PERMUTATION, STABILIZATION AND DECOMPOSITION

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Abstract. Informed by our understanding of the tt-geometry of permutation modules, we investigate the proper definition of the 'stable permutation category' of a finite group. Then we prove that this category decomposes over cyclic and generalized quaternion groups and only in those cases.

Dedicated to Dave Benson on the occasion of his seventieth birthday

1. Introduction

For the whole paper, G is a finite group and k a field of characteristic $p > 0$. We introduce the *stable permutation category* of G with coefficients in k

$$\mathrm{StPerm}(G;k).$$

It is to the usual stable module category $\mathrm{StMod}(kG)$, of kG-modules modulo projectives, what permutation modules are to general modules. However $\mathrm{StPerm}(G;k)$ is *not* defined as the additive quotient of the category of permutation modules by the subcategory of projectives. We explain in Remark 2.6 why this definition would not be very interesting. In fact, our stable permutation category $\mathrm{StPerm}(G;k)$ is not even a subcategory of $\mathrm{StMod}(kG)$ but rather an intermediate localization between the *derived category of permutation modules* $\mathrm{DPerm}(G;k)$ introduced in [BG23b] and the stable module category $\mathrm{StMod}(kG)$. Let us remind the reader.

We write $\mathrm{perm}(G;k)^\natural$ for the additive subcategory of p-permutation kG-modules inside the abelian category $\mathrm{mod}(kG)$ of finitely generated kG-modules (Recollection 2.3). The bounded homotopy category $\mathrm{K_b}(\mathrm{perm}(G;k)^\natural)$ is the compact part of $\mathrm{DPerm}(G;k)$. On the other hand, the bounded derived category $\mathrm{D_b}(kG) := \mathrm{D_b}(\mathrm{mod}(kG))$ is the compact part of $\mathrm{K}(\mathrm{Inj}(kG))$ by Krause [Kra05]. The stable permutation category $\mathrm{StPerm}(G;k)$ fits in a commutative square of localizations:

$$(1.1)\qquad \begin{array}{ccc} \mathrm{DPerm}(G;k) & \xrightarrow{\Upsilon} & \mathrm{K}(\mathrm{Inj}(kG)) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ \mathrm{StPerm}(G;k) & \xrightarrow{\bar{\Upsilon}} & \mathrm{StMod}(kG) \end{array} \qquad \begin{array}{ccc} \mathrm{K_b}(\mathrm{perm}(G;k)^\natural) & \xrightarrow{\Upsilon} & \mathrm{D_b}(kG) \\ {\scriptstyle p}\downarrow{\scriptstyle \natural} & & \downarrow{\scriptstyle q} \\ \mathrm{stperm}(G;k) & \xrightarrow{\bar{\Upsilon}} & \mathrm{stmod}(kG). \end{array}$$

The left-hand square displays big categories and the right-hand one the corresponding compact parts. In each square the left-hand column involves permutation modules, the right-hand column involves arbitrary kG-modules, and the horizontal functors Υ and $\bar{\Upsilon}$ are induced by the canonical inclusion $\mathrm{perm}(G;k)^\natural \hookrightarrow \mathrm{mod}(kG)$. The vertical functor q is the finite localization away from perfect complexes, by

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Rickard [Ric89]. We *define* $\mathrm{StPerm}(G;k)$ as the finite localization of $\mathrm{DPerm}(G;k)$ away from so-called *equivariantly perfect* complexes; the latter are those bounded complexes of p-permutation modules that are not only perfect in the derived category $\mathrm{D_b}(kG)$ but such that *all their modular H-fixed points* (a. k. a. Brauer quotients) are perfect, for all p-subgroups $H \le G$. See details in Section 2.

We prove that this stable permutation category is also the localization of the derived permutation category $\mathrm{DPerm}(G;k)$ on the open complement of the finitely many closed points in its spectrum. Conjecturally, $\mathrm{StPerm}(G;k)$ is also the localization of $\mathrm{DPerm}(G;k)$ on its 'periodic locus' in the sense of [Gal25]. This fact is known to hold for a class of groups containing p-groups but remains a conjecture in full generality. See details in Section 3.

With the definition established and justified by these alternate interpretations, we investigate when this stable permutation category $\mathrm{StPerm}(G;k)$ is indecomposable as a tt-category. This question might come as a surprise since the other three types of categories in (1.1) are always indecomposable, for instance because the ring of endomorphisms of their $\otimes$-unit $\mathrm{End}(\mathbb{1}) = k$ is indecomposable. And yet the stable permutation category can indeed decompose, in very special cases.

1.2. **Theorem** (Theorem 6.1)**.** *The stable permutation category* $\mathrm{StPerm}(G;k)$ *is indecomposable unless the p-Sylow of G is cyclic or generalized quaternion.*

We obtain this result by proving that the spectrum is connected as a topological space. Conversely, for G with cyclic or generalized quaternion p-Sylow, the spectrum of $\mathrm{stperm}(G;k)$ is disconnected and this forces $\mathrm{StPerm}(G;k)$ to be a product of two tt-categories, or more. Amusingly, the connected components of the spectrum that appear in those cases are reminiscent of spectra of other tt-categories and this observation led us to *predict* what the decompositions of $\mathrm{StPerm}(G;k)$ should be at the categorical level. In both cases, our geometric guess was confirmed:

1.3. **Theorem** (Theorem 7.10)**.** *The stable permutation category* $\mathrm{StPerm}(C_{p^n};k)$ *of the cyclic group of order p^n is tt-equivalent to the product of the n usual stable module categories* $\mathrm{StMod}(kC_p) \times \cdots \times \mathrm{StMod}(kC_{p^i}) \times \cdots \times \mathrm{StMod}(kC_{p^n})$.

1.4. **Theorem** (Theorem 7.6)**.** *For $p = 2$ and $n \ge 3$, the stable permutation category* $\mathrm{StPerm}(Q_{2^n};k)$ *of the generalized quaternion group Q_{2^n} of order 2^n is tt-equivalent to the product* $\mathrm{StPerm}(D_{2^{n-1}};k) \times \mathrm{StMod}(kQ_{2^n})$ *of the stable permutation category of the dihedral group $D_{2^{n-1}}$ of order 2^{n-1} and the usual stable module category of Q_{2^n}. Both of these factors are indecomposable.*

The organization of the paper is the following. Section 2 gives the definition of $\mathrm{StPerm}(G;k)$, with equivalent formulations and motivation from the perspective of tt-geometry in Section 3. We gather some basic properties in Section 4, including Mackey 2-functoriality. We also prove that the modular fixed-points (Brauer quotients) survive on stable permutation categories and we prove a 'Colimit Theorem' reducing the description of the spectrum of $\mathrm{StPerm}(G;k)$ to elementary abelian *subquotients* of G, analogous to the result for $\mathrm{DPerm}(G;k)$ in [BG25a]. In Section 6, we prove Theorem 1.2, after some group-theoretic preparation in Section 5. In Section 7 we give the decompositions of Theorems 1.3 and 1.4. In fact, similar decompositions hold for any finite group G whose p-Sylow is cyclic or generalized quaternion.

2. Definition via equivariantly perfect complexes

We write $\mathrm{mod}(kG)$ for the abelian category of finitely generated kG-modules and write $\mathrm{D_b}(kG)$ for its bounded derived category. Before turning to permutations, let us remind the reader of the usual stable *module* category.

2.1. *Recollection.* The *stable module category* $\mathrm{stmod}(kG)$ is the additive quotient of the category $\mathrm{mod}(kG)$ by the subcategory of projective modules. (Additive quotients by a subcategory are obtained by keeping the same objects and by modding out the morphisms that factor via an object of the subcategory.) It is triangulated by Happel [Hap88]. In fact, by Rickard [Ric89], we have a triangular-equivalence

$$\mathrm{stmod}(kG) \cong \mathrm{D_b}(kG)/\,\mathrm{D_{perf}}(kG) \tag{2.2}$$

with the Verdier quotient of the derived category $\mathrm{D_b}(kG)$ by the thick subcategory of perfect complexes. The category $\mathrm{D_b}(kG)/\,\mathrm{D_{perf}}(kG)$ is effortlessly triangulated, as any Verdier quotient. In contrast, we need kG to be Frobenius to obtain the equivalence (2.2) and to show that the additive quotient $\mathrm{stmod}(kG)$ is triangulated.

The problem we want to address is to find the correct analogue of the stable module category $\mathrm{stmod}(kG)$ in the context of *permutation modules*. We assume that the reader is aware of the importance of permutation modules in representation theory and beyond, *e.g.* in the theory of motives. See [BG23b] if necessary.

