ON THE SURJECTIVITY OF THE MAP OF SPECTRA ASSOCIATED TO A TENSOR-TRIANGULATED FUNCTOR

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ABSTRACT. We prove a few results about the map $\mathrm{Spc}(F)$ induced on tensor-triangular spectra by a tensor-triangulated functor F. First, F is conservative if and only if $\mathrm{Spc}(F)$ is surjective on closed points. Second, if F detects tensor-nilpotence of morphisms then $\mathrm{Spc}(F)$ is surjective on the whole spectrum. In fact, surjectivity of $\mathrm{Spc}(F)$ is equivalent to F detecting the nilpotence of some class of morphisms, namely those morphisms which are nilpotent on their cone.

1. Introduction

Hypotheses 1.1. Throughout the paper, $F\colon \mathcal{K} \to \mathcal{L}$ is a tensor-triangulated functor between essentially small tensor-triangulated categories $\mathcal{K}$ and $\mathcal{L}$. Assume that $\mathcal{K}$ is *rigid*, *i.e.* every object has a dual (Remark 2.1).

Consider the induced map on spectra

$$\varphi = \mathrm{Spc}(F)\colon\ \mathrm{Spc}(\mathcal{L}) \to \mathrm{Spc}(\mathcal{K})$$

in the sense of tensor-triangular geometry [Bal05, Bal10b, Ste16]. Our first result is a characterization of conservativity of F.

Theorem 1.2. *Under Hypotheses 1.1, the following properties are equivalent:*

(a) *The functor $F\colon \mathcal{K} \to \mathcal{L}$ is conservative,* i.e. *it detects isomorphisms.*

(b) *The induced map $\varphi\colon\ \mathrm{Spc}(\mathcal{L}) \to \mathrm{Spc}(\mathcal{K})$ is surjective on closed points,* i.e. *for every closed point $\mathcal{P}$ in $\mathrm{Spc}(\mathcal{K})$, there exists $\mathcal{Q}$ in $\mathrm{Spc}(\mathcal{L})$ such that $\varphi(\mathcal{Q}) = \mathcal{P}$.*

We can remove the assumption that $\mathcal{K}$ is rigid, at the cost of replacing (a) by:

(a') F detects $\otimes$-nilpotence of objects, *i.e.* $F(x) = 0 \Rightarrow x^{\otimes n} = 0$ for some $n \geq 1$.

Our main results are dedicated to surjectivity of φ on the whole of $\mathrm{Spc}(\mathcal{K})$.

Theorem 1.3. *Under Hypotheses 1.1, suppose that the functor $F\colon \mathcal{K} \to \mathcal{L}$ detects $\otimes$-nilpotence of morphisms,* i.e. *every $f\colon x \to y$ in $\mathcal{K}$ such that $F(f) = 0$ satisfies $f^{\otimes n} = 0$ for some $n \geq 1$. Then the induced map $\varphi\colon\ \mathrm{Spc}(\mathcal{L}) \to \mathrm{Spc}(\mathcal{K})$ is surjective.*

This result is clearly a corollary of (b)$\Rightarrow$(a) in the following more technical result:

Theorem 1.4. *Under Hypotheses 1.1, the following properties are equivalent:*

(a) *The morphism $\varphi\colon\ \mathrm{Spc}(\mathcal{L}) \to \mathrm{Spc}(\mathcal{K})$ is surjective.*

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(b) *The functor $F\colon \mathcal{K}\to\mathcal{L}$ detects $\otimes$-nilpotence of morphisms which are already $\otimes$-nilpotent on their cone, i.e. every $f\colon x\to y$ in $\mathcal{K}$ such that $F(f)=0$ and such that $f^{\otimes m}\otimes\mathrm{cone}(f)=0$ for some $m\geq 1$ satisfies $f^{\otimes n}=0$ for some $n\geq 1$.*

At this point, the Devinatz-Hopkins-Smith [DHS88] Nilpotence Theorem might come to some readers' mind. This celebrated result asserts that a morphism between finite objects in the topological stable homotopy category SH must be $\otimes$-nilpotent if it vanishes on complex cobordism. Hopkins and Smith used the Nilpotence Theorem in the subsequent work [HS98] to prove the Chromatic Tower Theorem. A reformulation of the latter, in terms of $\mathrm{Spc}(\mathrm{SH}^c)$, can be found in [Bal10a, § 9]. From the Nilpotence Theorem it follows that every prime of SH^c is the kernel of some Morava K-theory. This implication is analogous to the surjectivity of Theorem 1.3 in the special case of SH.

Let us stress however that the scope of Theorems 1.2 and 1.3 is broader than the topological example. In fact, SH plays among general tensor-triangulated categories the same role that $\mathbb{Z}$ plays among general commutative rings. Commutative algebra is not only the study of $\mathbb{Z}$, and tt-geometry is not only the study of SH. For the reader who never heard of tensor-triangulated categories and yet had the fortitude to read thus far, let us recall that tt-categories also appear in algebraic geometry (*e.g.* derived categories of schemes), in representation theory (*e.g.* derived and stable categories of finite groups), in noncommutative topology (*e.g.* KK-categories of C^*-algebras), in motivic theory (*e.g.* stable $\mathbb{A}^1$-homotopy and derived categories of motives), and in equivariant analogues (*e.g.* equivariant stable homotopy theory). A good introduction can be found in [HPS97, § 1.2]. Tensor-triangular geometry is an umbrella theory for all those examples. In particular, computing $\mathrm{Spc}(\mathcal{K})$ is *the* fundamental problem for every tt-category $\mathcal{K}$ out there; see [Bal05, Thm. 4.10].

After this motivational digression, let us return to the development of our results. It is interesting to know whether the converse of Theorem 1.3 holds true in glorious generality: Does surjectivity of $\mathrm{Spc}(F)$ alone guarantee that F detects $\otimes$-nilpotence of morphisms? By Theorem 1.4, this problem can be reduced as follows.

