphism $\phi : P \xrightarrow{\sim} P^{\#}[2k]$ and hence the class $[P] \in K_0(R)$ satisfies $\tau([P]) = [P]$. So this class $[P]$ defines an element in $\hat{H}^0(\mathbb{Z}/2,K_0(R))$. This assignment $(P,\phi) \mapsto [P]$ defines a map $\alpha' : \mathrm{W}^{2k}(R_\sigma\text{–proj}) \to \hat{H}^0(\mathbb{Z}/2,K_0(R))$. The map α is obtained by composition with the obvious map $q : \hat{H}^0(\mathbb{Z}/2,K_0(R)) \longrightarrow \hat{H}^0(\mathbb{Z}/2,\widetilde{K_0}(R))$:

$$\begin{array}{ccc} \mathrm{W}^{2k}(R_\sigma\text{–proj}) & \xrightarrow{\quad\alpha\quad} & \hat{H}^0(\mathbb{Z}/2,\widetilde{K_0}(R))\,. \\ & {}_{\alpha'}\searrow \quad \nearrow_{q} & \\ & \hat{H}^0(\mathbb{Z}/2,K_0(R)) & \end{array} \tag{9}$$

Let us also recall from [HS, App. A] the definition of the map

$$\beta : \hat{H}^0(\mathbb{Z}/2,\widetilde{K_0}(R)) \longrightarrow \mathrm{W}^{2k+1}(R_\sigma\text{–free})\,.$$

Take a class $[P] \in K_0(R)$ such that $\tilde{\tau}([P]) = [P]$ in $\widetilde{K_0}(R)$. Then the image in $\mathrm{W}^{2k+1}(R_\sigma\text{−free})$ is the class of the space over $P \oplus P^{\#}[2k+1]$ with the hyperbolic form. Note that it is really a hyperbolic space in $\mathrm{K}^{\mathrm{b}}(R\text{−proj})$ but not in $\mathrm{K}^{\mathrm{b}}(R\text{−free})$, since the lagrangian does not exist in this subcategory. So its Witt class only becomes zero in $\mathrm{W}^{2k+1}(R_\sigma\text{−proj})$, as predicted by the cofinality exact sequence (6).

4. Simple artinian rings

In this section we describe the Tate cohomology groups $\hat{H}^i(\mathbb{Z}/2, \widetilde{K_0}(S))$ for $i = 0, 1$, of a *simple artinian ring* S with an involution σ and such that $\frac{1}{2} \in S$. We also compute the shifted Witt groups of S with respect to the involution σ.

Let $S = \mathrm{M}_\ell(D)$, for a positive integer ℓ and a division algebra D. Then $K_0(S) \cong \mathbb{Z}$ and diagram (7) becomes:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \cdots \\ & & \downarrow{\scriptstyle \cdot\ell} & & \downarrow{\scriptstyle \cdot\ell} & & \downarrow{\scriptstyle \cdot\ell} & & \\ \cdots & \longrightarrow & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \xrightarrow{0} & \cdots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \cdots & \longrightarrow & \widetilde{K_0}(S) & \xrightarrow{1-\tilde{\tau}} & \widetilde{K_0}(S) & \xrightarrow{1+\tilde{\tau}} & \widetilde{K_0}(S) & \longrightarrow & \cdots \end{array} \tag{10}$$

So, we have $\hat{H}^0(\mathbb{Z}/2, K_0(S)) = \mathbb{Z}/2$ and $\hat{H}^1(\mathbb{Z}/2, K_0(S)) = 0$. Therefore, the Tate cohomology long exact sequence (8) takes the following shape:

$$0 \to \hat{H}^1(\mathbb{Z}/2, \widetilde{K_0}(S)) \to \mathbb{Z}/2 \to \mathbb{Z}/2 \to \hat{H}^0(\mathbb{Z}/2, \widetilde{K_0}(S)) \to 0 \tag{11}$$

where the map $\mathbb{Z}/2 \to \mathbb{Z}/2$ is given by multiplication by ℓ. Hence:

Proposition 4.1. *Let $S \simeq \mathrm{M}_\ell(D)$ be a simple artinian ring with involution, for a division algebra D. The Tate cohomology groups $\hat{H}^0(\mathbb{Z}/2, \widetilde{K_0}(S))$ and $\hat{H}^1(\mathbb{Z}/2, \widetilde{K_0}(S))$ are both zero for ℓ odd, and both isomorphic to $\mathbb{Z}/2$ for ℓ even.* □

The following theorem due to Ranicki holds for *semi-simple* rings as well. It is also a special case of [BW, Prop. 5.2].

Theorem 4.2 (Ranicki [R])**.** *Let (S, σ) be a semi-simple ring with involution. Then*

$$\mathrm{W}^{2k+1}(S_\sigma\text{−proj}) = 0\,.$$

Corollary 4.3. *Let (S, σ) be a semi-simple ring with involution. Then*

$$\mathrm{W}^{2k+1}(S_\sigma\text{−free}) = \operatorname{coker}\big(\mathrm{W}^{2k}(S_\sigma\text{−proj}) \xrightarrow{\alpha} \hat{H}^0(\mathbb{Z}/2, \widetilde{K_0}(S))\big)\,.$$

Proof. Immediate from Thm. 4.2 and from the cofinality exact sequence (6). □

We now want to explicitly compute the groups $\mathrm{W}^1(S_\sigma\text{−free})$ and $\mathrm{W}^3(S_\sigma\text{−free})$.

For any left S-module P, recall that $\pi_P : P \to P^{**}$ is the natural isomorphism of S-modules defined by $(\pi_P(x))(f) = \sigma(f(x))$ for all $x \in P$ and all $f \in P^*$.

Notation 4.4. Let P, Q be left S-modules. For a homomorphism $f : P \to Q^*$, it is convenient to abbreviate by f^t the *transpose* of f. This morphism $f^t : Q \to P^*$ is defined by

$$f^t := f^* \circ \pi_Q\,.$$

It is not difficult to see that $f^{t\,t} = f$. In particular, when $Q = P$, we obtain a map $f \mapsto f^t$ which is an automorphism ${}^t : \mathrm{Hom}_S(P, P^*) \to \mathrm{Hom}_S(P, P^*)\,.$

Let N be a simple left module over S and N^* be its dual considered as a left S-module via the involution σ on S. Note that N being simple forces N^* to be simple as well. Since S is a simple artinian ring, these two modules are isomorphic.

Proposition 4.5. *Let (S,σ) be a simple artinian ring with involution and N be a simple left S-module. Then, there exists an $\epsilon \in \{\pm 1\}$ and an isomorphism $f : N \xrightarrow{\sim} N^*$ such that $f = \epsilon f^t$.*

Proof. Since $\frac{1}{2} \in S$, any $f \in \mathrm{Hom}_S(N, N^*)$ can be written as $f = \frac{f+f^t}{2} + \frac{f-f^t}{2}$. There exists an isomorphism $f \in \mathrm{Hom}_S(N, N^*)$, so in particular $f \neq 0$. This implies that one of $\frac{f+f^t}{2}$ or $\frac{f-f^t}{2}$ is non-zero and is in turn an isomorphism. □

Since N is a simple left S-module, our division algebra D such that $S \simeq \mathrm{M}_\ell(D)$ can be taken to be $\mathrm{End}_S(N)$.