2.3. *Recollection.* The full additive subcategory $\mathrm{perm}(G;k)$ of $\mathrm{mod}(kG)$ of *permutation* kG-modules consists of those isomorphic to $k(X)$ for a finite G-set X. We slightly enlarge it by including direct summands $\mathrm{perm}(G;k)^\natural \subseteq \mathrm{mod}(kG)$; the latter are called *p-permutation* or *trivial source* kG-modules. (If G is a p-group then $\mathrm{perm}(G;k)^\natural = \mathrm{perm}(G;k)$.) As $\mathrm{perm}(G;k)^\natural$ is just an additive category, its bounded derived category is simply its bounded homotopy category, no need to invert quasi-isomorphisms. We shall denote it $\mathcal{K}(G)$ as we did in [BG25a]:

$$\mathcal{K}(G) := \mathrm{D_b}(\mathrm{perm}(G;k)^\natural) = \mathrm{K_b}(\mathrm{perm}(G;k)^\natural).$$

The 'big' *derived category of permutation modules* $\mathrm{DPerm}(G;k)$ of [BG25a] is a rigidly-compactly generated tt-category whose compact part is the above $\mathcal{K}(G)$. This 'Ind-completion' can be realized for instance inside the unbounded homotopy category $\mathrm{K}(\mathrm{Perm}(G;k))$ of not necessarily finitely generated permutation modules, as the localizing subcategory generated by $\mathrm{perm}(G;k)$. Its rigid-compact objects is indeed the subcategory $\mathrm{DPerm}(G;k)^c = \mathrm{thick}(\mathrm{perm}(G;k)) = \mathcal{K}(G)$.

Alternatively, $\mathrm{DPerm}(G;k)$ is the homotopy category of modules over the Bredon cohomology spectrum $H\underline{k}$ in genuine G-spectra, see [Fuh25].

2.4. *Remark.* The inclusion $\mathrm{perm}(G;k)^\natural \subseteq \mathrm{mod}(kG)$ does not yield an inclusion on derived categories. Quite the opposite, the canonical functor, that we denote $\Upsilon\colon \mathcal{K}(G) \to \mathrm{D_b}(kG)$, is actually a Verdier quotient by [BG23a, Theorem 5.13]. As $\mathrm{stmod}(kG)$ is a further Verdier quotient of $\mathrm{D_b}(kG)$, we immediately obtain:

2.5. **Corollary.** *The thick subcategory of* $\mathrm{stmod}(kG)$ *generated by permutation modules is the whole* $\mathrm{stmod}(kG)$. □

2.6. *Remark.* Since free modules are permutation, the category $\mathrm{perm}(G;k)^\natural$ contains the subcategory $\mathrm{proj}(kG)$ of finitely generated projective kG-modules. One can then form the *additive* quotient

$$\frac{\mathrm{perm}(G;k)^\natural}{\mathrm{proj}(kG)}. \tag{2.7}$$

This additive category has been called the 'stable category of p-permutation modules' in (unpublished) literature. We shall avoid this terminology. The additive quotient (2.7) is clearly a full subcategory of the stable module category $\mathrm{stmod}(kG)$. However (2.7) is not triangulated in any evident way. In fact (2.7) generates the whole $\mathrm{stmod}(kG)$ as a thick triangulated subcategory by Corollary 2.5.

In view of the above discussion, we propose to define $\mathrm{stperm}(G;k)$ as a suitable localization of the bounded derived category of p-permutation modules $\mathcal{K}(G)$. We present justifications in the subsequent Section 3.

2.8. *Recollection.* Let $H \le G$ be a p-subgroup and write $G/\!\!/H = N_G(H)/H$ for the Weyl group of H in G. There exists (see [BG25a, § 5]) a tensor functor

$$\Psi^H\colon\ \mathrm{perm}(G;k)^\natural \to \mathrm{perm}(G/\!\!/H;k)^\natural$$

characterized by the property that $\Psi^H(kX) \cong k(X^H)$ for every finite G-set X. It is called *modular H-fixed-points* or *Brauer quotient*. It induces a tt-functor on homotopy categories (applying Ψ^H degreewise on complexes) that we still denote $\Psi^H\colon \mathcal{K}(G) \to \mathcal{K}(G/\!\!/H)$. Composed with the canonical localization of Remark 2.4 we obtain a tt-functor to the derived category of the Weyl group

$$\check{\Psi}^H\colon \mathcal{K}(G) \xrightarrow{\Psi^H} \mathcal{K}(G/\!\!/H) \overset{\Upsilon}{\twoheadrightarrow} \mathrm{D_b}(k(G/\!\!/H)).$$

2.9. *Definition.* A bounded complex $C \in \mathcal{K}(G)$ of p-permutation kG-modules is called *equivariantly perfect* if its image under modular H-fixed-points functor $\Psi^H(C)$ is a perfect complex over the Weyl group of H, for every p-subgroup $H \le G$. This means that $\Psi^H(C)$ is *quasi-isomorphic* to a bounded complex of finitely generated projective $k(G/\!\!/H)$-modules, or in formula $\check{\Psi}^H(C) \in \mathrm{D_{perf}}(k(G/\!\!/H))$, for every H. Let us denote by

$$\mathcal{K}_{\text{eq-perf}}(G) = \mathrm{DPerm}(G;k)^c_{\text{eq-perf}}$$

the subcategory of $\mathcal{K}(G) = \mathrm{DPerm}(G;k)^c$ consisting of equivariantly perfect complexes. Equivalently,

$$\mathcal{K}_{\text{eq-perf}}(G) = \mathrm{Ker}\left(\mathcal{K}(G) \xrightarrow{(\Psi^H)} \prod_{H\le G} \mathrm{D_b}(k(G/\!\!/H)) \xrightarrow{(2.2)} \prod_{H\le G} \mathrm{stmod}(k(G/\!\!/H))\right)$$

can be written as the kernel of a tt-functor (where H runs through the (conjugacy classes of) p-subgroups). In particular, it is a tt-ideal.

2.10. *Example.* Every bounded complex of projective kG-modules is equivariantly perfect. Indeed, $\Psi^H(kG)$ is kG for $H = 1$ and zero for $H \neq 1$.

2.11. *Example.* Let $G = C_p$ and $C = 0 \to k \to kC_p \to kC_p \to k \to 0$ the usual acyclic complex. For $H = 1$, the complex $\Psi^1(C) = C$ is quasi-isomorphic to zero hence it is perfect as complex of kG-modules. And for $H = C_p$, the complex $\Psi^{C_p}(C) = 0 \to k \to 0 \to 0 \to k \to 0$ is perfect over the trivial group $G/\!\!/H = 1$, as every complex of k-vector spaces is. It follows that C is equivariantly perfect.

2.12. *Remark.* Let us dispel any possible confusion between the equivariantly perfect complexes of Definition 2.9 and two adjacent notions

$$\mathrm{K_b}(\mathrm{proj}(kG)) \ \subseteq\ \mathcal{K}_{\text{eq-perf}}(G) \ \subseteq\ \Upsilon^{-1}(\mathrm{D_{perf}}(kG)).$$

(a) Every bounded complex of projective kG-modules is equivariantly perfect (Example 2.10). It follows that, inside $\mathcal{K}(G)$, we have an inclusion of tt-ideals $\langle kG\rangle = \mathrm{K_b}(\mathrm{proj}(kG)) \subseteq \mathcal{K}_{\text{eq-perf}}(G)$. It is usually a proper inclusion, as shown in Example 2.11.

(b) By definition (take $H = 1$ in Definition 2.9), every equivariantly perfect complex is perfect as a complex of kG-modules. The converse is true for groups with very small p-Sylow but is false in general. Indeed, suppose that G has order divisible by p^2. Pick H a copy of C_p in the center of a p-Sylow of G. Our assumption implies that p divides the order of $G/\!\!/H$. Let $C = {}^{\otimes}\mathrm{Ind}_1^G(0 \to k \xrightarrow{1} k \to 0)$ be the 'Koszul object' of [BG25a, 3.15], where it is denoted $\mathrm{kos}_G(1)$. This complex is acyclic, hence trivially perfect as a complex of kG-modules. However, $\Psi^H(C)$ generates $\mathcal{K}(G/\!\!/H)$ as a tt-ideal, by [BG25a, Lemma 5.21 (for $K = 1$)]. Since $G/\!\!/H$ has order divisible by p, the category $\mathcal{K}(G/\!\!/H)$ cannot consist only of perfect complexes (for $\mathrm{stmod}(k(G/\!\!/H)) \neq 0$). Therefore $\Psi^H(C)$ is not perfect over $G/\!\!/H$.

We are now ready for our central definition.

2.13. *Definition.* The *stable permutation category* of G is the idempotent-completion of the Verdier quotient of the derived permutation category $\mathcal{K}(G) = \mathrm{K_b}(\mathrm{perm}(G;k)^\natural)$ by the tt-ideal of equivariantly perfect complexes (Definition 2.9)

$$\mathrm{stperm}(G;k) = \left(\frac{\mathcal{K}(G)}{\mathcal{K}_{\text{eq-perf}}(G)}\right)^\natural.$$

It is therefore an idempotent-complete rigid tensor-triangulated category in such a way that the $\natural$-localization functor $p\colon \mathcal{K}(G) \to \mathrm{stperm}(G;k)$ is a tt-functor.