Question 1.5. Under Hypotheses 1.1, if $\varphi\colon\ \mathrm{Spc}(\mathcal{L})\to\mathrm{Spc}(\mathcal{K})$ is surjective and if $f\colon x\to y$ satisfies $F(f)=0$, is f necessarily $\otimes$-nilpotent on its cone?

We do not know any counter-example. In fact, we can give a positive answer under the assumption that $F\colon\mathcal{K}\to\mathcal{L}$ admits a right adjoint. Since $\mathcal{K}$ and $\mathcal{L}$ are essentially small (typically the 'compact' objects of some big ambient category), existence of such a right adjoint is rather restrictive. In the context of [BDS16], it would be equivalent to having 'Grothendieck-Neeman' duality. To give an example, this right adjoint exists in the case of a finite separable extension, see [Bal16b]. The following are generalizations of some of the results in [Bal16a].

Theorem 1.6. *Under Hypotheses 1.1, suppose that $F\colon\mathcal{K}\to\mathcal{L}$ admits a right adjoint $U\colon\mathcal{L}\to\mathcal{K}$. Then the map $\varphi\colon\ \mathrm{Spc}(\mathcal{L})\to\mathrm{Spc}(\mathcal{K})$ is surjective if and only if the functor $F\colon\mathcal{K}\to\mathcal{L}$ detects $\otimes$-nilpotence of morphisms.*

Again, this is a special case of a sharper, slightly more technical result.

Theorem 1.7. *Under Hypotheses 1.1, suppose that $F\colon\mathcal{K}\to\mathcal{L}$ admits a right adjoint $U\colon\mathcal{L}\to\mathcal{K}$ and consider the image $U(\mathbb{1})\in\mathcal{K}$ of the $\otimes$-unit. Then the image of the map $\varphi\colon\ \mathrm{Spc}(\mathcal{L})\to\mathrm{Spc}(\mathcal{K})$ is exactly the support of the object $U(\mathbb{1})$:*

$$\mathrm{im}(\mathrm{Spc}(F))=\mathrm{supp}(U(\mathbb{1}))\,.$$

An example of the latter, not covered by the separable extensions of [Bal16a], can be obtained by 'modding out' coefficients in motivic categories, see [VSF00, Chap. 5]. For instance, if $\mathcal{K} = \mathrm{DM}_{\mathrm{gm}}(X;\mathbb{Z}) \xrightarrow{F} \mathrm{DM}_{\mathrm{gm}}(X;\mathbb{Z}/p) = \mathcal{L}$ then we have $\mathrm{im}(\mathrm{Spc}(F)) = \mathrm{supp}(\mathbb{Z}/p)$. From these techniques, one can easily reduce the computation of the (yet unknown) spectrum of the integral derived category of geometric motives $\mathrm{DM}_{\mathrm{gm}}(X,\mathbb{Z})$ to the case of field coefficients:

$$\mathrm{Spc}(\mathrm{DM}_{\mathrm{gm}}(X;\mathbb{Z})) = \mathrm{im}(\mathrm{Spc}(\mathrm{DM}_{\mathrm{gm}}(X;\mathbb{Q}))) \sqcup \bigsqcup_p \mathrm{im}(\mathrm{Spc}(\mathrm{DM}_{\mathrm{gm}}(X;\mathbb{Z}/p)))\,.$$

These considerations will be pursued elsewhere.

In the presence of a 'big' ambient category, our condition of detecting $\otimes$-nilpotence could also be related to conservativity, as discussed in [MNN17, Thm. 4.19].

Let us now state a direct consequence of Theorem 1.3, that was apparently never noticed despite its importance and simplicity. It is the case where F is faithful.

Corollary 1.8. *Suppose that $\mathcal{K} \subset \mathcal{L}$ is a rigid tensor-triangulated subcategory. Then every prime $\mathcal{P} \in \mathrm{Spc}(\mathcal{K})$ is the intersection of a prime $\mathcal{Q} \in \mathrm{Spc}(\mathcal{L})$ with $\mathcal{K}$.*

A special sub-case of interest is that of 'cellular' subcategories, *i.e.* those $\mathcal{K} \subseteq \mathcal{L}$ generated by a collection of 'nice' objects of $\mathcal{L}$, typically $\otimes$-invertible ones (spheres). Such cellular subcategories $\mathcal{K}$ are commonly studied when the ambient $\mathcal{L}$ appears out-of-reach of known methods. For instance, Dell'Ambrogio [Del10] used this approach for equivariant KK-theory, and later with Tabuada [DT12] for non-commutative motives. Peter [Pet13] discusses the case of mixed Tate motives. Similarly, Heller-Ormsby [HO16] consider cellular subcategories in their recent study of tt-geometry in stable motivic homotopy theory. In all cases, Corollary 1.8 says that whatever can be detected via these cellular subcategories $\mathcal{K}$ is actually relevant information about the bigger and more mysterious ambient category $\mathcal{L}$. In particular, surjectivity of the comparison homomorphisms introduced in [Bal10a] can be tested on the cellular subcategory:

Corollary 1.9. *Let $u \in \mathcal{L}$ be a $\otimes$-invertible object and $\mathcal{K}$ the full thick triangulated subcategory of $\mathcal{L}$ generated by $\{\, u^{\otimes n} \mid n \in \mathbb{Z} \,\}$, which is supposed rigid ([1]). Note that the graded rings $R^\bullet_{\mathcal{K},u}$ and $R^\bullet_{\mathcal{L},u}$ associated to u are the same in $\mathcal{K}$ and in $\mathcal{L}$:*

$$R^\bullet_{\mathcal{K},u} \overset{def}{=} \mathrm{Hom}_{\mathcal{K}}(\mathbb{1}, u^{\otimes \bullet}) = \mathrm{Hom}_{\mathcal{L}}(\mathbb{1}, u^{\otimes \bullet}) \overset{def}{=} R^\bullet_{\mathcal{L},u}\,.$$