Definition 4.6. Choose a simple S-module N and choose $f : N \xrightarrow{\sim} N^*$ as in Proposition 4.5. Set $D := \mathrm{End}_S(N)$ and define on it an involution $\sigma_f : D \to D$ by

$$\sigma_f(\alpha) = f^{-1} \circ \alpha^* \circ f$$

for all $\alpha \in \mathrm{End}_S(N) = D$. It is an involution because $f^t = \pm f$.

Lemma 4.7. *With this notation, exactly one of the two following cases can occur:*

(a) *For all simple S-modules N and for all isomorphisms $f : N \xrightarrow{\sim} N^*$ with $f = \pm f^t$, we have $\sigma_f = \mathrm{id}_D$ on the division algebra $D = \mathrm{End}_S(N)$.*

(b) *For all simple S-modules N and for all isomorphisms $f : N \xrightarrow{\sim} N^*$ with $f = \pm f^t$, we have $\sigma_f \neq \mathrm{id}_D$ on the division algebra $D = \mathrm{End}_S(N)$.*

Moreover, in case (a), D must be a field and for all isomorphisms $S \simeq \mathrm{M}_\ell(D)$, the involution induced by σ on $\mathrm{M}_\ell(D)$ restricts to the identity on D.

Proof. For $i = 1, 2$, let $f_i : N \xrightarrow{\sim} N^*$ be such that $f_i = \epsilon_i f_i^t$. Consider the unit $u := f_1^{-1} \circ f_2 \in D$. For every $\alpha \in D$ we have $\sigma_{f_2}(\alpha) = u^{-1} \cdot \sigma_{f_1}(\alpha) \cdot u$. If now the involution σ_{f_1} is the identity on D then D is commutative and hence $\sigma_{f_2} = \sigma_{f_1}$ by the above equation, that is, $\sigma_{f_2} = \mathrm{id}$ as claimed in (a). This shows that $\sigma_f = \mathrm{id}$ independently of the choice of f. To see independence of this property with respect to the simple module, first note that two such simple modules are isomorphic, then transport one f from one module to the other and finally use what we just proved. This establishes the dichotomy between (a) and (b).

For the "moreover part", it therefore suffices to prove the following claim:

Consider a field D and some involution σ_0 on $S_0 := \mathrm{M}_\ell(D)$. Let $N_0 := D^\ell$, so that $D \cong \mathrm{Hom}_{S_0}(N_0, N_0)$, via the identification sending $d \in D$ to multiplication by d. Suppose that for some (and hence any) isomorphism $f : N_0 \xrightarrow{\sim} N_0^*$ as in 4.5 we have $\sigma_f = \mathrm{id}_D$. Then, we claim that σ_0 is also the identity on $D \hookrightarrow \mathrm{M}_\ell(D)$.

To see this, first note that D being the center of S_0, we must have $\sigma_0(D) \subset D$. Consider an element $\alpha \in D = \mathrm{Hom}_{S_0}(N_0, N_0)$ and its dual $\alpha^* \in \mathrm{Hom}_{S_0}(N_0^*, N_0^*)$. For any $h \in N_0^*$ and any $y \in N_0$, we have by definition, $\alpha^*(h)(y) = h(\alpha(y)) = h(\alpha \cdot y) = \alpha \cdot h(y)$. Since $\alpha \in D$ belongs to the center of S_0, we have further that $\alpha \cdot h(y) = h(y) \cdot \alpha = (\sigma_0(\alpha) \cdot h)(y)$ by definition of the action of S_0 on $h \in N_0^*$. Since the above holds for any $y \in N_0$, we have proved for all $h \in N_0^*$ that:

$$\alpha^*(h) = \sigma_0(\alpha) \cdot h \tag{12}$$

We apply this equation to $h = f(z)$ for $z \in N_0$ where f is our S_0-isomorphism $f : N_0 \xrightarrow{\sim} N_0^*$. Since by assumption $\alpha = \sigma_f(\alpha)$, we have:

$$\alpha \cdot z = \sigma_f(\alpha)(z) \overset{\text{def}}{=} f^{-1}(\alpha^*(f(z))) \overset{(12)}{=} f^{-1}(\sigma_0(\alpha) \cdot f(z)) = \sigma_0(\alpha) \cdot z$$

for any $z \in N_0$. Hence $\alpha = \sigma_0(\alpha)$, for all $\alpha \in D$, which is the claim. □

We recall the following well-known result which is a consequence of the Skolem-Noether theorem, see for instance [KMRT, 2.19].

Theorem 4.8. *Let $\Bbbk$ be a field and ℓ be a positive integer. For any involution σ on $\mathrm{M}_\ell(\Bbbk)$ such that σ is identity on $\Bbbk$, there exists $u \in \mathrm{GL}_\ell(\Bbbk)$ such that $u^t = \pm u$ and $\sigma(a) = u^{-1} \cdot a^t \cdot u$ for $a \in \mathrm{M}_\ell(\Bbbk)$. Furthermore, such a matrix u is uniquely determined up to multiplication by invertible elements of $\Bbbk$. (In particular, symmetry or skew-symmetry of u does not depend on the choice of u.)*

Definition 4.9. We summarize the various cases as follows.

(a) Suppose that there exists a field $\Bbbk$ and an isomorphism of rings with involution $S \simeq \mathrm{M}_\ell(\Bbbk)$ where the latter is equipped with the involution $a \mapsto u^{-1} \cdot a^t \cdot u$ for some $u = \pm u^t \in \mathrm{GL}_\ell(\Bbbk)$. Then we say that σ is *central*. Furthermore, central involutions are of the following two types :

(a.1) the involution σ on S is *split-orthogonal* if $u = u^t$;

(a.2) the involution σ on S is *split-symplectic* if $u = -u^t$.

By Skolem-Noether, this does not depend on the choice of the isomorphism $S \simeq \mathrm{M}_\ell(\Bbbk)$, if such an isomorphism exists. Since $\frac{1}{2} \in S$, both cases (a.1) and (a.2) cannot occur simultaneously.

(b) We say that an involution σ on S is *non-central* if it is not central.

Remark 4.10. By definition, an involution σ on a simple artinian ring $S \simeq \mathrm{M}_\ell(D)$ is exactly one of split-orthogonal, split-symplectic or non-central. When σ is split-orthogonal or split-symplectic, D is a field. Note that the converse need not be true, as the example of $S = \mathbb{C}$ with complex conjugation immediately shows. In Lemma 4.7, the involution is split-orthogonal or split-symplectic exactly if we are in case (a) and it is non-central exactly if we are in case (b).