Let $\mathrm{StPerm}(G;k) = \mathrm{DPerm}(G;k)/\,\mathrm{Loc}(\mathcal{K}_{\text{eq-perf}}(G))$ be the corresponding finite localization of the 'big' derived permutation category $\mathrm{DPerm}(G;k)$ at the tt-ideal $\mathcal{K}_{\text{eq-perf}}(G)$ of its compacts $\mathrm{DPerm}(G;k)^c = \mathcal{K}(G)$. We baptize $\mathrm{StPerm}(G;k)$ the *'big' stable permutation category* of the group G, with coefficients in k. By Neeman-Thomason localization [Nee92], we can identify $\mathrm{StPerm}(G;k)^c$ with $\mathrm{stperm}(G;k)$ – hence the idempotent-completion in the latter.

2.14. *Remark.* We do not know whether the idempotent-completion is necessary in the definition of $\mathrm{stperm}(G;k)$. The definition of $\mathrm{stmod}(kG)$, either as an additive (Recollection 2.1) or triangulated quotient through Rickard's Theorem (2.2), does not involve an idempotent-completion because one can show that it is already idempotent-complete. It is conceivable that the same holds true for permutation modules but it would require an argument.

2.15. **Proposition.** *We have the commutative diagrams of tt-functors* (1.1). *The stable module category* $\mathrm{StMod}(G;k)$ *is the finite localization of the stable permutation category* $\mathrm{StPerm}(G;k)$ *away from the tt-ideal of compacts consisting of complexes of p-permutation modules that are perfect as complexes of kG-modules.*

Proof. By Remark 2.12 (b), the equivariantly perfect complexes are perfect. This means that the kernel of p is sent to zero by $q \circ \Upsilon$ and therefore $q \circ \Upsilon$ localizes along p, to yield the unique factorization $\bar{\Upsilon}$. The kernel of the localization $\bar{\Upsilon}$ is then the image under p of the kernel of $q \circ \Upsilon$, that is, the localizing subcategory generated by complexes of p-permutation modules that are perfect in $\mathrm{D_b}(kG)$. □

3. Justifications of the definition

We want to justify our definition of the stable permutation category given in Section 2, from various tt-geometric perspectives. We remind the reader of the stable module category precedent.

3.1. *Recollection.* By [BCR97] the spectrum of $\mathrm{D_b}(kG)$ is homeomorphic, via the comparison map, to the homogeneous spectrum of the cohomology (see also [Bal10, Proposition 8.5]):

$$\mathrm{Spc}(\mathrm{D_b}(kG)) \xrightarrow{\sim} \mathrm{Spec^h}(\mathrm{H}^\bullet(G,k)). \tag{3.2}$$

This space is local: Its unique closed point is the tt-ideal $\mathcal{M} = (0)$ of $\mathrm{D_b}(kG)$ on the left-hand side of (3.2), which corresponds to $\mathrm{H}^+(G,k)$ on the right-hand side. In the language of projective algebraic geometry, $\mathrm{H}^+(G,k)$ is the 'irrelevant' ideal and removing it yields the usual projective variety $\mathrm{Proj}(\mathrm{H}^\bullet(G,k))$. At the level of the derived category $\mathrm{D_b}(kG)$, the closed point $\mathcal{M} = (0)$ is exactly the support of the tt-ideal $\mathrm{D_{perf}}(kG)$ of perfect complexes. Removing this support, the open complement $\mathrm{Spc}(\mathrm{D_b}(kG)) \smallsetminus \{\mathcal{M}\}$ becomes the spectrum of the corresponding localization $\mathrm{D_b}(kG)/\,\mathrm{D_{perf}}(kG)$. And the latter is the stable module category by (2.2). Away from the 'irrelevant' closed points, the homeomorphism (3.2) restricts to

$$\mathrm{Spc}(\mathrm{stmod}(kG)) \xrightarrow{\sim} \mathrm{Proj}(\mathrm{H}^\bullet(G,k)). \tag{3.3}$$

In summary, the stable module category $\mathrm{stmod}(kG)$ is obtained by a localization of the usual derived category $\mathrm{D_b}(kG)$ away from the irrelevant ideal, *i.e.* away from the unique closed point of its spectrum.

A similar pattern holds for the stable permutation category, with 'local' replaced by 'semi-local'. Let us recall some tt-geometry from [BG25a].

3.4. *Recollection.* The main result of [BG25a, § 7] identified the spectrum of the tt-category $\mathcal{K}(G) = \mathrm{DPerm}(G;k)^c$ of Recollection 2.3. This spectrum admits a stratification by locally closed subsets indexed by conjugacy classes of p-subgroups

$$\mathrm{Spc}(\mathcal{K}(G)) = \coprod_{(H)\in\mathrm{Sub}_p(G)/G} \mathcal{V}_{G/\!\!/H} \tag{3.5}$$

where $\mathcal{V}_{G/\!\!/H} \cong \mathrm{Spec^h}(\mathrm{H}^*(G/\!\!/H;k)) \cong \mathrm{Spc}(\mathrm{D_b}(k(G/\!\!/H)))$ is the (extended) cohomological support variety associated with the Weyl group $G/\!\!/H$, also known as the cohomological open in $\mathrm{Spc}(\mathcal{K}(G/\!\!/H))$. More precisely, for each p-subgroup $H \le G$, the tt-functor $\check{\Psi}^H\colon \mathcal{K}(G) \to \mathrm{D_b}(k(G/\!\!/H))$ of Recollection 2.8 induces a continuous map $\check{\psi}^H = \mathrm{Spc}(\check{\Psi}^H)$ that we prove to be injective $\check{\psi}^H\colon \mathrm{Spc}(\mathrm{D_b}(k(G/\!\!/H))) \hookrightarrow \mathrm{Spc}(\mathcal{K}(G))$ and whose image is our $\mathcal{V}_{G/\!\!/H}$ in (3.5). In particular, the closed point (0) of each $\mathrm{Spc}(\mathrm{D_b}(k(G/\!\!/H)))$ as in Recollection 2.1 gives a closed point

$$\mathcal{M}(H) = \mathcal{M}_G(H) := \check{\psi}^H(0) = (\check{\Psi}^H)^{-1}(0) = \mathrm{Ker}(\check{\Psi}^H\colon \mathcal{K}(G) \to \mathrm{D_b}(k(G/\!\!/H)))$$

that belongs to the stratum $\mathcal{V}_{G/\!\!/H}$. We proved in [BG25a, Corollary 7.31] that these $\mathcal{M}(H)$ are all the closed points of $\mathrm{Spc}(\mathcal{K}(G))$. In other words, $\mathcal{K}(G)$ is 'semi-local' in the sense that its spectrum admits only finitely many closed points $\mathcal{M}(H)$, one for every G-conjugacy class of p-subgroups $H \in \mathrm{Sub}_p(G)$.

3.6. **Proposition.** *An object* $C \in \mathcal{K}(G) = \mathrm{DPerm}(G;k)^c$ *is equivariantly perfect in the sense of Definition 2.9 if and only if its support is contained in the subset* $\{\, \mathcal{M}(H) \,|\, H \in \mathrm{Sub}_p(G) \,\}$ *of closed points of* $\mathrm{Spc}(\mathcal{K}(G))$.

Proof. As with any tt-functor the support of the image $\check{\Psi}^H(C)$ of an object C is the preimage $(\check{\psi}^H)^{-1}(\mathrm{supp}(C))$ of its support under the induced map $\check{\psi}^H = \mathrm{Spc}(\check{\Psi}^H)$. It follows from (3.5) that C has support in $\{\, \mathcal{M}(H) \,|\, H \in \mathrm{Sub}_p(G) \,\}$ if and only if every $\check{\Psi}^H(C) \in \mathrm{D_b}(k(G/\!\!/H))$ has support in the closed point (0) of $\mathrm{Spc}(\mathrm{D_b}(G/\!\!/H))$, which is equivalent to $\check{\Psi}^H(C)$ being perfect over $G/\!\!/H$ by Recollection 2.1. □

3.7. *Remark.* Definition 2.13 is an instance of a very general tt-construction. For any Thomason subset $Y \subseteq \mathrm{Spc}(\mathcal{K})$ of the spectrum of a tt-category $\mathcal{K}$, we form the *localization of* $\mathcal{K}$ *on the complement* $U = Y^{\complement}$ by localization-idempotent-completion

$$\mathcal{K}|_U := (\mathcal{K}/\mathcal{K}_Y)^\natural$$

We typically do this for Y closed with quasi-compact open complement U. This tt-category $\mathcal{K}|_U$ comes with a $\natural$-localization tt-functor $\mathcal{K} \to \mathcal{K}|_U$ that we sometimes refer to as restriction on U. It induces an embedding $\mathrm{Spc}(\mathcal{K}|_U) \hookrightarrow \mathrm{Spc}(\mathcal{K})$ whose image is U. In short, $\mathrm{Spc}(\mathcal{K}|_U) = U$, as it should be.