If the comparison map $\rho^\bullet_{\mathcal{K},u}$ for $\mathcal{K}$ (recalled below) is surjective for the 'cellular' subcategory $\mathcal{K}$ then the comparison map $\rho^\bullet_{\mathcal{L},u}$ for the ambient $\mathcal{L}$ is also surjective:

$$\begin{tikzcd}
\mathrm{Spc}(\mathcal{L}) \arrow[r, two heads, "Cor.\ 1.8"] \arrow[d, "\rho^\bullet_{\mathcal{L},u}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{Spc}(\mathcal{K}) \arrow[d, "\rho^\bullet_{\mathcal{K},u}"] & \ni & \mathcal{P} \arrow[d, mapsto, "\rho^\bullet_{\mathcal{K},u}"'] \\
\mathrm{Spec}^\bullet(R^\bullet_{\mathcal{L},u}) \arrow[r, equal] & \mathrm{Spec}^\bullet(R^\bullet_{\mathcal{K},u}) & \ni & \rho^\bullet_{\mathcal{K},u}(\mathcal{P}) \overset{def}{=} \{\, f \in R^\bullet_{\mathcal{K},u} \mid \mathrm{cone}(f) \notin \mathcal{P} \,\}\,.
\end{tikzcd}$$

For an introduction to these comparison maps and their importance, the reader is invited to consult the above references [Bal10a, Del10, DT12, HO16] or [San13].

[1] This is automatic if $\mathcal{L}$ lives in a 'big' ambient category with internal hom, where rigid objects are closed under triangles. See [HPS97, Thm. A.2.5 (a)].

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2. The proofs

The tensor $\otimes\colon \mathcal{K}\times\mathcal{K} \longrightarrow \mathcal{K}$ is exact in each variable and $\mathbb{1}$ stands for the $\otimes$-unit in $\mathcal{K}$. Recall that a *tt-ideal* $\mathcal{J} \subseteq \mathcal{K}$ is a triangulated, thick, $\otimes$-ideal subcategory, *i.e.* it is non-empty, is closed under taking cones, direct summands and under tensoring by any object of $\mathcal{K}$. For $\mathcal{E} \subseteq \mathcal{K}$, we denote by $\langle \mathcal{E} \rangle \subseteq \mathcal{K}$ the tt-ideal it generates.

A proper tt-ideal $\mathcal{P} \subsetneq \mathcal{K}$ is *prime* if $x \otimes y \in \mathcal{P}$ implies $x \in \mathcal{P}$ or $y \in \mathcal{P}$. The *spectrum* $\mathrm{Spc}(\mathcal{K}) = \{ \mathcal{P} \subset \mathcal{K} \,|\, \mathcal{P} \text{ is prime} \}$ has a topology whose basis of open is given by the subsets $U(x) = \{ \mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \,|\, x \in \mathcal{P} \}$, for every $x \in \mathcal{K}$. The closed complement $\mathrm{supp}(x) = \{ \mathcal{P} \in \mathrm{Spc}(\mathcal{K}) \,|\, x \notin \mathcal{P} \}$ is called the *support* of the object x. A tensor-triangulated functor $F\colon \mathcal{K} \to \mathcal{L}$ induces a continuous map $\varphi = \mathrm{Spc}(F)\colon \mathrm{Spc}(\mathcal{L}) \to \mathrm{Spc}(\mathcal{K})$ given explicitly by $\varphi(\mathcal{Q}) = F^{-1}(\mathcal{Q})$, for every prime $\mathcal{Q} \subset \mathcal{L}$.

Remark 2.1. Our assumption that the tensor category $\mathcal{K}$ is *rigid*, means that there exists an exact functor called the *dual*

$$(-)^\vee \colon \mathcal{K}^{\mathrm{op}} \longrightarrow \mathcal{K}$$

that provides an adjoint to tensoring with any object $x \in \mathcal{K}$ as follows:

$$\begin{tikzcd} \mathcal{K} \arrow[d, shift right=2, "x\otimes-"'] \arrow[d, phantom, "\dashv"] \\ \mathcal{K} \arrow[u, shift right=2, "x^\vee\otimes-"'] \end{tikzcd} \tag{2.2}$$

Some authors call such objects x *strongly dualizable*, *e.g.* [HPS97]. The adjunction (2.2) comes with units (coevaluation) and counits (evaluation)

$$\eta_x \colon \mathbb{1} \to x^\vee \otimes x \qquad \text{and} \qquad \epsilon_x \colon x \otimes x^\vee \to \mathbb{1} \tag{2.3}$$

which satisfy the relation

$$(\epsilon_x \otimes x) \circ (x \otimes \eta_x) = 1_x \,. \tag{2.4}$$

It follows from (2.4) that x is a direct summand of $x \otimes x^\vee \otimes x \cong x^{\otimes 2} \otimes x^\vee$.

It is a general fact that any tensor functor $F\colon \mathcal{K} \to \mathcal{L}$ preserves rigidity, since we can use $F(x^\vee)$ as $F(x)^\vee$ with $F(\eta_x)$ and $F(\epsilon_x)$ as units and counits. See for instance [FHM03, Prop. 3.1]. In particular, although we do not assume $\mathcal{L}$ rigid, every object we use below will be rigid as long as it comes from $\mathcal{K}$.

Remark 2.5. In a not-necessarily rigid tt-category, an object x with empty support, $\mathrm{supp}(x) = \varnothing$, is $\otimes$-nilpotent, *i.e.* $x^{\otimes n} = 0$ for some $n \geq 1$. See [Bal05, Cor. 2.4]. When x is rigid, $x^{\otimes n} = 0$ forces $x = 0$ since x is a summand of $x^{\otimes n} \otimes (x^\vee)^{\otimes(n-1)}$.