Lemma 4.11. *Let σ be a non-central involution on S and let N be a simple S-module. Then, for both possible $\epsilon \in \{\pm 1\}$, there exists an isomorphism $f : N \xrightarrow{\sim} N^*$ such that $f = \epsilon f^t$.*

Proof. Note that an involution satisfying (a) of Lemma 4.7 is central. So our non-central involution must satisfy case (b). Consider any isomorphism $f : N \xrightarrow{\sim} N^*$ such that $f = \pm f^t$, see Prop. 4.5. Then $\sigma_f \neq \mathrm{id}_D$ where $D = \mathrm{End}_S(N)$ by the above discussion. If $f = \epsilon f^t$ we are done. Assume that $f = -\epsilon f^t$. Then, using $\frac{1}{2}$, since $\sigma_f \neq \mathrm{id}$, there exists a non-zero $\alpha \in D$ with $\sigma_f(\alpha) = -\alpha$. Set $g := \alpha^* \circ f : N \xrightarrow{\sim} N^*$. Then $g^t = f^t \, \alpha = -\epsilon \, f \, \alpha = \epsilon \, f \, \sigma_f(\alpha) = \epsilon \, f \, f^{-1} \, \alpha^* \, f = \epsilon \, g$. So g is an ϵ-symmetric isomorphism $N \xrightarrow{\sim} N^*$ as wanted. □

Lemma 4.12. *Let (S, σ) be a simple artinian ring with involution and let $k = 0$ or 1. Then the homomorphism $\alpha' : \mathrm{W}^{2k}(S_\sigma\text{–proj}) \to \hat{H}^0(\mathbb{Z}/2, K_0(S)) = \mathbb{Z}/2$ of diagram (9) is surjective in any of the following cases :*

(a) *if the involution is non-central;*

(b) *if $k = 0$ and the involution is split-orthogonal;*

(c) *if $k = 1$ and the involution is split-symplectic.*

In particular, the homomorphism $\alpha : \mathrm{W}^{2k}(S_\sigma\text{–proj}) \to \hat{H}^0(\mathbb{Z}/2, \widetilde{K_0}(S))$ is also surjective in these cases.

Proof. By the description of $\alpha' : \mathrm{W}^{2k}(S_\sigma\text{–proj}) \to \hat{H}^0(\mathbb{Z}/2, K_0(S))$ as essentially being $[P, \varphi] \mapsto [P]$, see 3.2, and since $K_0(S) \simeq \mathbb{Z}$ is generated by the class of the simple S-module N, to prove surjectivity of α' it suffices to prove that there exists an ϵ-symmetric space (N, φ) over N for $\epsilon = (-1)^k$. This is done as follows.

Part (a). By Lemma 4.11 there exist two isomorphisms $f_k : N \to N^*$ for $k = 0, 1$ such that ${f_k}^t = (-1)^k \, f_k$, and we can simply choose $\varphi := f_k$.

Parts (b) and (c). Let $S = \mathrm{M}_\ell(\Bbbk)$ for some field $\Bbbk$ and some positive integer ℓ. Then $N := \Bbbk^\ell$ is the simple S-module. (Elements of N are thought of as column-vectors in the matrix notation below.) By Definition 4.9, there exists $u \in \mathrm{M}_\ell(\Bbbk)$ with $u = \epsilon\, u^t$ for $\epsilon = (-1)^k$ such that $\sigma(x) = u^{-1} x^t u$ for every $x \in \mathrm{M}_\ell(\Bbbk)$. Define the homomorphism $\varphi : N \to N^*$ by $w \mapsto (v \mapsto v w^t u)$. It is easy to check that φ is well-defined, non-zero and S-linear. Under φ^t, we have $w \mapsto \big(v \mapsto \varphi^*(\pi_N(w))(v)\big)$. For any $v, w \in N$, we have $\varphi^*(\pi_N(w))(v) = \pi_N(w)(\varphi(v)) = \sigma(\varphi(v)(w)) = \sigma(w v^t u) = u^{-1} u^t v w^t u = \epsilon\, v w^t u = \epsilon\, \varphi(w)(v)$. Hence $\varphi^t = \epsilon\, \varphi$ is a $(-1)^k$-symmetric isomorphism $N \overset{\sim}{\to} N^*$ as wanted.

The final claim follows from $\alpha = q \circ \alpha'$, see (9), and from the surjectivity of $q : \hat{H}^0(\mathbb{Z}/2, K_0(S)) \to \hat{H}^0(\mathbb{Z}/2, \widetilde{K_0}(S))$, see (8). □

Lemma 4.13. *Let $\Bbbk$ be a field and σ be an involution on $S := \mathrm{M}_\ell(\Bbbk)$.*

(a) *If σ is split-orthogonal, then* $\mathrm{W}^2(S_\sigma\text{−proj}) = 0$.
(b) *If σ is split-symplectic, then* $\mathrm{W}^0(S_\sigma\text{−proj}) = 0$.

Proof. This is a result about classical Witt groups and we only recall the proof for the reader's convenience. Let $\sigma(-) = u^{-1} \cdot (-)^t \cdot u$ for $u \in S$ a unit such that $u = \epsilon \cdot u^t$ with $\epsilon = \pm 1$, see 4.8. Denote by $t : S \to S$ the involution given by matrix transposition. Then by "scaling", see [Kn, Rem I.5.8.2, p. 28], for $\delta = \pm 1$, we have

$$\mathrm{W}^{\delta}_{\mathrm{us}}(S, \sigma) \overset{\sim}{\to} \mathrm{W}^{\epsilon \cdot \delta}_{\mathrm{us}}(S, t).$$

On the other hand, by Morita equivalence, see [Kn, Thm. I.9.3.5, p. 56], we have

$$\mathrm{W}^{-}_{\mathrm{us}}(S, t) \overset{\sim}{\to} \mathrm{W}^{-}_{\mathrm{us}}(\Bbbk) = 0\,.$$

Hence for $\epsilon = 1$, that is, if σ is split-orthogonal on S, we have for $\delta = -1$

$$\mathrm{W}^2(S_\sigma\text{−proj}) = \mathrm{W}^{-}_{\mathrm{us}}(S, \sigma) \overset{\sim}{\to} \mathrm{W}^{-}_{\mathrm{us}}(S, t) = 0$$

and for $\epsilon = -1$, that is, if σ is split-symplectic on S, we have for $\delta = +1$

$$\mathrm{W}^0(S_\sigma\text{−proj}) = \mathrm{W}^{+}_{\mathrm{us}}(S, \sigma) \overset{\sim}{\to} \mathrm{W}^{-}_{\mathrm{us}}(S, t) = 0\,.$$

This proves the Lemma. □

Theorem 4.14. *Let $S = \mathrm{M}_\ell(D)$ be a simple artinian ring with an involution σ. Then*

(a) *If σ is split-orthogonal,* $\mathrm{W}^1(S_\sigma\text{−free}) = 0$ *whereas*

$$\mathrm{W}^3(S_\sigma\text{−free}) \simeq \begin{cases} 0 & \text{if } \ell \text{ is odd} \\ \mathbb{Z}/2 & \text{if } \ell \text{ is even.} \end{cases}$$

(b) *If σ is split-symplectic,* $\mathrm{W}^1(S_\sigma\text{−free}) \cong \mathbb{Z}/2$ *and* $\mathrm{W}^3(S_\sigma\text{−free}) = 0$.

