EXAMPLES OF TENSOR-TRIANGULATED CATEGORIES

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Abstract. As an introduction to tensor-triangular geometry, we review examples of tensor-triangulated categories occurring in a range of different areas. Our goal is not to provide detailed constructions and definitions but to underline the repetition of the same structures throughout areas.

Contents

1. Commutative algebra and algebraic geometry

Let us start in commutative algebra. Let A be a commutative ring. Consider

$$\mathcal{K}=\mathrm{D}^{\mathrm{perf}}(A) \quad\subseteq\quad \mathcal{T}=\mathrm{D}(A) \tag{1.1}$$

where $\mathrm{D}^{\mathrm{perf}}(A)$ denotes the category of perfect complexes and $\mathrm{D}(A)$ is the derived category of all A-modules. Perfect complexes $\mathrm{D}^{\mathrm{perf}}(A)\cong\mathrm{K}^{\mathrm{b}}(A\text{-}\mathrm{proj})$ can be realized as the category of bounded complexes of finitely generated projective A-modules, with maps of complexes up to homotopy. The big derived category is obtained by inverting quasi-isomorphisms on unbounded complexes of A-modules.

This extends to algebraic geometry. Let X be a quasi-compact and quasi-separated scheme (a purely topological condition, asking for the underlying space of X to admits a basis of quasi-compact open). One generalizes (1.1)

$$\mathcal{K}=\mathrm{D}^{\mathrm{perf}}(X) \quad\subseteq\quad \mathcal{T}=\mathrm{D}(X) \tag{1.2}$$

in an essentially straightforward way. The big triangulated category, $\mathrm{D}(X)$, is the derived category of complexes of $\mathcal{O}_X$-modules with quasi-coherent homology. The small one, $\mathrm{D}^{\mathrm{perf}}(X)$, consists of those complexes that are perfect on each affine open of X. Neeman [Nee92a] proves that perfect complexes are the *compact* objects of $\mathrm{D}(X)$, those objects $c\in\mathcal{T}$ such that $\mathrm{Hom}_{\mathcal{T}}(c,-)\colon\mathcal{T}\to\mathrm{Ab}$ preserves coproducts. The fact that the 'big' category $\mathcal{T}$ is generated by the 'small' subcategory $\mathcal{K}=\mathcal{T}^c$ of compact objects is a recurring feature of all examples.

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One can wonder why we care about these structures. Derived categories have been tremendously useful in algebraic geometry, for instance through the power of homological algebra. But perhaps one of their first uses was to formulate what we now call *Grothendieck duality*. Let us mention it since it connects with our geometric motivations.

1.3. *Example.* To illustrate Grothendieck duality in an embryonic form, consider the ring $A = \mathbb{Z}$ and its quotient $k = \mathbb{Z}/p\mathbb{Z}$. A finite-dimensional k-vector space V can be viewed as an A-module i_*V. Although the dual of V as a vector space has the same dimension as V, the dual of i_*V over A is zero: $\mathrm{Hom}_A(i_*V, A) = 0$ simply because i_*V is a torsion A-module. Take $V = k$ for instance. Then i_*V is in fact (quasi-isomorphic to) a perfect complex. In the derived category of A, we have $i_*V \cong (\cdots 0 \to A \xrightarrow{p} A \to 0 \cdots)$, with the two A's in homological degree 1 and 0. The dual $(-)^\vee$ of the latter, as a perfect complex, is $(\cdots 0 \to A \xrightarrow{p} A \to 0 \cdots)$, with the two A's now in homological degree 0 and -1. So $(i_*V)^\vee \cong i_*(V^\vee)[-1]$ and this relation is true for any perfect complex over k. The shift by 1 comes from the difference of Krull dimensions between $\mathrm{Spec}(A)$ and $\mathrm{Spec}(k)$. Playing instead with $A = k[X_1, \ldots, X_n]$ and $k = A/\langle X_1, \ldots, X_n\rangle$ would yield a shift by n for instance.

Full-scale Grothendieck duality relies even more on derived categories than is apparent in the above example. Indeed, the exceptional inverse image functor $f^!$ and dualizing complexes only make sense at the derived level.

In any case, the elementary Example 1.3 already illustrates two things:

(1) There are phenomena that occur at the level of derived categories that simply do not make sense at the level of modules.

(2) Some geometric information can be read on the derived category, like the relative dimension of the morphism $i\colon \mathrm{Spec}(k) \to \mathrm{Spec}(A)$ in our example.

1.4. *Remark.* An area that grew hand-in-hand with derived categories is K-theory. Indeed, in the paper that is considered by some as the birthplace of K-theory [BS58], Grothendieck constructs the push-forward on K_0 by means of the derived direct-image functor f_* akin to the i_* we considered in Example 1.3. Decades later, Thomason's famous higher K-theory paper [TT90] also relies in a critical way on improvements in the theory of perfect complexes. The relationship between triangulated categories and K-theory has not been entirely rosy though. Notoriously, Neeman [Nee92b] showed that Waldhausen's K-theory cannot be recovered from homotopy categories and it was not until the recent work of Muro-Raptis [MR17] on K-theory of derivators that the two subjects reconciled to live happily ever after.