We begin with Theorem 1.2, which is relatively straightforward. We only need a few standard facts from basic tt-geometry, which do not use rigidity, namely:

(A) Given a $\otimes$-multiplicative class S of objects in $\mathcal{K}$ (*i.e.* $\mathbb{1} \in S$ and $x, y \in S \Rightarrow x \otimes y \in S$) and a tt-ideal $\mathcal{J} \subset \mathcal{K}$ such that $\mathcal{J} \cap S = \varnothing$, then there exists a prime $\mathcal{P} \in \operatorname{Spc}(\mathcal{K})$ such that $\mathcal{J} \subseteq \mathcal{P}$ and $\mathcal{P} \cap S = \varnothing$. This fact uses that $\mathcal{K}$ is essentially small and is proven in [Bal05, Lemma 2.2].

(B) A point $\mathcal{P} \in \operatorname{Spc}(\mathcal{K})$ is closed if and only $\mathcal{P}$ is a *minimal* prime for inclusion in $\mathcal{K}$ (*i.e.* $\mathcal{P}' \subseteq \mathcal{P} \Rightarrow \mathcal{P}' = \mathcal{P}$). See [Bal05, Prop. 2.9].

(C) Any non-empty closed subset, for instance $\overline{\{\mathcal{P}\}}$ for a point $\mathcal{P}$, or $\operatorname{supp}(x)$ for a non-trivial object x, contains a closed point. See [Bal05, Cor. 2.12].

(D) For $F \colon \mathcal{K} \to \mathcal{L}$ and $\varphi = \operatorname{Spc}(F) \colon \operatorname{Spc}(\mathcal{L}) \to \operatorname{Spc}(\mathcal{K})$, and every object $x \in \mathcal{K}$, we have $\operatorname{supp}(F(x)) = \varphi^{-1}(\operatorname{supp}(x))$ in $\operatorname{Spc}(\mathcal{L})$. See [Bal05, Prop. 3.6].

Proof of Theorem 1.2. Suppose that $F \colon \mathcal{K} \to \mathcal{L}$ is conservative and let $\mathcal{P} \in \operatorname{Spc}(\mathcal{K})$ be a closed point, *i.e.* a minimal prime. Consider its complement $S = \mathcal{K} \smallsetminus \mathcal{P}$. Since $\mathcal{P}$ is prime, S is $\otimes$-multiplicative in $\mathcal{K}$ and does not contain zero. Since F is a conservative tensor functor, the same holds for the class $F(S)$ in $\mathcal{L}$. (Recall that for a triangulated functor F, conservativity is equivalent to $F(x) = 0 \Rightarrow x = 0$, since a morphism is an isomorphism if and only if its cone is zero.) By the general fact (A) recalled above, for the $\otimes$-multiplicative class $F(S)$ and for the tt-ideal $\mathcal{J} = 0$ in $\mathcal{L}$, there exists a prime $\mathcal{Q} \in \operatorname{Spc}(\mathcal{L})$ such that $\mathcal{Q} \cap F(S) = \varnothing$. This relation implies that $F^{-1}(\mathcal{Q}) \subseteq \mathcal{P}$. By minimality of the closed point $\mathcal{P}$, see (B), this inclusion $F^{-1}(\mathcal{Q}) \subseteq \mathcal{P}$ forces $\mathcal{P} = F^{-1}(\mathcal{Q}) = \varphi(\mathcal{Q})$.

Conversely, suppose that $\varphi \colon \operatorname{Spc}(\mathcal{L}) \to \operatorname{Spc}(\mathcal{K})$ is surjective on closed points and let $x \in \mathcal{K}$ be such that $F(x) = 0$. We want to show that $x = 0$. Suppose *ab absurdo* that $x \neq 0$. Then we have $\operatorname{supp}(x) \neq \varnothing$. By (C), we know that there exists a closed point $\mathcal{P} \in \operatorname{supp}(x)$, which by assumption belongs to the image of φ, say $\mathcal{P} = \varphi(\mathcal{Q})$. But then $\mathcal{Q} \in \varphi^{-1}(\operatorname{supp}(x)) = \operatorname{supp}(F(x))$ by (D). This last statement contradicts $\operatorname{supp}(F(x)) = \operatorname{supp}(0) = \varnothing$. So $x = 0$ as claimed. ☐

Remark 2.6. The proof also gives a statement for $\mathcal{K}$ not rigid. In that case, the property $\operatorname{supp}(x) = \varnothing$ does not necessarily imply that $x = 0$ but that x is $\otimes$-nilpotent, as an object. See Remark 2.5. Surjectivity of φ onto closed points is therefore equivalent to F detecting $\otimes$-nilpotence of objects. See Theorem 1.2 (a').

Remark 2.7. In complete generality, if a closed point $\mathcal{P} \in \operatorname{Spc}(\mathcal{K})$ belongs to the image of $\varphi \colon \operatorname{Spc}(\mathcal{L}) \to \operatorname{Spc}(\mathcal{K})$, say $\mathcal{P} = \varphi(\mathcal{Q})$, then $\mathcal{P}$ is also the image of a *closed* point $\mathcal{Q}'$, which can be chosen in the closure of $\mathcal{Q}$. Indeed, there exists a closed point $\mathcal{Q}' \in \overline{\{\mathcal{Q}\}}$ by (C) and continuity of φ implies $\varphi(\mathcal{Q}') \in \overline{\{\mathcal{P}\}} = \{\mathcal{P}\}$.

* * *

We now turn to the slightly more tricky Theorem 1.4. Let us clarify the following:

Definition 2.8. A morphism $f \colon x \to y$ is called *$\otimes$-nilpotent* if $f^{\otimes n} \colon x^{\otimes n} \to y^{\otimes n}$ is zero for some $n \geq 1$. We say that $f \colon x \to y$ is *$\otimes$-nilpotent on an object z* in $\mathcal{K}$ if there exists $n \geq 1$ such that $f^{\otimes n} \otimes z$ is the zero morphism $x^{\otimes n} \otimes z \to y^{\otimes n} \otimes z$. In particular, f is *$\otimes$-nilpotent on its cone* if there exists $n \geq 1$ such that $f^{\otimes n} \otimes \operatorname{cone}(f) = 0$.