(c) *If σ is non-central,* $\mathrm{W}^1(S_\sigma\text{−free}) = 0$ *and* $\mathrm{W}^3(S_\sigma\text{−free}) = 0$.

For "split-orthogonal", "split-symplectic" and "non-central" see Definition 4.9.

Proof. By 4.3, we know that for $k = 0$ or 1, the Witt group $\mathrm{W}^{2k+1}(S_\sigma\text{−free})$ that we want to determine is the cokernel of the following map:

$$\alpha : \mathrm{W}^{2k}(S_\sigma\text{−proj}) \longrightarrow \hat{H}^0\big(\mathbb{Z}/2, \widetilde{K_0}(S)\big)\,, \tag{13}$$

where, by Proposition 4.1, the right-hand group $\hat{H}^0(\mathbb{Z}/2, \widetilde{K_0}(S))$ is zero or $\mathbb{Z}/2$. By Lemma 4.12, we know that this map is surjective in some cases and this immediately gives us (c) as well as the vanishing of W^1 in (a) and of W^3 in (b). We are left with $\mathrm{W}^3(S_\sigma\text{−free})$ when σ is split-orthogonal and $\mathrm{W}^1(S_\sigma\text{−free})$ when σ is split-symplectic. In these cases, Lemma 4.13 tells us that the left-hand group of (13) is zero and therefore the cokernel of α is simply $\hat{H}^0\big(\mathbb{Z}/2, \widetilde{K_0}(S)\big)$. Then, the result follows from Proposition 4.1 : this group is zero for ℓ odd and $\mathbb{Z}/2$ for ℓ even. Note that in case (b) a split-symplectic involution can only exist on $\mathrm{M}_\ell(D)$ for ℓ even. □

5. Semi-simple rings

In this section we start by computing the Tate cohomology groups $\hat{H}^i(\mathbb{Z}/2, \widetilde{K_0}(S))$ for $i = 0, 1$, of a *semi-simple ring* S with an involution σ and such that $\frac{1}{2} \in S$. We then describe the shifted Witt groups of S with respect to the involution σ.

Let (S, σ) be a semi-simple ring with an involution. By the Wedderburn-Artin Theorem (see for instance [L, 3.5]), $S \xrightarrow{\sim} \mathrm{M}_{\ell_1}(D_1) \times \cdots \times \mathrm{M}_{\ell_r}(D_r)$. We denote by σ again the involution induced by the above isomorphism on $\mathrm{M}_{\ell_1}(D_1) \times \cdots \times \mathrm{M}_{\ell_r}(D_r)$. We consider the effect of σ on the simple artinian ring $\mathrm{M}_{\ell_i}(D_i)$ for each i, $1 \le i \le r$. We have the following two possibilities :

(a) $\sigma(\mathrm{M}_{\ell_i}(D_i)) \subset \mathrm{M}_{\ell_i}(D_i)$. In this case σ induces an involution on $\mathrm{M}_{\ell_i}(D_i)$.
(b) $\sigma(\mathrm{M}_{\ell_i}(D_i)) \not\subset \mathrm{M}_{\ell_i}(D_i)$. In this case, as $\sigma(\mathrm{M}_{\ell_i}(D_i))$ is a simple submodule of $\mathrm{M}_{\ell_1}(D_1) \times \cdots \times \mathrm{M}_{\ell_r}(D_r)$, there exists j, $1 \le j \le r$ such that $\sigma(\mathrm{M}_{\ell_i}(D_i)) = \mathrm{M}_{\ell_j}(D_j)$, see [L, 3.8]. Hence σ acts as the switch involution on $\mathrm{M}_{\ell_i}(D_i) \times \sigma(\mathrm{M}_{\ell_i}(D_i)) \xrightarrow{\sim} \mathrm{M}_{\ell_i}(D_i) \times (\mathrm{M}_{\ell_i}(D_i))^{op}$.

So we have the following :

Notation 5.1. Any semi-simple ring with involution decomposes as

$$S = \prod_{i=1}^{n} A_i \quad \times \quad \prod_{j=1}^{m} B_j \times B_j^{op}$$

where, for $1 \le i \le n$ and $1 \le j \le m$, A_i and B_j are simple artinian rings such that σ maps A_i to itself and is the switch involution on $B_j \times B_j^{op}$. The factors $B_j \times B_j^{op}$ will turn out to play no significant role in the sequel. For $1 \le i \le n$, we write

$$A_i = \mathrm{M}_{\ell_i}(D_i)$$

for integers $\ell_i \ge 1$ and division algebras D_i.

Definition 5.2. Let S and σ be as above. We respectively denote by $\mathbf{so}(S)$, $\mathbf{ss}(S)$ and $\mathbf{nc}(S)$ the number of indices i, $1 \le i \le n$ such that σ is a split-orthogonal, a split-symplectic or a non-central involution on A_i as defined in 4.9.

Remark 5.3. We start by describing diagram (7) in our situation. Since $K_0(S) \cong \mathbb{Z}^n \oplus \mathbb{Z}^{2m}$, this diagram becomes :

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} & & \\
\cdots & \longrightarrow & \mathbb{Z}^n \oplus \mathbb{Z}^{2m} & \xrightarrow{1-\tau} & \mathbb{Z}^n \oplus \mathbb{Z}^{2m} & \xrightarrow{1+\tau} & \mathbb{Z}^n \oplus \mathbb{Z}^{2m} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & \widetilde{K_0}(S) & \xrightarrow{1-\tilde{\tau}} & \widetilde{K_0}(S) & \xrightarrow{1+\tilde{\tau}} & \widetilde{K_0}(S) & \longrightarrow & \cdots
\end{array}$$

where we can describe τ and ϕ explicitly as follows. For each $1 \le j \le m$, write the simple artinian rings B_j as $B_j = \mathrm{M}_{\ell'_j}(E_j)$ for division algebras E_j. Then:

$$\tau = \begin{pmatrix} \mathrm{Id}_n & 0 \\ 0 & J_m \end{pmatrix} \text{ where } J_m = \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & 0 & \\ & & \ddots & & \\ & 0 & & 0 & 1 \\ & & & 1 & 0 \end{pmatrix} \in \mathrm{M}_{2m}(\mathbb{Z}) \text{ and } \phi = \begin{pmatrix} \ell_1 \\ \vdots \\ \ell_n \\ \ell'_1 \\ \ell'_1 \\ \vdots \\ \ell'_m \\ \ell'_m \end{pmatrix}.$$