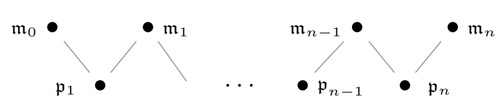
When $\mathcal{K} = \mathcal{T}^c$ is the compact part of a 'big' tt-category $\mathcal{T}$, it follows by [Nee92] that $\mathcal{T}|_U := \mathcal{T}/\,\mathrm{Loc}(\mathcal{K}_Y)$ remains a 'big' tt-category whose compact part is the idempotent-completion of the corresponding Verdier quotient of compacts $\mathcal{K}/\mathcal{K}_Y$, in other words $(\mathcal{T}|_U)^c = (\mathcal{T}^c)|_U$.

We apply this to $\mathcal{T} = \mathrm{DPerm}(G;k)$ and $\mathcal{K} = \mathcal{T}^c = \mathcal{K}(G)$ as in Recollection 2.3 and to $U \subset \mathrm{Spc}(\mathcal{K}(G))$ the open complement of the closed points. It is quasi-compact for $\mathrm{Spc}(\mathcal{K}(G))$ is noetherian by [BG25a, Proposition 9.1]. The restriction of $\mathcal{T}$ on U is $\mathcal{T}|_U = \mathrm{StPerm}(G;k)$, with compact part $\mathcal{K}(G)|_U = \mathrm{stperm}(G;k)$ by Proposition 3.6. It follows that the spectrum of the stable permutation category is the open U obtained from the semi-local $\mathrm{Spc}(\mathcal{K}(G))$ by 'puncturing out' the finitely many closed points:

3.8. **Corollary.** *The spectrum of* $\mathrm{stperm}(G;k)$ *is homeomorphic to the open subspace of* $\mathrm{Spc}(\mathcal{K}(G))$ *complement of the closed points* $\{\, \mathcal{M}(H) \,|\, H \in \mathrm{Sub}_p(G) \,\}$. □

3.9. *Example.* Let $G = C_{p^n}$ for $n \geq 1$. Write $1 = H_n < \cdots < H_1 < H_0 = G$ for the subgroups of G, with H_i of index p^i. Then by [BG25a, Proposition 8.3] the spectrum of $\mathcal{K}(C_{p^n})$ is the following space with $2n+1$ points:

(3.10) $W^n =$



The spectrum of the derived category of any non-trivial cyclic p-group (e.g., the quotients G/H_i for $i > 0$) is a Sierpiński space $\mathrm{Spc}(\mathrm{D_b}(C_{p^i})) = \{\mathfrak{p}, \mathfrak{m}\}$ with $\mathfrak{p}$ open and $\mathfrak{m}$ closed. The stratum $\mathcal{V}_{G/H_i}$ of (3.5) is the pairs $\{\mathfrak{p}_i, \mathfrak{m}_i\}$ of (3.10) for $i = 1 \ldots, n$ and the left-most stratum boils down to $\mathcal{V}_{G/G} = \{\mathfrak{m}_0\}$ for $i = 0$. Here $\mathfrak{m}_i = \mathrm{Ker}(\check{\Psi}^{H_i}) = \mathcal{M}(H_i)$ is the closed point corresponding to the p-subgroup H_i. Removing all those closed points leaves a finite and discrete subspace

$$\mathrm{Spc}(\mathrm{stperm}(C_{p^n};k)) = \{\mathfrak{p}_1\} \sqcup \ldots \sqcup \{\mathfrak{p}_n\}.$$

which is the spectrum of the stable permutation category by Corollary 3.8. By general tt-geometry [Bal07], the fact that the spectrum of $\mathcal{K} = \mathrm{stperm}(C_{p^n};k)$

is disconnected $\mathrm{Spc}(\mathcal{K}) = U_1 \sqcup \cdots \sqcup U_n$ forces the rigid idempotent-complete tt-category $\mathcal{K}$ to be the product $\mathcal{K} \cong \mathcal{K}|_{U_1} \times \cdots \times \mathcal{K}|_{U_n}$ of its local categories on each U_i. We shall compute those components in Theorem 7.10.

3.11. *Example.* Let $G = V_4 = C_2^{\times 2}$ be the Klein-four group and $p = 2$. The stratification (3.5) consists of 5 strata: a singleton (for $H = V_4$), two Sierpinski spaces (for each $H < V_4$ of index 2), and an "extended" projective line $\bar{\mathbb{P}}^1_k$, that is, a projective line with a unique closed point added on top (for $H = 1$). Removing all the closed points, Corollary 3.8 tells us that $\mathrm{Spc}(\mathrm{stperm}(G; k))$ is a $\mathbb{P}^1_k$ with three extra points (for the three strata $\mathcal{V}_{G/H}$ with $G/H \simeq C_2$). We recover in this way a space that we displayed in [BG25a, § 18]:

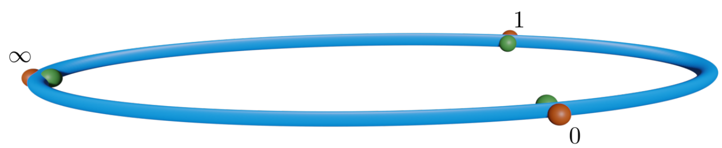


FIGURE 1. Artist rendering of $\mathrm{Spc}(\mathrm{stperm}(V_4; k))$.

3.12. *Remark.* More generally, for any elementary abelian p-group G, the strata $\mathcal{V}_{G/H}$ in (3.5) for $H \le G$ are extended projective spaces $\bar{\mathbb{P}}^{n-1}_k$, where n is the p-rank of G/H, see [BG25a, Example 8.6]. In fact, the space $\mathrm{Spc}(\mathrm{DPerm}(G; k)^c)$ has a natural structure of a Dirac scheme, see [BG25a, Corollary 15.6].

3.13. *Remark.* The punctured spectrum

$$\mathrm{Spc}(\mathrm{stmod}(kG)) \ \subset\ \mathrm{Spc}(\mathrm{D_b}(kG))$$

obtained by removing the closed point is precisely the *periodic locus* of $\mathrm{D_b}(kG)$, namely the open subset of those points $\mathcal{P} \in \mathrm{Spc}(\mathrm{D_b}(kG))$ such that the local category $\mathrm{D_b}(kG)/\mathcal{P}$ has a periodic suspension: $\Sigma^d \mathbb{1} \simeq \mathbb{1}$ for some $d \neq 0$. Similarly, we expect that the punctured spectrum

$$\mathrm{Spc}(\mathrm{stperm}(G; k)) \ \subset\ \mathrm{Spc}(\mathcal{K}(G))$$

obtained by removing the closed points is the periodic locus of $\mathcal{K}(G) = \mathrm{DPerm}(G; k)^c$. This characterization of the stable permutation category $\mathrm{stperm}(G; k)$ is established for p-groups in [Gal25, Theorem 8.5]. For general finite groups it remains conjectural at this stage.

4. General properties

Recall that the assignment $G \mapsto \mathrm{DPerm}(G; k)$ comes with restriction functors along any group homomorphisms $u\colon G \to G'$, that have an ambidextrous adjoint in case u is injective, namely (co)induction. This data forms a Mackey 2-functor in the sense of [BD20]. We claim that the 2-functor $G \mapsto \mathrm{StPerm}(G; k)$ inherits a structure of Mackey 2-functor but only for restriction along injective homomorphisms. The restriction and induction functors are compatible with those of DPerm under the localizations $\mathrm{DPerm}(G; k) \twoheadrightarrow \mathrm{StPerm}(G; k)$. This is reminiscent of what happens with the passage from the usual derived category to the stable module category, see [BD20, Example 4.2.6]. In the language of [BD20], this $G \mapsto \mathrm{StPerm}(G; k)$ is a Mackey 2-functor defined on the 2-category of finite groupoids and faithful 1-cells.

The analogous statements hold for the 2-functor $G \mapsto \mathrm{stperm}(G;k)$ of subcategories of compacts.

4.1. **Proposition.** *The equivariantly perfect complexes of p-permutation modules (Definition 2.9) are preserved by conjugation by elements in G, by restriction to subgroups and by induction from subgroups. They are also preserved by modular fixed-points functors with respect to every p-subgroup.*

Proof. For restriction Res^G_K to a subgroup $K \le G$, we can use its compatibility with modular fixed points ([BG25a, Proposition 5.15]): for every p-subgroup $H \le K$ we have $\check{\Psi}^H \circ \mathrm{Res}^G_K \cong \mathrm{Res}^{G/\!\!/H}_{K/\!\!/H} \circ \check{\Psi}^H$. Since $K/\!\!/H$ is a subgroup of $G/\!\!/H$, the restriction of a perfect complex over $G/\!\!/H$ remains perfect over $K/\!\!/H$. [BG25a, Proposition 5.15] also deals with conjugation, in the same way.

For modular fixed-point, the argument is similar, using [BG25a, Proposition 5.17] instead. Note that there is also a restriction to a subgroup involved in that case.