The following useful fact was already observed in [Bal10a, Prop. 2.12]:

Proposition 2.9. *Let $f \colon x \to y$ be a morphism in $\mathcal{K}$. Then*

$$\left\{\, z \in \mathcal{K} \;\middle|\; f \text{ is } \otimes\text{-nilpotent on } z \,\right\}$$

forms a tt-ideal, even if $\mathcal{K}$ is not rigid.

Closure under direct summands and $\otimes$ is clear from the definition. The trick for closure under cones, is that if $f^{\otimes n_i}\otimes z_i=0$ for $i=1,2$ and if $z_1\to z_2\to z_3\to\Sigma z_1$ is an exact triangle, then $f^{\otimes(n_1+n_2)}\otimes z_3$ will vanish. This is the place where the same statement would fail with 'f vanishes on z' (instead of 'f $\otimes$-nilpotent on z').

Proposition 2.10. *Let $\xi\colon w\to\mathbb{1}$ be a morphism in $\mathcal{K}$ (not necessarily rigid) such that $\xi\otimes\operatorname{cone}(\xi)=0$. Then the cone of $\xi^{\otimes n}$ generates the same tt-ideal, for all n:*

$$\langle\operatorname{cone}(\xi)\rangle=\big\{\,z\in\mathcal{K}\,\big|\,\xi\textit{ is }\otimes\textit{-nilpotent on }z\,\big\}=\langle\operatorname{cone}(\xi^{\otimes n})\rangle\,.$$

Proof. The assumption $\xi\otimes\operatorname{cone}(\xi)=0$ implies that the object $\operatorname{cone}(\xi)$ belongs to $\big\{\,z\in\mathcal{K}\,\big|\,\xi\text{ is }\otimes\text{-nilpotent on }z\,\big\}$, which is a tt-ideal by Proposition 2.9. On the other hand, if the morphism $\xi^{\otimes n}\otimes z$ is zero then the exact triangle

$$w^{\otimes n}\otimes z\xrightarrow{\xi^{\otimes n}\otimes z=0}z\longrightarrow\operatorname{cone}(\xi^{\otimes n})\otimes z\longrightarrow\Sigma w^{\otimes n}\otimes z$$

implies that z is a summand of $\operatorname{cone}(\xi^{\otimes n})\otimes z$. Hence z belongs to $\langle\operatorname{cone}(\xi^{\otimes n})\rangle$. Finally, in the Verdier quotient $\mathcal{K}/\langle\operatorname{cone}(\xi)\rangle$, the morphism ξ is an isomorphism, hence so is $\xi^{\otimes n}$. Therefore $\operatorname{cone}(\xi^{\otimes n})\in\langle\operatorname{cone}(\xi)\rangle$. In short, we have obtained

$$\langle\operatorname{cone}(\xi)\rangle\subseteq\big\{\,z\in\mathcal{K}\,\big|\,\xi^{\otimes n}\otimes z=0\text{ for some }n\geq1\,\big\}\subseteq\cup_{n\geq1}\langle\operatorname{cone}(\xi^{\otimes n})\rangle\subseteq\langle\operatorname{cone}(\xi)\rangle\,.$$

This proves the claim. Compare [Bal10a, § 2]. □

We can now establish the key observation of the paper:

Corollary 2.11. *Let $x\in\mathcal{K}$ be a rigid object in a (not necessarily rigid) tt-category $\mathcal{K}$. Choose ξ_x a 'homotopy fiber' of the coevaluation morphism η_x of (2.3), i.e. choose an exact triangle in $\mathcal{K}$*

$$w_x\xrightarrow{\xi_x}\mathbb{1}\xrightarrow{\eta_x}x^\vee\otimes x\longrightarrow\Sigma w_x\tag{2.12}$$

for a morphism ξ_x. Then the tt-ideal $\langle x\rangle$ generated by our object is exactly the subcategory on which ξ_x is $\otimes$-nilpotent:

$$\langle x\rangle=\big\{\,z\in\mathcal{K}\,\big|\,\xi_x^{\otimes n}\otimes z=0\textit{ for some }n\geq1\,\big\}\,.\tag{2.13}$$

Moreover, for every $n\geq1$ the morphism $\xi_x^{\otimes n}$ is $\otimes$-nilpotent on its cone.

Proof. Consider the exact triangle obtained by tensoring (2.12) with x:

$$x\otimes w_x\xrightarrow{x\otimes\xi_x}x\xrightarrow{x\otimes\eta_x}x\otimes x^\vee\otimes x\xrightarrow{x\otimes\zeta_x}\Sigma x\otimes w_x$$

By the unit-counit relation (2.4), the morphism $x\otimes\eta_x$ is a monomorphism. This forces $x\otimes\xi_x=0$. Hence $\xi_x\otimes\operatorname{cone}(\xi_x)\simeq\xi_x\otimes x^\vee\otimes x=0$ and we can apply Proposition 2.10 to $\xi=\xi_x$. It gives us (2.13) since $\langle\operatorname{cone}(\xi_x)\rangle=\langle x^\vee\otimes x\rangle=\langle x\rangle$ by rigidity of x. The 'moreover part' also follows from Proposition 2.10 where we proved that ξ is $\otimes$-nilpotent on $\operatorname{cone}(\xi^{\otimes n})$. □

The above result allows us to translate questions about tt-ideals into a $\otimes$-nilpotence problem. We isolate a surjectivity argument that we shall use twice.

Lemma 2.14. *Under Hypotheses 1.1, choose for every $x\in\mathcal{K}$ an exact triangle as in (2.12). Let $\mathcal{P}\in\operatorname{Spc}(\mathcal{K})$ be a prime. Suppose that $\mathcal{P}$ satisfies the following technical condition:*

$$\textit{For all }x\in\mathcal{P},\ \textit{all }s\in\mathcal{K}\smallsetminus\mathcal{P}\textit{ and all }n\geq1,\ \textit{we have }F(\xi_x^{\otimes n}\otimes s)\neq0.\tag{2.15}$$

Then $\mathcal{P}$ belongs to the image of $\varphi\colon\operatorname{Spc}(\mathcal{L})\to\operatorname{Spc}(\mathcal{K})$.