Using this, it is easy to see that $\hat{H}^0(\mathbb{Z}/2, K_0(S)) = (\mathbb{Z}/2)^n$ and $\hat{H}^1(\mathbb{Z}/2, K_0(S)) = 0$. So, the factors $B_j \times B_j^{op}$ do not contribute to these Tate cohomology groups. The above diagram also induces a long exact cohomology sequence as in (8):

$$0 \to \hat{H}^1(\mathbb{Z}/2, \widetilde{K_0}(S)) \to \mathbb{Z}/2 \to (\mathbb{Z}/2)^n \to \hat{H}^0(\mathbb{Z}/2, \widetilde{K_0}(S)) \to 0 \tag{14}$$

where the map $\mathbb{Z}/2 \to (\mathbb{Z}/2)^n$ sends 1 to the vector $(\ell_1, \dots, \ell_n)$. Hence we have $\hat{H}^0(\mathbb{Z}/2, \widetilde{K_0}(S)) = (\mathbb{Z}/2)^{n-1}$ or $(\mathbb{Z}/2)^n$ and $\hat{H}^1(\mathbb{Z}/2, \widetilde{K_0}(S)) = 0$ or $\mathbb{Z}/2$ depending on the existence of an odd ℓ_i. So, we have proved:

Proposition 5.4. *Let S be a semi-simple ring with an involution σ, written as $S = \prod_1^n A_i \times \prod_1^m (B_j \times B_j^{op})$ with $A_i = \mathrm{M}_{\ell_i}(D_i)$ as in 5.1 above. Then the Tate cohomology groups are*

$$\hat{H}^0(\mathbb{Z}/2, \widetilde{K_0}(S)) = \frac{(\mathbb{Z}/2)^n}{(\ell_1, \dots, \ell_n)} = \begin{cases} (\mathbb{Z}/2)^{n-1} & \text{if } \ell_i \text{ is odd for some } i \\ (\mathbb{Z}/2)^n & \text{if } \ell_i \text{ is even for all } i \end{cases}$$

and

$$\hat{H}^1(\mathbb{Z}/2, \widetilde{K_0}(S)) = \begin{cases} 0 & \text{if } \ell_i \text{ is odd for some } i \\ \mathbb{Z}/2 & \text{if } \ell_i \text{ is even for all } i\,. \end{cases}$$

□

Remark 5.5. Regarding the Witt groups of projective modules, recall that the odd-indexed ones vanish by Thm. 4.2 and that all of them vanish for the switch involutions, that is, for $1 \le j \le m$, by 1.13, the group $\mathrm{W}^*((B_j \times B_j)_\sigma\text{–proj})$ is zero. Hence decomposition 5.1 gives (for $*$ even) $\mathrm{W}^*(S_\sigma\text{–proj}) = \bigoplus_{i=1}^n \mathrm{W}^*((A_i)_\sigma\text{–proj})$. We now want to collect some results on the shifted Witt groups of free modules over (S, σ).

Theorem 5.6. *Let (S, σ) be a semi-simple ring with involution such that $\frac{1}{2} \in S$. Using the decomposition 5.1 of S as $S = \prod_1^n A_i \times \prod_1^m (B_j \times B_j^{op})$ with $A_i = \mathrm{M}_{\ell_i}(D_i)$ and the notation for the numbers $\mathbf{so}(S)$ and $\mathbf{ss}(S)$ of split-orthogonal and split-symplectic involutions from 5.2, we have*

$$\mathrm{W}^1(S_\sigma\text{–free}) \;\simeq\; (\mathbb{Z}/2)^{\mathbf{ss}(S)}$$

whereas

$$\mathrm{W}^3(S_\sigma\text{–free}) \;\simeq\; \begin{cases} (\mathbb{Z}/2)^{\mathbf{so}(S)} & \text{if } \ell_i \text{ is even whenever } \sigma \text{ is split-orthogonal on } A_i \\ (\mathbb{Z}/2)^{\mathbf{so}(S)-1} & \text{if } \exists\, i \text{ with } \ell_i \text{ odd and } \sigma \text{ split-orthogonal on } A_i\,. \end{cases}$$

Proof. From Cor. 4.3, the group $\mathrm{W}^{2k+1}(S_\sigma\text{−free})$ is the cokernel of the homomorphism $\alpha : \mathrm{W}^{2k}(S_\sigma\text{−proj}) \to \hat{H}^0\big(\mathbb{Z}/2, \widetilde{K_0}(S)\big)$. Let us contemplate this in the following commutative diagram, see (9) for the commutative triangle on the left :

$$\begin{array}{ccccc} \mathrm{W}^{2k}(S_\sigma\text{−proj}) & \xrightarrow{\quad\alpha\quad} & \hat{H}^0\big(\mathbb{Z}/2, \widetilde{K_0}(S)\big) & \overset{\beta}{\twoheadrightarrow} & \mathrm{W}^{2k+1}(S_\sigma\text{−free})\,. \\ & {\scriptstyle\alpha'}\searrow & \nearrow\!\!\!\!\nearrow{\scriptstyle q} & & \\ & (\mathbb{Z}/2)^n \simeq \hat{H}^0\big(\mathbb{Z}/2, K_0(S)\big) & & & \end{array} \tag{15}$$

The isomorphism $(\mathbb{Z}/2)^n \simeq \hat{H}^0\big(\mathbb{Z}/2, K_0(S)\big)$ is established in Rem. 5.3 and it is also shown in (14) that q is surjective with kernel generated by the element $(\ell_1, \ldots, \ell_n) \in (\mathbb{Z}/2)^n$. It follows that $\mathrm{W}^{2k+1}(S_\sigma\text{−free})$ is a quotient of $(\mathbb{Z}/2)^n$, more precisely :

$$\mathrm{W}^{2k+1}(S_\sigma\text{−free}) \simeq (\mathbb{Z}/2)^n \,/\, M^k$$

where $M^k \subset (\mathbb{Z}/2)^n$ is the subspace generated by $(\ell_1, \ldots, \ell_n)$ and by the image $\mathrm{im}(\alpha')$ of the map α'. Hence, killing $\mathrm{im}(\alpha')$ first and then $(\ell_1, \ldots, \ell_n)$, we have :

$$\mathrm{W}^{2k+1}(S_\sigma\text{−free}) \simeq \mathrm{coker}(\alpha') \,/\, \overline{(\ell_1, \ldots, \ell_n)} \tag{16}$$

where $\overline{(\ell_1, \ldots, \ell_n)}$ is the image of $(\ell_1, \ldots, \ell_n)$ via the surjection $(\mathbb{Z}/2)^n \to \mathrm{coker}(\alpha')$. Now, this $\mathrm{coker}(\alpha')$ is easy to describe from the following commutative diagram obtained by projecting S to the product $\prod_{i=1}^n A_i$:

$$\begin{array}{ccccc} \mathrm{W}^{2k}(S_\sigma\text{−proj}) & \xrightarrow{\quad\alpha'\quad} & \hat{H}^0\big(\mathbb{Z}/2, K_0(S)\big) & \xleftarrow{\quad\simeq\quad} & (\mathbb{Z}/2)^n \\ \downarrow{\scriptstyle\simeq} & & \downarrow & & \| \\ \bigoplus_{i=1}^{n} \mathrm{W}^{2k}((A_i)_\sigma\text{−proj}) & \xrightarrow[\oplus\,\alpha'_i]{} & \bigoplus_{i=1}^{n} \hat{H}^0\big(\mathbb{Z}/2, K_0(A_i)\big) & \xleftarrow{\quad\simeq\quad} & (\mathbb{Z}/2)^n \end{array}$$

where the notation α'_i of course means : α' for the ring A_i. This diagram commutes by naturality, both of the construction of α' in 3.2 and of the computation of $\hat{H}^0\big(\mathbb{Z}/2, K_0(-)\big)$ in 5.3. The isomorphism on the left comes from Rem. 5.5. So, the image $\mathrm{im}(\alpha')$ is the sum $\oplus_{i=1}^n \mathrm{im}(\alpha'_i)$, that is, it sits "diagonally" in $(\mathbb{Z}/2)^n$. Now, the i^{th} map $\alpha'_i : \mathrm{W}^{2k}((A_i)_\sigma\text{−proj}) \to \hat{H}^0\big(\mathbb{Z}/2, K_0(A_i)\big)$ is zero when σ is split-orthogonal on A_i and $k = 1$, or, when σ is split-symplectic on A_i and $k = 0$ (Lemma 4.13) ; the map α'_i is surjective in all other cases (Lemma 4.12). In other words,

$$\mathrm{coker}(\alpha') \simeq \begin{cases} (\mathbb{Z}/2)^{\mathbf{ss}(S)} & \text{for } k = 0 \\ (\mathbb{Z}/2)^{\mathbf{so}(S)} & \text{for } k = 1 \end{cases}$$

and the vector $(\ell_1, \ldots, \ell_n) \in (\mathbb{Z}/2)^n$ is sent to its obvious image in $\mathrm{coker}(\alpha')$ obtained by keeping only those ℓ_i for the indices $1 \le i \le n$ such that σ is either split-symplectic or split-orthogonal on A_i for $k = 0$ or 1 respectively. But in the split-symplectic cases, ℓ_i is necessarily even. The result follows from this and (16). ☐

Corollary 5.7. *With notation as above, suppose furthermore that S is commutative. Then*

$$\mathrm{W}^1(S_\sigma\text{−free}) \;=\; 0 \qquad \textit{and} \qquad \mathrm{W}^3(S_\sigma\text{−free}) = \begin{cases} 0 & \textit{if } \mathbf{so}(S) = 0 \\ (\mathbb{Z}/2)^{\mathbf{so}(S)-1} & \textit{if } \mathbf{so}(S) \ge 1\,. \end{cases}$$

where $\mathbf{so}(S)$ reduces here to the number of simple factors of S on which the involution is trivial.

Proof. Since S is commutative, so are its simple factors $A_i = \mathrm{M}_{\ell_i}(D_i)$ and so $\ell_i = 1$ for each $1 \le i \le n$. In particular, there do not exist split-symplectic involutions: $\mathbf{ss}(S) = 0$ and split-orthogonal involutions are trivial. Hence the result by Thm. 5.6. □

6. Commutative case: maximal ideals detected by W^3

We import the following result from L-theory:

Theorem 6.1 (Davis-Ranicki [DR, Cor. 5.2 (ii)]). *Let (R, σ) be a semi-local ring with involution. Let $\bar{\sigma}$ denote the involution on $\bar{R} = R/\mathrm{Rad}(R)$ induced by σ. Then the natural homomorphism*

$$\mathrm{W}^{2k+1}(R_\sigma\text{–free}) \longrightarrow \mathrm{W}^{2k+1}(\bar{R}_{\bar{\sigma}}\text{–free})$$

is injective.

Using our computations of Section 5, we deduce from the above the following result in the commutative case:

Theorem 6.2. *Let R be a commutative semi-local ring with an involution. Let as before $\mathbf{so}(\bar{R})$ be the number of simple factors of $\bar{R} = R/\mathrm{Rad}(R)$ on which the involution is trivial, see 5.2. Then*

(a) $\mathrm{W}^1(R_\sigma\text{–proj}) = 0$.
(b) *If* $\mathbf{so}(\bar{R}) \le 1$ *then* $\mathrm{W}^3(R_\sigma\text{–proj}) = 0$ *as well.*
(c) *If* $\mathbf{so}(\bar{R}) > 1$ *then* $\mathrm{W}^3(R_\sigma\text{–proj})$ *is a finitely generated* $\mathbb{Z}/2$*-module of rank at most* $\mathbf{so}(\bar{R}) - 1$.

In particular, the rank of $\mathrm{W}^3(R_\sigma\text{–proj})$ *is strictly less than* $\#\,\mathrm{Max}(R)$*, the number of maximal ideals in R.*

Proof. We first prove the theorem for R a commutative semi-local ring with $\mathrm{Spec}(R)$ *connected*. In this case, all finitely generated projective modules are free. Hence $\hat{H}^0(\mathbb{Z}/2, \widetilde{K_0}(R)) = \hat{H}^1(\mathbb{Z}/2, \widetilde{K_0}(R)) = 0$ and $\mathrm{W}^{2k+1}(R_\sigma\text{–free}) = \mathrm{W}^{2k+1}(R_\sigma\text{–proj})$. So, by Theorem 6.1, it is enough to consider the image of $\mathrm{W}^{2k+1}(R_\sigma\text{–proj})$ in $\mathrm{W}^{2k+1}(\bar{R}_{\bar{\sigma}}\text{–free})$. The result then follows by Corollary 5.7. The non-connected case follows easily from the above and the next Remark. □

Remark 6.3. For any two semi-local rings with involution (R_1, σ_1) and (R_2, σ_2)

$$\mathrm{W}^*((R_1 \times R_2)_{(\sigma_1 \times \sigma_2)}\text{–proj}) = \mathrm{W}^*((R_1)_{\sigma_1}\text{–proj}) \;\oplus\; \mathrm{W}^*((R_2)_{\sigma_2}\text{–proj})\,.$$

A commutative semi-local ring with involution R is a product of "connected" semi-local rings with involution and of switch-involutions. Indeed, such an R is isomorphic to a product $R \simeq R_1 \times \ldots \times R_n$ of semi-local rings with connected spectrum. The involution fixes some connected components and switches pairs of others. For $1 \le i \le n$, we then have $\sigma(R_i) \subset R_i$ or $\sigma(R_i) \subset R_j$ for some $j \neq i$. In the former case, σ induces an involution on R_i. In the latter case, we have an isomorphism $\sigma : R_i \simeq R_j$ (with inverse σ) and therefore the ring with involution $\big(R_i \times R_j\,,\, \sigma\big)$ is isomorphic to $R_i \times R_i$ with the switch involution. The Witt groups do not see those factors (see Example 1.13).