Finally, for induction, one can argue by means of supports, using Proposition 3.6. It suffices to check that, for every $K \le G$ and every $C \in \mathcal{K}(K)$, the support of the induced complex $\mathrm{Ind}^G_K(C)$ is equal to the image of the support of C under $\mathrm{Spc}(\mathrm{Res}^G_K)$. This is a general tt-fact about the image of a rigid object by a right adjoint to a separable extension [Bal16, Theorem 3.4 (c)]. It then suffices to use that $\mathrm{Spc}(\mathrm{Res}^G_K)$ maps closed points to closed points [BG25a, Lemma 11.9 (b)]. □

4.2. **Corollary.** *Let $H \le G$ be a subgroup. There is a well-defined coproduct-preserving tt-functor $\mathrm{Res}^G_H\colon \mathrm{StPerm}(G;k) \to \mathrm{StPerm}(H;k)$ compatible with the functor $\mathrm{Res}^G_H\colon \mathrm{DPerm}(G;k) \to \mathrm{DPerm}(H;k)$ under localization, meaning that the following diagram commutes up to isomorphism*

$$\begin{array}{ccc} \mathrm{DPerm}(G;k) & \xrightarrow{\mathrm{Res}^G_H} & \mathrm{DPerm}(H;k) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathrm{StPerm}(G;k) & \xrightarrow{\mathrm{Res}^G_H} & \mathrm{StPerm}(H;k). \end{array}$$

Restriction admits a two-sided adjoint $\mathrm{Ind}^G_H\colon \mathrm{StPerm}(H;k) \to \mathrm{StPerm}(G;k)$, compatible with $\mathrm{Ind}^G_H\colon \mathrm{DPerm}(H;k) \to \mathrm{DPerm}(G;k)$ under localization (as above).

Let $H \le G$ be a p-subgroup. There exists a tt-functor of 'stable' modular H-fixed-points $\Psi^H\colon \mathrm{StPerm}(G;k) \to \mathrm{StPerm}(G/\!\!/H;k)$, compatible with the original $\Psi^H\colon \mathrm{DPerm}(G;k) \to \mathrm{DPerm}(G/\!\!/H;k)$ under localization (as above again).

All the above functors preserve compact objects and give similar statements for stperm *instead of* StPerm, mutatis mutandis.

Proof. Once the original functors preserve the tt-ideals of compacts that we quotient out (equivariantly perfect complexes), these functors pass to the finite localizations of the 'big' categories in a unique way. Natural transformations are preserved, and therefore adjunctions follow. Since the original functors preserve compacts, so do the new ones on stable categories. □

4.3. **Corollary.** *The 2-functors $G \mapsto \mathrm{StPerm}(G;k)$ and $G \mapsto \mathrm{stperm}(G;k)$ (with respect to restriction) are Mackey 2-functors on the 2-category of finite group(oid)s and* faithful *1-cells.*

Proof. All properties, in particular Ambidexterity and the Mackey formula [BD20, Definition 1.1.7] are inherited by localization since all units and counits come from the Mackey 2-functor $\mathrm{DPerm}(-;k)$, as long as we only use restriction under faithful homomorphisms as in Corollary 4.2. We leave the details to the reader. □

4.4. **Corollary.** *For every subgroup $H \le G$ the category $\mathrm{StPerm}(H;k)$ is equivalent to the category of A^G_H-modules in $\mathrm{StPerm}(G;k)$ where the tt-ring $A^G_H = \mathrm{Ind}^G_H(\mathbb{1})$ is $k(G/H)$ with the usual multiplication coming from $\mathrm{perm}(G;k)$ (making all $\gamma \in G/H$ orthogonal idempotents). The same holds on subcategories of compacts.*

Proof. Again, this passes under localization or holds by general 2-Mackey theory [BD20, Theorem 2.4.1]. One obtains $\mathrm{Ind}^G_H(\mathbb{1}) = k(G/H)$ from DPerm. □

4.5. *Remark.* One can actually show that the Mackey 2-functor $\mathrm{StPerm}(-;k)$ is *cohomological* in the sense of [BD24], meaning that for every subgroup $H \le G$ the composite $\mathrm{Id} \to \mathrm{Ind}^G_H \mathrm{Res}^G \to \mathrm{Id}$, of the unit for the $\mathrm{Res} \dashv \mathrm{Ind}$ adjunction with the counit for $\mathrm{Ind} \dashv \mathrm{Res}$, is equal to multiplication by $[G\!:\!H]$. Again, this is immediate from the fact that the original $\mathrm{DPerm}(-)$ is itself cohomological and from the fact that the units and counits pass to the localization.

4.6. *Remark.* It follows from Remark 4.5 that restriction $\mathrm{Res}^G_H\colon \mathrm{StPerm}(G;k) \to \mathrm{StPerm}(H;k)$ is a faithful functor whenever the index $[G\!:\!H]$ is prime to p. Hence this restriction functor satisfies effective descent, in the classical sense. More precisely, the canonical Eilenberg-Moore functor $\mathcal{T} = \mathrm{StPerm}(G;k) \to \mathrm{Desc}_{\mathcal{T}}(A)$ to the descent category for $A = A^G_H$ in $\mathcal{T}$ is an equivalence by [Bal12, Corollary 3.1].

4.7. *Remark.* In the same vein, one can prove that the Mackey 2-functor StPerm inherits from DPerm the property of being a *Green* 2-functor, in the sense of Dell'Ambrogio [Del22]. This follows from the fact that $\mathrm{DPerm} \twoheadrightarrow \mathrm{StPerm}$ is not only a quotient by a localizing subcategory but by a tensor-ideal.

We can adapt the Colimit Theorem from the case of the derived permutation category [BG25a] to the stable permutation category.

4.8. *Recollection.* The category $\mathcal{E}_p(G)$ of elementary abelian subquotients of G is defined as follows. Its objects are pairs (H,K) where $K \trianglelefteq H$ is a section of G and H/K is elementary abelian. A morphism $(H,K) \to (H',K')$ is an element $g \in G$ such that $K' \le K^g$ and $H^g \le H'$. By [BG25a, Construction 11.4] each such morphism $g\colon (H,K) \to (H',K')$ yields a tt-functor

$$\mathcal{K}(g)\colon \mathcal{K}(H'/K') \xrightarrow{\Psi^{\bar{K}}} \mathcal{K}(H'/K^g) \xrightarrow{\mathrm{Res}} \mathcal{K}(H^g/K^g) \xrightarrow{\sim} \mathcal{K}(H/K) \tag{4.9}$$

combining modular fixed points with respect to the p-subgroup $\bar{K} := K^g/K' \lhd H'/K'$, restriction to the subgroup $H^g/K^g \le H'/K^g$ and conjugation by g. Applying Spc gives a continuous map $\mathrm{Spc}(\mathcal{K}(H/K)) \to \mathrm{Spc}(\mathcal{K}(H'/K'))$ and the colimit of this diagram of topological spaces is $\mathrm{Spc}(\mathcal{K}(G))$ by [BG25a, Theorem 11.10]:

$$\operatorname*{colim}_{(H,K)\in\mathcal{E}_p(G)} \mathrm{Spc}(\mathcal{K}(H/K)) \xrightarrow{\sim} \mathrm{Spc}(\mathcal{K}(G)). \tag{4.10}$$

The comparison maps from each $\mathrm{Spc}(\mathcal{K}(H/K))$ to $\mathrm{Spc}(\mathcal{K}(G))$ are similarly induced by the tt-functor $\mathcal{K}(G) \xrightarrow{\Psi^K} \mathcal{K}(G/\!\!/K) \xrightarrow{\mathrm{Res}} \mathcal{K}(H/K)$.

We now prove the analogous statement for the stable permutation category.

4.11. **Theorem.** *Modular fixed points and restrictions yield as above a homeomorphism of topological spaces*

$$\operatorname*{colim}_{(H,K)\in\mathcal{E}_p(G)} \mathrm{Spc}(\mathrm{stperm}(H/K;k)) \xrightarrow{\sim} \mathrm{Spc}(\mathrm{stperm}(G;k)). \tag{4.12}$$

Proof. In view of Corollary 3.8, it suffices to verify that for each of the various functors f^* discussed above (namely modular fixed points, restriction and conjugation) with induced map $f=\mathrm{Spc}(f^*)$ on spectra, we have $f^{-1}(\{\text{closed points}\}) = \{\text{closed points}\}$. For then the homeomorphism (4.12) is the restriction of the homeomorphism (4.10) to the open complement of the closed points.

The inclusion $\subseteq$ follows from Propositions 3.6 and 4.1. The converse inclusion is [BG25a, Lemma 11.9 (b)]. □

5. Bottleneck subgroups

In this short preparatory section, P is a p-group. We discuss when P admits a 'bottleneck' subgroup H by which we mean a proper non-trivial subgroup $1\neq H<P$ that cannot be surrounded by a section $K'<H<H'$ with $K'\trianglelefteq H'$ and H'/K' elementary abelian of p-rank at least two. In other words, we discuss when every subgroup H' containing H as a subgroup of index p has only H as index-p subgroup.

Bottleneck subgroups can exist at the top and at the bottom of a p-group:

5.1. *Remark.* It is well-known that a p-group that contains a *unique index-p subgroup H* must be cyclic. Indeed, H must contain any central element of order p and, modding it out, one concludes by induction on the order of the group.