Proof. Consider the complement $S = \mathcal{K} \smallsetminus \mathcal{P}$. Let $\mathcal{J} \subseteq \mathcal{L}$ be the tt-ideal generated by $F(\mathcal{P})$, just viewed as a class of objects in $\mathcal{L}$. We claim that $\mathcal{J} = \langle F(\mathcal{P})\rangle$ equals

$$\mathcal{J}' := \left\{\, y \in \mathcal{L} \,\middle|\, \text{there exists } x \in \mathcal{P} \text{ such that } y \in \langle F(x)\rangle \,\right\}.$$

Indeed, since we have $F(\mathcal{P}) \subseteq \mathcal{J}' \subseteq \mathcal{J}$ directly from the definitions, it suffices to show that $\mathcal{J}'$ is a tt-ideal. It is clearly thick and a $\otimes$-ideal. For closure under cones, if $y_1 \to y_2 \to y_3 \to \Sigma y_1$ is exact in $\mathcal{L}$ and $y_i \in \langle F(x_i)\rangle$ for $x_i \in \mathcal{P}$ and $i = 1, 2$, then $y_3 \in \langle y_1, y_2\rangle \subseteq \langle F(x_1), F(x_2)\rangle = \langle F(x_1 \oplus x_2)\rangle$ and $x_1 \oplus x_2$ still belongs to $\mathcal{P}$.

Now, for every object $x \in \mathcal{K}$, the tt-functor $F\colon \mathcal{K} \to \mathcal{L}$ sends an exact triangle over the unit η_x as in (2.12) to an exact triangle in $\mathcal{L}$:

$$F(w_x) \xrightarrow{F(\xi_x)} \mathbb{1} \xrightarrow{\eta_{F(x)}} F(x)^\vee \otimes F(x) \xrightarrow{F(\zeta_x)} \Sigma F(w_x)\,.$$

Here we use that $F(\eta_x) = \eta_{F(x)}$ which is another way of saying that F preserves duals. See Remark 2.1. Using this last exact triangle in Corollary 2.11 for the rigid object $F(x)$ in the tt-category $\mathcal{L}$, we see that

$$\langle F(x)\rangle = \left\{\, y \in \mathcal{L} \,\middle|\, F(\xi_x)^{\otimes n} \otimes y = 0 \text{ for some } n \geq 1 \,\right\}.$$

Combining this with the description of $\mathcal{J} = \langle F(\mathcal{P})\rangle$ as $\mathcal{J}'$ above, we obtain

$$\langle F(\mathcal{P})\rangle = \left\{\, y \in \mathcal{L} \,\middle|\, F(\xi_x)^{\otimes n} \otimes y = 0 \text{ for some } n \geq 1 \text{ and some } x \in \mathcal{P} \,\right\}.$$

It follows that if $s \in S = \mathcal{K} \smallsetminus \mathcal{P}$ then $F(s)$ cannot belong to $\mathcal{J} = \langle F(\mathcal{P})\rangle$. Indeed, if $F(s) \in \langle F(\mathcal{P})\rangle$ then by the above there exists $x \in \mathcal{P}$ and $n \geq 1$ such that $0 = F(\xi_x)^{\otimes n} \otimes F(s) \cong F(\xi_x^{\otimes n} \otimes s)$ since F is a $\otimes$-functor. This contradicts (2.15).

In short, we have shown that the $\otimes$-multiplicative class $F(S) = F(\mathcal{K} \smallsetminus \mathcal{P})$ does not meet the tt-ideal $\mathcal{J} = \langle F(\mathcal{P})\rangle$, in the tt-category $\mathcal{L}$. By the existence trick (A) again, there exists a prime $\mathcal{Q}$ satisfying the following two relations: $\mathcal{J} \subseteq \mathcal{Q}$ and $F(S) \cap \mathcal{Q} = \varnothing$. Unpacking the definition of $S = \mathcal{K} \smallsetminus \mathcal{P}$ and $\mathcal{J} = \langle F(\mathcal{P})\rangle$, these two relations mean respectively $\mathcal{P} \subseteq F^{-1}(\mathcal{Q})$ and $F^{-1}(\mathcal{Q}) \subseteq \mathcal{P}$. Hence $\mathcal{P} = F^{-1}(\mathcal{Q}) = \varphi(\mathcal{Q})$ as wanted. □

We are now ready to prove our main result.

Proof of Theorem 1.4.

(a)⇒(b): Suppose that $\varphi\colon \mathrm{Spc}(\mathcal{L}) \to \mathrm{Spc}(\mathcal{K})$ is surjective and let $f\colon x \to y$ be a morphism such that $F(f) = 0$ and which is $\otimes$-nilpotent on its cone, say $f^{\otimes m} \otimes \mathrm{cone}(f) = 0$. It follows from the exact triangle $x \xrightarrow{f} y \to \mathrm{cone}(f) \to \Sigma x$ in $\mathcal{K}$ and from $F(f) = 0$ that $F(\mathrm{cone}(f)) \simeq F(y) \oplus \Sigma F(x)$ in $\mathcal{L}$. Taking supports, we have $\mathrm{supp}(F(\mathrm{cone}(f))) = \mathrm{supp}(F(x)) \cup \mathrm{supp}(F(y))$. By (D), this translates into

$$\varphi^{-1}(\mathrm{supp}(\mathrm{cone}(f))) = \varphi^{-1}(\mathrm{supp}(x)) \cup \varphi^{-1}(\mathrm{supp}(y)) = \varphi^{-1}(\mathrm{supp}(x) \cup \mathrm{supp}(y))\,.$$

Since φ is surjective, this implies $\mathrm{supp}(\mathrm{cone}(f)) = \mathrm{supp}(x) \cup \mathrm{supp}(y)$. Therefore $x, y \in \langle \mathrm{cone}(f)\rangle$. But we assumed that f is $\otimes$-nilpotent on $\mathrm{cone}(f)$ and it follows from Proposition 2.9 that f is also $\otimes$-nilpotent on x and on y. This means that there exists $n \geq 1$ such that $f^{\otimes n} \otimes x = 0\colon x^{\otimes(n+1)} \to y^{\otimes n} \otimes x$. But then $f^{\otimes(n+1)}$ decomposes as

$$\begin{tikzcd} x^{\otimes n+1} \arrow[r, "f^{\otimes n} \otimes x = 0"'] \arrow[rr, bend left, "f^{\otimes(n+1)}"] & y^{\otimes n} \otimes x \arrow[r, "y^{\otimes n} \otimes f"'] & y^{\otimes(n+1)} \end{tikzcd}$$

and is therefore also zero, that is, $f^{\otimes(n+1)} = 0$ as wanted.