For the rest of this section, we assume that (R, σ) is a commutative semi-local ring with involution *such that* $\mathrm{Spec}(R)$ *is connected*, that is, R is not decomposable as a product $R = R_1 \times R_2$. It is easy to reduce general considerations for commutative semi-local rings to connected ones as explained above. So, we have R–proj $= R$–free and we can use the notation

$$\mathrm{W}^*(R) := \mathrm{W}^*(R_\sigma\text{–proj}) = \mathrm{W}^*(R_\sigma\text{–free})\,.$$

We now want to produce non-zero elements in $\mathrm{W}^3(R)$. We first define explicit Witt classes and then check their non-vanishing, as well as other relations between them, in the group $\mathrm{W}^3(\bar{R}_{\bar\sigma}-\text{free})$, via the injection of Theorem 6.1 :

$$\mathrm{W}^3(R) = \mathrm{W}^3(R_\sigma-\text{free}) \hookrightarrow \mathrm{W}^3(\bar{R}_{\bar\sigma}-\text{free})\,.$$

Definition 6.4. Let $\mathfrak{m} \in \mathrm{Max}(R)$ be a maximal ideal of R. We say that $\mathfrak{m}$ is *detected* by a pair of elements $a, b \in R$ if the following hold :

(a) σ is the identity on $R/\mathfrak{m}$,
(b) $\sigma(a)\cdot b + \sigma(b)\cdot a = 0$ in R,
(c) $a \in \mathfrak{m}$ and $a \notin \mathfrak{m}'$ for all $\mathfrak{m}' \in \mathrm{Max}(R)\setminus\{\mathfrak{m}\}$,
(d) $b \notin \mathfrak{m}$ and $b \in \mathfrak{m}'$ for all $\mathfrak{m}' \in \mathrm{Max}(R)\setminus\{\mathfrak{m}\}$.

Lemma 6.5. *If* $\mathfrak{m}$ *is detected by* $a, b \in R$ *then the following :*

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} a \\ b\end{pmatrix}} R^2 \xrightarrow{\begin{pmatrix}\sigma(b) & \sigma(a)\end{pmatrix}} R \longrightarrow 0$$

is an exact sequence. In other words, we can construct the class $[\,1\,,\,a\,,\,b\,]^{(3)} \in \mathrm{W}^3(R)$ *as in Definition 1.2.*

Proof. Composition is zero by condition (b) of 6.4. Exactness follows by localization at each maximal ideal, where either a or b is a unit. □

Notation 6.6. The set $\{\mathfrak{m} \in \mathrm{Max}(R)\,|\,\text{the involution is trivial on } R/\mathfrak{m}\}$ has $\mathbf{so}(\bar{R})$ elements by definition. Let us identify $(\mathbb{Z}/2)^{\mathbf{so}(\bar{R})}$ with the free $\mathbb{Z}/2$-module over this set and let us denote by $e_\mathfrak{m}$ the basis element of $(\mathbb{Z}/2)^{\mathbf{so}(\bar{R})}$ corresponding to $\mathfrak{m}$. We denote further by $\overline{e_\mathfrak{m}}$ the class of $e_\mathfrak{m}$ in the quotient $(\mathbb{Z}/2)^{\mathbf{so}(\bar{R})}/(1,\ldots,1)$.

Proposition 6.7. *The isomorphism of Theorem 5.6 reads*

$$\mathrm{W}^3(\bar{R}_{\bar\sigma}-\text{free}) \;\simeq\; \frac{(\mathbb{Z}/2)^{\mathbf{so}(\bar{R})}}{(1,\ldots,1)}\,. \tag{17}$$

Use notation 6.6. Suppose that $\mathfrak{m} \in \mathrm{Max}(R)$ *is detected by* $a,\, b \in R$ *and consider the Witt class* $[\,1\,,\,a\,,\,b\,]^{(3)} \in \mathrm{W}^3(R)$ *and its image in* $\mathrm{W}^3(\bar{R}_{\bar\sigma}-\text{free})$*. The above isomorphism* (17) *maps the image of* $[\,1\,,\,a\,,\,b\,]^{(3)}$ *to the class* $\overline{e_\mathfrak{m}}$*.*

Proof. The description of $\mathrm{W}^3(\bar{R}_{\bar\sigma}-\text{free})$ simply comes from Theorem 5.6 and the fact that $\ell_i = 1$ for all i in the commutative case.

Consider the 3-space $(Q_\bullet, \psi)$ whose Witt class defines $[\,1\,,\,a\,,\,b\,]^{(3)}$, see Definition 1.2. We have

$$Q_\bullet = \qquad \cdots 0 \longrightarrow 0 \longrightarrow R^1 \xrightarrow{(a)} R^1 \longrightarrow 0 \longrightarrow 0 \cdots$$

By the very choice of the element $a \in R$, see 6.4 (c), this complex becomes over $\bar{R}$

$$\bar{Q}_\bullet := Q_\bullet \otimes_R \bar{R} = \qquad \cdots 0 \longrightarrow 0 \longrightarrow R/\mathfrak{m} \xrightarrow{0} R/\mathfrak{m} \longrightarrow 0 \longrightarrow 0 \cdots$$

since for all other factors $R/\mathfrak{m}'$ the class of $a \in R/\mathfrak{m}'$ is invertible, that is, the corresponding complex is zero. With this, it is easy to check that $(\bar{Q}_\bullet, \bar\psi)$ simply becomes over $\bar{R}$ a hyperbolic space $P \oplus P^{\#}[3]$ over the object $P = (R/\mathfrak{m})[1]$, in the sense of the 3-shifted duality of course. We now have to refer to the proof of Theorem 5.6, indeed to diagram (15) which is the source of the identification of $\mathrm{W}^3(\bar{R}_{\bar\sigma}-\text{free})$ as a quotient of $(\mathbb{Z}/2)^{\mathbf{so}(S)}$. The claim is that the image in $\mathrm{W}^3(\bar{R}_{\bar\sigma}-\text{free})$ of the element $e_\mathfrak{m}$ of $(\mathbb{Z}/2)^{\mathbf{so}(\bar{R})} \subset (\mathbb{Z}/2)^n = \hat{H}^0\big(\mathbb{Z}/2, K_0(\bar{R})\big)$ is the above 3-space $(\bar{Q}_\bullet, \bar\psi)$. This follows immediately from the description of β given in [HS], see 3.2. □

Corollary and Definition 6.8. *Assume that a maximal ideal* $\mathfrak{m}$ *is detected by* $a, b \in R$. *Then the class* $[\,1\,,\,a\,,\,b\,]^{(3)}$ *in* $\mathrm{W}^3(R)$ *is independent of the choice of* $a, b \in R$ *which detect* $\mathfrak{m}$. *We shall therefore denote it by*

$$\omega(\mathfrak{m}) := [\,1\,,\,a\,,\,b\,]^{(3)} \in \mathrm{W}^3(R)$$

and simply say that the maximal ideal $\mathfrak{m}$ *is* detected by W^3.