5.2. *Remark.* At the other end, a p-group that has a *unique subgroup of order p* must be either a cyclic group or a generalized quaternion group Q_{2^n} of order 2^n. This is more involved, see [Bro82, Theorem 4.3, p. 99]. Recall that for $n\geq 3$, the *generalized quaternion group* is presented as follows

$$Q_{2^n} = \langle x,y \mid x^{2^{n-1}}=1,\ y^2=x^{2^{n-2}}, yxy^{-1}=x^{-1}\rangle. \tag{5.3}$$

Its unique subgroup of order 2 is the center $Z(Q_{2^n})=\langle y^2\rangle$.

Note that in those examples, both in the cyclic case and in the generalized quaternion case, the unique subgroup of order p is central.

We are going to show that these are the only examples.

5.4. **Lemma.** *Let P be a p-group that contains a non-trivial proper subgroup H with the following property: For every subgroup $H'\leq P$ containing H with $[H'\!:\!H]=p$, the group H' is cyclic. Then P is cyclic or generalized quaternion.*

Proof. By induction, we can assume the result for every p-group of order $|P|/p$. Recall that for every proper subgroup H of a p-group P, there exists $H<H'\leq P$ with $[H'\!:\!H]=p$. It follows from our assumption (H' being cyclic) that H is cyclic.

Observation: Let $z\in Z(P)$ be a central element of order p. Then $z\in H$.

Indeed, if z was not in H then the subgroup $H'=\langle H,z\rangle\simeq H\times C_p$ would contradict our hypothesis. Let us pick such a $z\in Z(P)\cap H$.

We distinguish two cases. First, suppose that $H=\langle z\rangle$. Then $H\leq Z(P)$ is central. For every element $g\in P$ of order p, the subgroup $H':=\langle z,g\rangle$ cannot be bigger than H otherwise it would be isomorphic to $C_p\times C_p$ and contradict our hypothesis again. It follows that $g\in H$. Consequently, H is the only subgroup of P of order p. By Remark 5.2 we obtain that P is cyclic or generalized quaternion.

The second option is that H is strictly larger than the central cyclic subgroup $\langle z\rangle$. Consider then the quotient $\bar{P} = P/\langle z\rangle$ and $\bar{H} = H/\langle z\rangle \lneq \bar{P}$. One easily verifies that $\bar{H}$ still satisfies the same hypothesis in $\bar{P}$ as H did in P. By induction hypothesis, we know that $\bar{P}$ is either cyclic or generalized quaternion, and in particular admits a unique subgroup of order p, which is central and therefore necessarily in $\bar{H}$ by the above Observation (applied to $\bar{P}$ and $\bar{H}$).

Let $g \in P$ be an element of order p. We claim that $g \in H$. Its class $\bar{g} \in \bar{P}$ modulo $\langle z\rangle$ has order 1 or p hence belongs to $\bar{H}$ by the above discussion. It follows from this and from $\langle z\rangle \leq H$ that $g \in H$ as claimed.

Consequently, H contains every subgroup of P of order p and since the cyclic group H has a unique subgroup of order p the same holds for P. We conclude that P must be cyclic or generalized quaternion by Remark 5.2. □

We shall use the above result in the following form:

5.5. **Proposition.** *Let P be a p-group that is neither cyclic nor generalized quaternion and let $1 \neq H < P$ be a proper non-trivial subgroup. Then there exist a section $K' \trianglelefteq H' \leq P$ and proper inclusions $K' < H < H'$, with H'/K' elementary abelian (necessarily of p-rank at least two).*

Proof. By Lemma 5.4 there exists a *non-cyclic* subgroup $H' > H$ with $[H' : H] = p$. Since H' is not cyclic, it has more than one subgroup of index p (Remark 5.1). We get the result with $K' = \bigcap_{[H' : N]=p} N$ the Frattini subgroup of H'. □

6. Indecomposability

We can apply Theorem 4.11 and Proposition 5.5 to prove a first result about the stable permutation category, namely deciding when it decomposes.

6.1. **Theorem.** *Let G be a group, whose p-Sylow is neither cyclic, nor generalized quaternion (for $p = 2$). Then* $\mathrm{stperm}(G; k)$ *is indecomposable as a tensor-triangulated category, that is, its spectrum is connected.*

Proof. The fact that $\mathrm{Spc}(\mathcal{K})$ disconnected implies $\mathcal{K} \simeq \mathcal{K}_1 \times \mathcal{K}_2$ when $\mathcal{K}$ is rigid is direct from [Bal07, Theorem 2.11]. Let P be a p-Sylow of G. Since the tt-functor $\mathrm{Res}^G_P\colon \mathrm{stperm}(G; k) \to \mathrm{stperm}(P; k)$ is faithful by Remark 4.5, the continuous map $\mathrm{Spc}(\mathrm{Res}^G_P)$ is surjective ([Bal18]) and we can therefore reduce the theorem to the case where $G = P$ is a p-group.

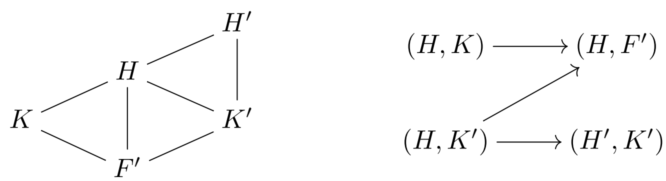
By Theorem 4.11, the space $\mathrm{Spc}(\mathrm{stperm}(G))$ is covered by the images of the spaces $\mathrm{Spc}(\mathrm{stperm}(H/K))$ for $(H, K) \in \mathcal{E}_p(G)$ in the category of elementary abelian subquotients of G recalled in Recollection 4.8. These spaces are non-empty as long as H/K is non-trivial.

We know from [BG25a, Proposition 15.11] that for every non-trivial elementary abelian p-group E (for instance any H/K above) the spectrum $\mathrm{Spc}(\mathrm{K_b}(\mathrm{perm}(E; k))$ is irreducible: it has a generic point. The closed points in $\mathrm{Spc}(\mathrm{K_b}(\mathrm{perm}(E; k))$ are of height equal to the p-rank [BG25a, Remark 15.15]. Hence, after removing them, the open $\mathrm{Spc}(\mathrm{stperm}(E; k))$ remains irreducible and in particular connected. It follows that any continuous image of that space $\mathrm{Spc}(\mathrm{stperm}(E; k))$ remains connected. We are going to see that all those images inside $\mathrm{Spc}(\mathrm{stperm}(G; k))$ are connected by showing they overlap gradually.

Let us look at the 'top' elementary abelian section, namely (G, F) where $F \triangleleft G$ is the Frattini subgroup of our p-group G. Let us denote by $A \subseteq \mathrm{Spc}(\mathrm{stperm}(G; k))$

the connected component that contains the image of $\mathrm{Spc}(\mathrm{stperm}(G/F;k))$. We want to show that A contains every other image of $\mathrm{Spc}(\mathrm{stperm}(H/K;k))$ for every other non-trivial section $(H,K) \in \mathcal{E}_p(G)$. We proceed by induction on $[G:H]$. As long as $H = G$, the image of $\mathrm{Spc}(\mathrm{stperm}(H/K;k))$ is contained in A since the Frattini is the smallest: $F \le K$ gives $\mathrm{Im}(\mathrm{Spc}(\Psi^K)) \subseteq \mathrm{Im}(\mathrm{Spc}(\Psi^F))$. So we can assume $[G\!:\!H] > 1$ and that the image of every (H',K') with $[G\!:\!H'] < [G\!:\!H]$ is contained in A.

Let $(H,K) \in \mathcal{E}_p(G)$ with H/K non-trivial and $H \lneq G$ proper. By Proposition 5.5 there exists $(H',K') \in \mathcal{E}_p(G)$ with strict inclusions $K' < H < H'$. Let $F' < H$ be the Frattini subgroup of H. We have the following left-hand side diagram of subgroups of G, in which a line indicates a normal subgroup with elementary abelian quotient:



Those inclusions yield the morphisms (all given by the identity $1 \in G$) in $\mathcal{E}_p(G)$ displayed on the right-hand side above. Since all $E \in \{H/K,\ H/F',\ H/K',\ H'/K'\}$ are non-trivial, the spectra $\mathrm{Spc}(\mathrm{stperm}(E;k))$ are non-empty. And since they are all connected, we conclude that the image of $\mathrm{Spc}(\mathrm{stperm}(H/K;k))$ for the section (H,K) belongs to the same connected component as that of (H',K'). Since H' is larger than H, we conclude by induction hypothesis that this connected component is indeed A. □

7. Decomposition over cyclic and generalized quaternion

We want to prove a converse to Theorem 6.1. If the p-Sylow of G is cyclic or generalized quaternion (for $p = 2$ in the latter case) then $\mathrm{StPerm}(G;k)$ will decompose and we want to identify the components.