(b)⇒(a): Suppose that $F\colon \mathcal{K} \to \mathcal{L}$ detects $\otimes$-nilpotence of those morphisms which are already zero on their cone. Let $\mathcal{P} \in \operatorname{Spc}(\mathcal{K})$ be a prime and let us show that property (2.15) in Lemma 2.14 is satisfied. Let $g = \xi_x^{\otimes n} \otimes s$ be the morphism in (2.15) for some objects $x \in \mathcal{P}$ and $s \in \mathcal{K} \smallsetminus \mathcal{P}$ and for $n \geq 1$. Suppose *ab absurdo* that $F(g) = 0$. The cone of $g = \xi_x^{\otimes n} \otimes s$ is simply $\operatorname{cone}(\xi_x^{\otimes n}) \otimes s$. By Corollary 2.11, $\xi_x^{\otimes n}$ is $\otimes$-nilpotent on its cone. Hence g is $\otimes$-nilpotent on its cone as well. We can therefore apply our assumption (b) to g and deduce from the (absurd) assumption $F(g) = 0$ that $g = \xi_x^{\otimes n} \otimes s$ is $\otimes$-nilpotent. In other words, ξ_x is $\otimes$-nilpotent on $s^{\otimes m}$ for some $m \geq 1$. By Corollary 2.11 again, this implies that $s^{\otimes m}$ belongs to $\langle x \rangle \subseteq \mathcal{P}$, and therefore $s \in \mathcal{P}$ since $\mathcal{P}$ is prime, a contradiction with the choice of s in $S = \mathcal{K} \smallsetminus \mathcal{P}$. In short, we have verified property (2.15) of Lemma 2.14 for the prime $\mathcal{P}$, which tells us that $\mathcal{P}$ belongs to the image of φ as claimed. □

* * *

Let us now prove Theorems 1.6 and 1.7. We therefore assume the existence of an adjoint $U\colon \mathcal{L} \to \mathcal{K}$ to our tensor-triangulated functor F:

$$
\begin{tikzcd}
\mathcal{K} \arrow[d, shift right, "F"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{L} \arrow[u, shift right, "U"']
\end{tikzcd}
\tag{2.16}
$$

By general theory, U must satisfy a projection formula

$$
U(F(x) \otimes z) \cong x \otimes U(z) \tag{2.17}
$$

for all $x \in \mathcal{K}$ and $z \in \mathcal{L}$. The latter is an easy consequence of rigidity of x and the adjunctions (2.2) and (2.16). See for instance [FHM03, Prop. 3.2].

Proof of Theorem 1.7. Let $\mathcal{P} \in \operatorname{Spc}(\mathcal{K})$. We need to show that $\mathcal{P} \in \operatorname{im}(\varphi)$ if and only if $\mathcal{P} \in \operatorname{supp}(U(\mathbb{1}))$. The latter means $U(\mathbb{1}) \notin \mathcal{P}$.

Suppose first that $\mathcal{P} = \varphi(\mathcal{Q})$ for some $\mathcal{Q} \in \operatorname{Spc}(\mathcal{L})$. Then $\mathcal{P} = F^{-1}(\mathcal{Q})$. To show $U(\mathbb{1}) \notin \mathcal{P}$ it therefore suffices to show that $FU(\mathbb{1}) \notin \mathcal{Q}$. This is easy since, by the unit-counit relation for (2.16), the object $FU(\mathbb{1}_{\mathcal{L}}) \cong FUF(\mathbb{1}_{\mathcal{K}})$ admits $F(\mathbb{1}_{\mathcal{K}}) \cong \mathbb{1}_{\mathcal{L}}$ as a direct summand and $\mathbb{1}$ cannot belong to any prime.

The reverse inclusion is the interesting one. So, let $\mathcal{P} \in \operatorname{supp}(U(\mathbb{1}))$, meaning $U(\mathbb{1}) \notin \mathcal{P}$. Let us show that $\mathcal{P}$ satisfies condition (2.15) of Lemma 2.14. Take objects $x \in \mathcal{P}$ and $s \in \mathcal{K} \smallsetminus \mathcal{P}$, and suppose *ab absurdo* that $F(g) = 0$ where $g = \xi_x^{\otimes n} \otimes s$ for some $n \geq 1$ as before. By the projection formula (2.17) for $z = \mathbb{1}$, the property $UF(g) = U(0) = 0$ implies $g \otimes U(\mathbb{1}) = 0$. Consequently we have an exact triangle

$$
w_x^{\otimes n} \otimes s \otimes U(\mathbb{1}) \xrightarrow{g \otimes U(\mathbb{1}) = 0} s \otimes U(\mathbb{1}) \longrightarrow \operatorname{cone}(g) \otimes U(\mathbb{1}) \longrightarrow \Sigma w_x^{\otimes n} \otimes s \otimes U(\mathbb{1})
$$

in $\mathcal{K}$. This proves that $s \otimes U(\mathbb{1})$ is a direct summand of $\operatorname{cone}(g) \otimes U(\mathbb{1}) \in \langle \operatorname{cone}(g) \rangle \subseteq \langle \operatorname{cone}(\xi_x^{\otimes n}) \rangle$. By Proposition 2.10, the latter is contained in $\langle x \rangle \subseteq \mathcal{P}$. In short, we have $s \otimes U(\mathbb{1}) \in \mathcal{P}$. Since $\mathcal{P}$ is prime this forces $s \in \mathcal{P}$ or $U(\mathbb{1}) \in \mathcal{P}$, which are both absurd. So we have proven (2.15) for $\mathcal{P}$ and we conclude by Lemma 2.14 again. □