1.5. *Remark.* The expectations for the derived category $\mathrm{D}(X)$ to be a rich invariant of the scheme X were lowered by Mukai [Muk81] when he gave examples of non-isomorphic schemes $X \not\simeq X'$ with equivalent derived categories $\mathrm{D}(X) \simeq \mathrm{D}(X')$, as triangulated categories (for instance any abelian variety X and its dual X'). A key realization came with Thomason's classification [Tho97] of so-called thick tensor-ideals of $\mathrm{D}^{\mathrm{perf}}(X)$, where it became apparent that the tensor structure on $\mathrm{D}(X)$ plays an important role. So $\mathrm{D}(X)$ is not only a triangulated category but a tensor-triangulated category and Mukai's equivalences $\mathrm{D}(X) \simeq \mathrm{D}(X')$ do *not* preserve the tensor structure. It is one of the first results of tensor-triangular geometry that a *tensor*-triangulated equivalence $\mathrm{D}(X) \simeq \mathrm{D}(X')$, or just $\mathrm{D}^{\mathrm{perf}}(X) \simeq \mathrm{D}^{\mathrm{perf}}(X')$, does force a scheme isomorphism $X \simeq X'$.

2. Modular representation theory

Let G be a finite group and k be a field of coefficients. By Maschke, we know that if k has characteristic zero or prime to the order $|G|$ of the group then kG is semisimple. The hard case, when we assume that $\mathrm{char}(k) = p > 0$ divides $|G|$, is the topic of *modular representation theory.*

The non-semisimplicity of kG is measured by the additive quotient of the category of kG-modules by the subcategory of projectives. These are the *stable module categories*:

$$\mathcal{K} = \mathrm{stab}(kG) \quad \subseteq \quad \mathcal{T} = \mathrm{Stab}(kG). \tag{2.1}$$

The objects of $\mathcal{T}$ (resp. $\mathcal{K}$) are the (finitely generated) kG-modules; their maps are the kG-linear ones modulo the ones that factor via a projective module.

A great feature of the Krull-Schmidt property is that it passes from the abelian category $kG\text{-}\mathrm{mod}$ to the quotient $\mathrm{stab}(kG)$. So studying objects of $\mathrm{stab}(kG)$ and their decomposition into sums of indecomposables can be understood as studying non-projective indecomposable kG-modules. Obviously, understanding those stable module categories is a core question of modular representation theory and no *ad hoc* motivation is necessary.

Note that one does not expect to recover the group G from $\mathrm{Stab}(kG)$ already because $\mathrm{Stab}(kG) = 0$ for all groups G of order prime to p. In fact, we also have equivalences $\mathrm{Res}^G_H \colon \mathrm{Stab}(kG) \xrightarrow{\sim} \mathrm{Stab}(kH)$ for so-called *strongly p-embedded* subgroups $H \le G$ (*i.e.* when $H^g \cap H$ has order prime to p for all $g \in G \smallsetminus H$), like for instance $C_2 < S_3$ for $p = 2$. Still, such equivalences simply means that G and H have basically the *same* modular representation theory.

2.2. *Remark.* The tensor product (over k) of representations $M \otimes_k N$ with diagonal G-action $g\cdot(m\otimes n) := (gm)\otimes(gn)$ passes to the quotient by projectives and defines a tensor product $\otimes$ on $\mathrm{Stab}(kG)$. It then becomes interesting to describe kG-modules M that induce equivalences $M \otimes - \colon \mathrm{Stab}(kG) \xrightarrow{\sim} \mathrm{Stab}(kG)$, *i.e.* those that are $\otimes$-invertible in $\mathrm{Stab}(kG)$. For instance, every indecomposable N would then come together with its translates $M^{\otimes n} \otimes N$ for all $n \in \mathbb{Z}$. Of course kG-modules M that are of dimension one over k, that is, those that are $\otimes$-invertible in $kG\text{-}\mathrm{mod}$ remain $\otimes$-invertible in $\mathrm{Stab}(kG)$ but being $\otimes$-invertible in $\mathrm{Stab}(kG)$ is more flexible. In $kG\text{-}\mathrm{mod}$, it boils down to the fact that $M^\vee \otimes M \simeq k \oplus P$ for P projective, a property that has sadly been called *endotrivial* since $M^\vee \otimes M \cong \mathrm{End}_k(M)$. The study of endotrivial kG-modules has been a central topic of modular representation theory for the last decades and this is an area where the ideas of tensor-triangulated categories have brought some applications.