We are going to use a general fact about 'big' tt-categories $\mathcal{T}$, *i.e.* compactly-rigidly generated tensor-triangulated categories. Recall from Remark 3.7 that $\mathcal{T}|_U := \mathcal{T}/\mathrm{Loc}(\mathcal{T}^c_Y)$ when $U = Y^{\complement}$ is the complement of a Thomason subset (*e.g.* a quasi-compact open). Any coproduct-preserving tt-functor $f^*\colon \mathcal{T} \to \mathcal{S}$ between big tt-categories induces a continuous map $f := \mathrm{Spc}(f^*)\colon \mathrm{Spc}(\mathcal{S}^c) \to \mathrm{Spc}(\mathcal{T}^c)$ and the functor f^* sends $\mathcal{T}_Y$ into $\mathcal{S}_{f^{-1}(Y)}$ by [BF11, Theorem 6.3]. In particular $f^{-1}(U)^{\complement} = f^{-1}(U^{\complement})$ is Thomason in $\mathrm{Spc}(\mathcal{S}^c)$. It follows that f^* induces a tt-functor on the localizations $\bar{f}^*\colon \mathcal{T}|_U \to \mathcal{S}|_{f^{-1}(U)}$, that we can denote $f^*|_U$.

7.1. **Proposition.** *With above notation for $f^*\colon \mathcal{T} \to \mathcal{S}$ and $U \subseteq \mathrm{Spc}(\mathcal{T}^c)$, suppose that f^* is fully faithful. Then the induced functor*

$$f^*|_U\colon \mathcal{T}|_U \to \mathcal{S}|_{f^{-1}(U)}$$

remains fully faithful. Moreover, suppose in addition that $\mathcal{S}$ is generated as a tensor-triangulated category by $f^*(\mathcal{T})$ and some collection of objects supported outside of $f^{-1}(U)$, that is, objects in $\mathrm{Loc}(\mathcal{S}^c_{f^{-1}(U)^{\complement}})$. Then $f^*|_U$ is an equivalence.

Proof. The full-faithfulness of $f^*|_U$ is [BG25b, Lemma 5.1]. For the moreover part, the generators supported outside of $f^{-1}(U)$ become zero in $\mathcal{S}|_{f^{-1}(U)}$ by definition. It follows that $f^*|_U$ is also essentially surjective. □

7.2. *Remark.* We are going to apply the above to f^* given by inflation along $G \twoheadrightarrow G/N$ for a particular normal subgroup $N \trianglelefteq G$. Note that inflation does not descend to stable permutation categories, very much for the same reasons as with the usual stable module category: It does not preserve equivariantly perfect complexes. (For instance $k(G/N)$ is inflated from a G/N-perfect but if N admits a non-trivial p-subgroup $H \leq N$ the object $\Psi^H(k(G/N)) \simeq k^{[G:N]}$ is not perfect.) However, the version of inflation that we shall use is the one on derived permutation categories $\mathrm{Infl}_G^{G/N} \colon \mathrm{DPerm}(G/N;k) \to \mathrm{DPerm}(G;k)$, which is fine [BG25a, § 4].

7.3. **Proposition.** *For every normal subgroup $N \trianglelefteq G$, the coproduct-preserving tt-functor $\mathrm{Infl}_G^{G/N} \colon \mathrm{DPerm}(G/N;k) \to \mathrm{DPerm}(G;k)$ is fully faithful.*

Proof. As with any coproduct-preserving tt-functor one can verify full-faithfulness on compacts (check that the unit $\mathrm{Id} \to f_* f^*$ is an isomorphism and use cocontinuity of f^* and f_*). Here, the functor $\mathrm{Infl}_G^{G/N} \colon \mathrm{K_b}(\mathrm{perm}(G/N;k)^\natural) \to \mathrm{K_b}(\mathrm{perm}(G;k)^\natural)$ is fully faithful because so is $\mathrm{Infl}_G^{G/N} \colon \mathrm{perm}(G/N;k) \to \mathrm{perm}(G;k)$, the latter because it is the restriction of the usual restriction-of-scalars functor on modules $\mathrm{Infl}_G^{G/N} \colon \mathrm{mod}(k(G/N)) \to \mathrm{mod}(kG)$ along a ring epimorphism $kG \twoheadrightarrow k(G/N)$. □

Let us start discussing our examples of cyclic or generalized quaternions groups. We begin with the derived category of permutation $\mathcal{K}(G) = \mathrm{DPerm}(G;k)^c = \mathrm{K_b}(\mathrm{perm}(G;k)^\natural)$ as recalled in Recollections 2.3 and 3.4.

7.4. **Lemma.** *Let G be a p-group with a unique subgroup of order p (Remark 5.2). Let $N = Z(G) \triangleleft G$ be that subgroup.*

(a) *The spectrum $\mathrm{Spc}(\mathcal{K}(G))$ is the union of two closed subsets $Z_1 = \mathrm{Im}(\mathrm{Spc}(\Psi^N))$ and $Z_2 = \mathrm{supp}(k(G/N))$ whose intersection is exactly the closed point $\mathcal{M}(N)$.*

(b) *For the stable permutation category, the open subspace $\mathrm{Spc}(\mathrm{stperm}(G;k)) \subset \mathrm{Spc}(\mathcal{K}(G))$ of Corollary 3.8 is disconnected as follows*
$$\mathrm{Spc}(\mathrm{stperm}(G;k)) = \mathrm{Im}(\mathrm{Spc}(\Psi^N)) \ \sqcup \ \{*\}$$
where $\Psi^N \colon \mathrm{stperm}(G;k) \to \mathrm{stperm}(G/N;k)$ is (stable) modular N-fixed points and the singleton $\{*\}$ is the image of $\mathrm{Spc}(\mathrm{stmod}(G;k)) = \{*\}$.

(c) *There is an equivalence of tt-categories*
$$\mathrm{StPerm}(G;k) \xrightarrow{\sim} \mathrm{StPerm}(G/N;k) \times \mathrm{StMod}(G;k)$$
given by modular N-fixed-points $\Psi^N \colon \mathrm{StPerm}(G;k) \to \mathrm{StPerm}(G/N;k)$ and the canonical localization $\bar{\Upsilon} \colon \mathrm{StPerm}(G;k) \twoheadrightarrow \mathrm{StMod}(G;k)$. This decomposition preserves the compact objects and induces (b) *on spectra.*

Proof. Since N is normal the map $\psi^N := \mathrm{Spc}(\Psi^N) \colon \mathrm{Spc}(\mathcal{K}(G/N)) \to \mathrm{Spc}(\mathcal{K}(G))$ is a homeomorphism onto its closed image ([BG25a, Proposition 7.18]). For every non-trivial subgroup $H \leq G$ we have $N \trianglelefteq H$ and therefore the H-fixed-points functor Ψ^H factors via the N-fixed-points functor Ψ^N ([BG25a, Corollary 5.18]). Hence $Z_1 = \mathrm{Im}(\Psi^N)$ contains $\mathrm{Im}(\Psi^H)$ and in particular the stratum $\mathcal{V}_{G/\!\!/H}$ in (3.5). Thus the only stratum of $\mathrm{Spc}(\mathcal{K}(G))$ in (3.5) that is not contained in Z_1 is the cohomological open $\mathcal{V}_{G/\!\!/1} = \mathrm{Spc}(\mathrm{D_b}(kG))$. Since N is the maximal elementary abelian subgroup of G, or by direct computation of the cohomology, we see that $\mathrm{Spc}(\mathrm{Res}^G_N)$ yields a bijection of Sierpiński 2-point spaces $\mathrm{Spc}(\mathrm{D_b}(kN)) \overset{\sim}{\to} \mathrm{Spc}(\mathrm{D_b}(kG))$. Since $Z_2 = \mathrm{supp}(k(G/N))$ is as usual the image of $\mathrm{Spc}(\mathrm{Res}^G_N)$ it follows that Z_2 contains the last stratum $\mathcal{V}_G$ and therefore $Z_1 \cup Z_2$ is indeed the whole of $\mathrm{Spc}(\mathcal{K}(G))$ by (3.5). Their intersection $Z_1 \cap Z_2 = \mathrm{Im}(\psi^H) \cap Z_2$ can be computed by looking at the preimage of $Z_2 = \mathrm{supp}(k(G/N))$ under ψ^N, which is simply the support of $\Psi^N(k(G/N)) \cong k(G/N)$ in $\mathcal{K}(G/N)$. That free $k(G/N)$-module has support $\mathcal{M}_{G/N}(1)$ in $\mathrm{Spc}(\mathcal{K}(G/N))$. Its image under $\psi^N\colon \mathrm{Spc}(\mathcal{K}(G/N) \overset{\sim}{\to} Z_1$ is $\psi^H(\mathcal{M}_{G/N}(1)) = \mathcal{M}_G(N)$. This proves (a).