Proof of Theorem 1.6. In view of Theorem 1.7 it suffices to prove that $F\colon \mathcal{K} \to \mathcal{L}$ detects $\otimes$-nilpotence if and only if $\operatorname{supp}(U(\mathbb{1})) = \operatorname{Spc}(\mathcal{K})$, which means $\langle U(\mathbb{1}) \rangle = \mathcal{K}$. This is a standard argument, as in [Bal16a, Prop. 3.15] for instance. Let us outline it for completeness. The point is that $A := U(\mathbb{1})$ is a ring-object (for U is lax-monoidal). Let $J \xrightarrow{\xi} \mathbb{1} \xrightarrow{u} A \to \Sigma J$ be an exact triangle over the unit $u\colon \mathbb{1} \to A$

(the unit of the $F \dashv U$ adjunction at $\mathbb{1}$). We have $A \otimes \xi = 0$ (since $A \otimes u$ is a split monomorphism, retracted by multiplication $A \otimes A \to A$). A morphism $f\colon x \to y$ satisfies $F(f) = 0$ if and only if the composite $x \xrightarrow{f} y \xrightarrow{u \otimes y} A \otimes y$ is zero (by adjunction and the projection formula: $A \otimes - \simeq UF(-)$); this is in turn equivalent to the morphism $f\colon x \to y$ factoring via $\xi \otimes y\colon J \otimes y \to y$ (by the exact triangle $J \otimes y \xrightarrow{\xi \otimes y} y \xrightarrow{u \otimes y} A \otimes y \longrightarrow \Sigma J \otimes y$). So we are down to proving that $\xi\colon J \to \mathbb{1}$ is $\otimes$-nilpotent if and only if $\langle A \rangle = \mathcal{K}$. This is now immediate from Proposition 2.10, which says that $\langle A \rangle = \{ z \in \mathcal{K} \mid \xi \text{ is } \otimes\text{-nilpotent on } z \}$. Indeed, $\mathbb{1} \in \langle A \rangle$ if and only if ξ is $\otimes$-nilpotent on $\mathbb{1}$. $\square$

References

[Bal05] Paul Balmer. The spectrum of prime ideals in tensor triangulated categories. *J. Reine Angew. Math.*, 588:149–168, 2005.

[Bal10a] Paul Balmer. Spectra, spectra, spectra – tensor triangular spectra versus Zariski spectra of endomorphism rings. *Algebr. Geom. Topol.*, 10(3):1521–1563, 2010.

[Bal10b] Paul Balmer. Tensor triangular geometry. In *International Congress of Mathematicians, Hyderabad (2010), Vol. II*, pages 85–112. Hindustan Book Agency, 2010.

[Bal16a] Paul Balmer. The derived category of an étale extension and the separable Neeman-Thomason theorem. *J. Inst. Math. Jussieu*, 15(3):613–623, 2016.

[Bal16b] Paul Balmer. Separable extensions in tensor-triangular geometry and generalized Quillen stratification. *Ann. Sci. Éc. Norm. Supér. (4)*, 49(4):907–925, 2016.

[BDS16] Paul Balmer, Ivo Dell'Ambrogio, and Beren Sanders. Grothendieck-Neeman duality and the Wirthmüller isomorphism. *Compos. Math.*, 152(8):1740–1776, 2016.

[Del10] Ivo Dell'Ambrogio. Tensor triangular geometry and KK-theory. *J. Homotopy Relat. Struct.*, 5(1):319–358, 2010.

[DHS88] Ethan S. Devinatz, Michael J. Hopkins, and Jeffrey H. Smith. Nilpotence and stable homotopy theory. I. *Ann. of Math. (2)*, 128(2):207–241, 1988.

[DT12] Ivo Dell'Ambrogio and Gonçalo Tabuada. Tensor triangular geometry of non-commutative motives. *Adv. Math.*, 229(2):1329–1357, 2012.

[FHM03] H. Fausk, P. Hu, and J. P. May. Isomorphisms between left and right adjoints. *Theory Appl. Categ.*, 11:No. 4, 107–131, 2003.

[HO16] Jeremiah Heller and Kyle Ormsby. Primes and fields in stable motivic homotopy theory. Geometry & Topology, to appear. Preprint `arXiv:1608.02876`, 2016.

[HPS97] Mark Hovey, John H. Palmieri, and Neil P. Strickland. Axiomatic stable homotopy theory. *Mem. Amer. Math. Soc.*, 128(610), 1997.

[HS98] Michael J. Hopkins and Jeffrey H. Smith. Nilpotence and stable homotopy theory. II. *Ann. of Math. (2)*, 148(1):1–49, 1998.

[MNN17] Akhil Mathew, Niko Naumann, and Justin Noel. Nilpotence and descent in equivariant stable homotopy theory. *Adv. Math.*, 305:994–1084, 2017.

[Pet13] Tobias J. Peter. Prime ideals of mixed Artin-Tate motives. *Journal of K-theory*, 11(2):331–349, 2013.

[San13] Beren Sanders. Higher comparison maps for the spectrum of a tensor triangulated category. *Adv. Math.*, 247:71–102, 2013.

[Ste16] Greg Stevenson. A tour of support theory for triangulated categories through tensor triangular geometry. Preprint available at `http://arxiv.org/abs/1601.03595`, 2016.

[VSF00] Vladimir Voevodsky, Andrei Suslin, and Eric M. Friedlander. *Cycles, transfers, and motivic homology theories*, volume 143 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2000.

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