3. Stable homotopy theory

Historically, triangulated categories emerged in algebra, as discussed in Section 1, and simultaneously in topology. Simplifying the study of pointed topological spaces (say, compactly-generated ones, or simplicial sets) by working up to homotopy and by forcing suspension to be an equivalence leads to the *stable homotopy categories*

$$\mathcal{K} = \mathrm{SH}^{\mathrm{fin}} \quad \subset \quad \mathcal{T} = \mathrm{SH}\,. \tag{3.1}$$

The finiteness on the left-hand refers to taking *finite* CW-complexes and the large right-hand category is the homotopy category of spectra. Defining these precisely takes time but the reader can think of SH among tensor-triangulated categories as the analogue of $\mathbb{Z}$ among commutative rings: It is the 'initial' one. (Making such statement precise requires more structure than just triangulated categories.)

Here the motivation is somewhat different from Section 1. One is not trying to reconstruct a scheme from the invariant $\mathrm{D}(X)$ anymore. Understanding SH is a heavily watered-down but still very interesting version of the hopelessly complicated study of all topological spaces. In particular SH is additive thanks to stabilization, whereas Top certainly is not. The tensor product is given by the *smash product*, whose unit is the sphere spectrum S^0. The groups of homomorphisms from S^0 to its suspensions are the stable homotopy groups of spheres. Computing them is a core and very hard problem of topology, that connects with other parts of mathematics. So a first motivation for understanding SH as a whole is also to provide global structures that can ultimately shed new light on these groups.

The so-called *chromatic theory* of [DHS88, HS98] is such a global structure on SH and can be considered as an ancestor of tensor-triangular geometry.

Another motivation for the big category SH is Brown's representability theorem. It guarantees that all (suitably defined) 'cohomology theories' can be realized as $\mathrm{Hom}_{\mathrm{SH}}(-, E)$ for some fixed spectrum $E \in \mathrm{SH}$. It is important to allow arbitrary coproducts and thus to work with the big stable homotopy category SH and with spectra. Brown Representability is another theme traversing all examples and one of the motivations to study the big $\mathcal{T}$ and not merely its compact part $\mathcal{K}$.

There is a broader class of examples which is perhaps more in the spirit of Sections 1–2. Let G be a compact Lie group, for instance a finite one. There are versions of (3.1) based on G-spaces

$$\mathcal{K} = \mathrm{SH}^{\mathrm{fin}}(G) \quad \subset \quad \mathcal{T} = \mathrm{SH}(G). \tag{3.2}$$

Again, the precise constructions require some care. For specialists, we mean here the homotopy categories of genuine G-spectra. We invert G-homotopy equivalences (maps that are homotopy equivalences on all H-fixed points for all closed subgroups $H \leq G$) and we are stabilize with respect to all G-spheres.

In this class of examples, one can try to see how much of the combinatorial structure of G and its subgroups gets reflected in the tensor-triangulated structure of $\mathrm{SH}(G)$. Again, this is an area where tensor-triangular geometry sheds some light.

4. Motivic theory

Let S be a base noetherian scheme, typically the spectrum of a (perfect) field. Roughly speaking, *motivic* theory is the study of the cohomological properties of (smooth) schemes X over the base S, in which a new homotopy invariance is considered, *i.e.* with respect to the 'interval' $\mathbb{A}^1_S$. It comes in different flavors, most importantly, in analogy with (1.2), the *derived category of motives* over S

$$\mathcal{K} = \mathrm{DM}^{\mathrm{gm}}(S) \quad \subseteq \quad \mathcal{T} = \mathrm{DM}(S) \tag{4.1}$$

and, in analogy with (3.1), the $\mathbb{A}^1$*-stable homotopy category* over S

$$\mathcal{K} = \mathrm{SH}(S)^c \quad \subseteq \quad \mathcal{T} = \mathrm{SH}(S). \tag{4.2}$$

These categories contain in particular objects $[X]$ for every smooth scheme X over S, in such a way that $[\mathbb{A}^1 \times_S X] \simeq [X]$. The difference between the 'algebraic' (4.1) and the 'topological' (4.2) is that the 'coefficients' of the former are complexes of abelian groups whereas the 'coefficients' of the latter are spectra. In analogy with inverting all G-spheres in (3.2), we are here inverting another sphere coming from geometry, which amounts to inverting $\mathbb{P}^1$.

The importance of motivic theory is both conceptual and practical. It allowed Voevodsky to realize Grothendieck's vision of the abelian category of motives at a triangular level and it led him to prove the Bloch-Kato Conjecture.

We reach an interesting point where the tensor-triangulated categories have become substantially more complicated to construct and to apprehend as a whole than the ones of representation theory (2.1) for instance. This complexity is another motivation for tensor-triangular geometry as a way to provide general methods to analyze otherwise untractable structures. The tensor-triangular geometry of motivic examples is still at an early stage.

5. Further examples

Many of the examples listed above have variants that are highly interesting in their own right. A more detailed survey can be found in [Bal19].

The use of tensor-triangulated categories also appears in the KK-theory of C^*-algebras, equivariant or not. A survey can be found in [Del10].

Triangulated categories that ought to have a tensor structure (!) feature notably in homological mirror symmetry [Kon95].

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