Proof. This is immediate from Proposition 6.7 and from the injectivity of $\mathrm{W}^3(R) = \mathrm{W}^3(R_\sigma\text{-free}) \hookrightarrow \mathrm{W}^3(\bar{R}_{\bar{\sigma}}\text{-free})$, see Theorem 6.1, since the element $e_{\mathfrak{m}}$ is independent of the choice of a and b and $\overline{e_{\mathfrak{m}}}$ *a fortiori*. □

Corollary 6.9. *Suppose that all maximal ideals* $\mathfrak{m} \in \mathrm{Max}(R)$ *are detected by* W^3. *Then* $\mathrm{W}^3(R) \simeq (\mathbb{Z}/2)^{\#\,\mathrm{Max}(R)-1}$ *and the classes* $\{\omega(\mathfrak{m})\}_{\mathfrak{m}\in M}$ *generate* $\mathrm{W}^3(R)$ *with*

$$\sum_{\mathfrak{m}\in\mathrm{Max}(R)} \omega(\mathfrak{m}) = 0 \qquad \textit{in} \quad \mathrm{W}^3(R)\,.$$

Proof. This is again a direct consequence of Proposition 6.7 and of the similar properties for the elements $\overline{e_{\mathfrak{m}}} \in (\mathbb{Z}/2)^{\mathbf{so}(\bar{R})}/(1,\ldots,1)$, see 6.6. □

Theorem 6.10. *For any positive integer* $n \geq 1$. *Consider* $n+1$ *distinct odd primes* $p_0, \ldots, p_n$ *and let* $d = p_0 \cdot \ldots \cdot p_n$. *For each* $i = 0, \ldots, n$ *let* $\mathfrak{p}_i \in \mathrm{Spec}\,\big(\mathbb{Z}[\sqrt{d}]\big)$ *be a prime ideal above* $p_i\mathbb{Z}$. *Let the ring* R *be the semi-localization of* $\mathbb{Z}[\sqrt{d}]$ *around the primes* $\mathfrak{p}_0, \ldots, \mathfrak{p}_n$. *Define an involution* σ *on* R *by localization of the obvious involution on* $\mathbb{Z}[\sqrt{d}]$, *which sends* $\sqrt{d}$ *to* $-\sqrt{d}$.

Then, R *is a Dedekind domain with non-trivial involution in which all maximal ideals are detected by* W^3. *So in particular,*

$$\mathrm{W}^3(R_\sigma\text{-proj}) \cong (\mathbb{Z}/2)^n\,.$$

Proof. Since $p_0, \ldots, p_n$ are distinct odd primes and R is the semi-localization of $\mathbb{Z}[\sqrt{d}]$ around $\mathfrak{p}_0, \ldots, \mathfrak{p}_n$, we note that $\frac{1}{2} \in R$. The set of maximal ideals of R is $\mathrm{Max}(R) = \{\mathfrak{p}_i\}_{0\leq i\leq n}$. Let $0 \leq i \leq n$. It is easy to see that $\mathfrak{p}_i = \langle\, p_i, \sqrt{d}\,\rangle$. Consider the elements $a_i = p_i + \sqrt{d}$ and $b_i = p_0 \cdots p_{i-1} \cdot p_{i+1} \cdots p_n + \sqrt{d}$ in R. We claim that the pair (a_i, b_i) detects the maximal ideal $\mathfrak{p}_i$ for each $i = 0, \ldots, n$.

It is easy to see that σ is the identity on $R/\mathfrak{p}_i \cong \mathbb{Z}/p_i$. Now a direct computation shows that $\sigma(a_i)\cdot b_i + \sigma(b_i)\cdot a_i = 0$ in R. Since $p_0, \ldots, p_n$ are distinct odd primes, it is clear that $a_i \in \mathfrak{p}_i$ and $a_i \notin \mathfrak{p}_j$, for $j \neq i$. Again, $b_i \notin \mathfrak{p}_i$ and $b_i \in \mathfrak{p}_j$ for every $j \neq i$. Hence by 6.4 the pair $(a_i, b_i) \in R^2$ detects the maximal ideal $\mathfrak{p}_i$, for each $0 \leq i \leq n$. Since all maximal ideals of R are detected by W^3, we have by Corollary 6.9 that $\mathrm{W}^3(R_\sigma\text{-proj}) \cong (\mathbb{Z}/2)^{\#\,\mathrm{Max}(R)-1} = (\mathbb{Z}/2)^n$. Hence the result. □

We can also illustrate the non-vanishing of $\mathrm{W}^3(R)$ by a more geometric example:

Example 6.11. Consider the ring $\mathbb{R}[X,Y]/(X^2+Y^2-1)$. Let $\mathfrak{m}_{-1}$ and $\mathfrak{m}_1$ denote the maximal ideals in the above ring corresponding to the points $(-1,0)$ and $(1,0)$ respectively. So $\mathfrak{m}_{-1} = \langle\, x+1, y\,\rangle$ and $\mathfrak{m}_1 = \langle\, x-1, y\,\rangle$, where x and y denote the images of X and Y under the natural map from $\mathbb{R}[X,Y] \to \mathbb{R}[X,Y]/(X^2+Y^2-1)$. Let $R = (\mathbb{R}[X,Y]/(X^2+Y^2-1))_{((-1,0),(1,0))}$, the ring semi-localized at $\mathfrak{m}_{-1}$ and $\mathfrak{m}_1$. Consider the involution $\sigma : R \to R$ induced by $x \mapsto x$ and $y \mapsto -y$. Then σ is the identity on $R/\mathfrak{m}_{-1} \cong \mathbb{R}$ and also on $R/\mathfrak{m}_1$. Consider the elements $r = y+(1-x)$ and $s = y + (1+x)$ in R. It is easy to see that they satisfy the following relation: $\sigma(r)\cdot s + \sigma(s)\cdot r = 0$. Now $r \in \mathfrak{m}_1$ and $r \notin \mathfrak{m}_{-1}$. Similarly, $s \in \mathfrak{m}_{-1}$ and $s \notin \mathfrak{m}_1$. Hence by 6.4 $\mathfrak{m}_1$ is detected by the pair (r,s) and $\mathfrak{m}_{-1}$ is detected by the pair (s,r). Since both the maximal ideals of R are detected by W^3, by Corollary 6.9, we have

$$\mathrm{W}^3(R) \cong (\mathbb{Z}/2)^{\#\,\mathrm{Max}(R)-1} \cong \mathbb{Z}/2.$$

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