By Corollary 3.8, we can intersect the closed cover $\mathrm{Spc}(\mathcal{K}(G)) = Z_1 \cup Z_2$ with the open subset $\mathrm{Spc}(\mathrm{stperm}(G))$ of $\mathrm{Spc}(\mathcal{K}(G))$ and we get the disconnection:

$$\mathrm{Spc}(\mathrm{stperm}(G;k)) = U_1 \sqcup U_2 \quad \text{where} \quad U_i = Z_i \cap \mathrm{Spc}(\mathrm{stperm}(G;k)),\ i = 1, 2$$

since $Z_1 \cap Z_2$ is one of the closed points we remove in this process. It follows immediately from Corollary 4.2 that $U_1 = \mathrm{Im}(\psi^N) \cap \mathrm{Spc}(\mathrm{stperm}(G;k))$ remains the image of $\mathrm{Spc}(\Psi^N)$ for the functor Ψ^N descended to the stable permutation categories. As we saw above, Z_2 consisted of 3 points, $\mathcal{M}(N)$ and the two points of $\mathrm{Spc}(\mathrm{D_b}(kG))$, including $\mathcal{M}(1)$. Removing the two closed ones, $\mathcal{M}(N)$ and $\mathcal{M}(1)$, we are left with $U_2 = \{*\}$ the singleton from $\mathrm{Spc}(\mathrm{stmod}(kG))$. This gives (b).

By (b), the tt-category $\mathrm{stperm}(G;k)$ must decompose as $\mathcal{K}_1 \times \mathcal{K}_2$ with $\mathrm{Spc}(\mathcal{K}_1) = U_1 = \mathrm{Im}(\psi^N)$ and $\mathrm{Spc}(\mathcal{K}_2) = U_2 = \{*\}$; see [Bal07]. It is easy to identify the second component: It is the restriction of $\mathcal{K}(G)$ on the open singleton $\{*\} = \mathrm{Spc}(\mathrm{stmod}(kG))$ in $\mathrm{Spc}(\mathcal{K}(G))$. By Proposition 2.15 we know that this quotient is $\mathrm{stmod}(kG)$. The main claim is that $\Psi^N\colon \mathrm{stperm}(G;k) \to \mathrm{stperm}(G/N;k)$ induces an equivalence $\mathcal{K}_1 \overset{\sim}{\to} \mathrm{stperm}(G/N;k)$ on U_1. First note that this exists for the closed complement of U_1, namely the singleton $\{*\}$, is the support of $k(G/N)$ as discussed in the first part, and $\Psi^N(k(G/N)) = k(G/N)$ becomes zero in $\mathrm{stperm}(G/N;k)$ by Example 2.10. Furthermore, since the composite $\Psi^N \circ \mathrm{Infl}^{G/N}_G\colon \mathcal{K}(G/N) \to \mathcal{K}(G) \to \mathcal{K}(G/N)$ is isomorphic to the identity, it suffices to show that $\mathrm{Infl}^{G/N}_G$ descends to an equivalence $\mathrm{stperm}(G/N;k) \overset{\sim}{\to} \mathcal{K}_1$ to get the result. We need to be careful since inflation does not exist on the entire stable permutation categories.

We use the big tt-categories. Let us consider $f^* = \mathrm{Infl}^{G/N}_G\colon \mathrm{DPerm}(G/N;k) \to \mathrm{DPerm}(G;k)$ and $Y := \{\mathcal{M}_{G/N}(1)\}$ the closed point in the cohomological open of G/N, which is the support of the free $k(G/N)$-module $k(G/N)$. Again, we write $f = \mathrm{Spc}(f^*)\colon \mathrm{Spc}(\mathcal{K}(G)) \twoheadrightarrow \mathrm{Spc}(\mathcal{K}(G/N))$ for the induced map on spectra. The preimage $f^{-1}(Y)$ in $\mathrm{Spc}(\mathcal{K}(G))$ of the subset $Y = \mathrm{supp}(k(G/N))$ is as usual the support of the image object $\mathrm{Infl}^{G/N}_G(k(G/N))$ by the functor f^*, namely the permutation kG-module $k(G/N)$. The support of the latter is the closed subset Z_2 of (a). By fully-faithfulness of f^* (Proposition 7.3), we can apply Proposition 7.1 to get a well-defined fully-faithful functor

$$f^*|_{Y^\complement}\colon \mathrm{DPerm}(G/N;k)|_{Y^\complement} \to \mathrm{DPerm}(G;k)|_{Z_2^\complement} \tag{7.5}$$

from the localization of $\mathcal{T} = \mathrm{DPerm}(G/N;k)$ on the open $Y^\complement = \mathrm{Spc}(\mathcal{K}(G/N)) \smallsetminus \{\mathcal{M}_{G/N}(1)\}$ to the localization of $\mathcal{S} = \mathrm{DPerm}(G;k)$ on the open $Z_2^\complement = Z_1 \smallsetminus \{\mathcal{M}(N)\}$ of $\mathrm{Spc}(\mathcal{K}(G))$, where we use (a) in the last equality. We claim that this $f^*|_{Y^\complement}$ is an equivalence by the 'moreover part' of Proposition 7.1. Indeed, the generators $\{\, k(G/H) \mid H \le G \,\}$ of $\mathrm{DPerm}(G;k)$ are almost all in the essential image of f^*, that is, inflated from G/N. This is because N is contained in all non-trivial subgroups of G. Only the generator kG is not in the essential image of f^*. But kG has support in Z_2. So (7.5) is indeed an equivalence by Proposition 7.1.

Note that the two tt-categories in (7.5) have (homeomorphic) spectra consisting of $Y^{\complement} = \mathrm{Spc}(\mathcal{K}(G/N)) \smallsetminus \{\mathcal{M}_{G/N}(1)\}$ and $Z_2^{\complement} = \mathrm{Spc}(\mathcal{K}(G)) \smallsetminus \mathrm{supp}(k(G/N))$ respectively, which are homeomorphic under the restriction of $\psi^H = \mathrm{Spc}(\Psi^H)$ on $Y^{\complement}$, with inverse $\mathrm{Spc}(f^*)|_{Z_2^{\complement}}$. We can further localize these two categories by removing all the remaining closed points $\{\, \mathcal{M}_{G/N}(\bar{H}) \,|\, 1 \neq \bar{H} \leq G/N \,\}$ in the former, respectively their images $\{\, \mathcal{M}_G(H) \,|\, N \lneq H \leq G \,\}$ in the latter. In other words, on the left-hand side of (7.5) we localize $\mathrm{DPerm}(G/N; k)$ away from all its closed points, which gives the stable permutation category of G/N, and on the right-hand side of (7.5) we localize away from all the closed points and away from Z_2, which gives the stable permutation category of G localized away from (what remains of) Z_2. This is precisely the wanted localization $\mathrm{StPerm}(G; k)|_{U_1}$ since $U_1 = \mathrm{Spc}(\mathrm{stperm}(G; k)) \cap Z_2^{\complement}$ by construction. Consequently $f^* = \mathrm{Infl}_G^{G/N}$ yields a tt-equivalence

$$\mathrm{StPerm}(G/N; k) \xrightarrow{\sim} \mathrm{StPerm}(G; k)|_{U_1}.$$

Since Ψ^N is its left inverse, it is also an equivalence and we get (c). $\square$

7.6. **Theorem.** *Let $p = 2$ and let G be a finite group whose 2-Sylow is generalized quaternion of order 2^n, as in (5.3). Let $H \leq G$ be a cyclic subgroup of order 2, i.e. the center of some 2-Sylow of G. Then (stable) modular H-fixed-points Ψ^H (Corollary 4.2) and the canonical localization $\bar{\Upsilon}$ in (1.1) yield an equivalence*

$$(\Psi^N, \bar{\Upsilon})\colon \mathrm{StPerm}(G; k) \xrightarrow{\sim} \mathrm{StPerm}(G/\!\!/H; k) \times \mathrm{StMod}(k(G)))$$

corresponding to a decomposition $\mathrm{Spc}(\mathrm{stperm}(G; k)) = \mathrm{Im}(\mathrm{Spc}(\Psi^H)) \sqcup \{*\}$.

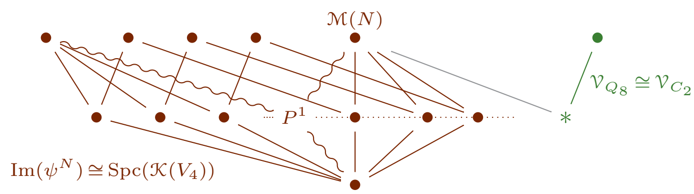
Proof. Let $P < G$ be a 2-Sylow of G containing H, necessarily as its center (Remark 5.2). We have $P/H \leq G/\!\!/H$ and the following diagram commutes (as follows from [BG25a, Proposition 5.15] by localization):

$$\begin{array}{ccc} \mathrm{StPerm}(G; k) & \xrightarrow{(\Psi^H, \bar{\Upsilon})} & \mathrm{StPerm}(G/\!\!/H; k) \times \mathrm{StMod}(k(G)) \\ \Big\downarrow {\scriptstyle \mathrm{Res}^G_P} & & \Big\downarrow {\scriptstyle \mathrm{Res}^{G/\!\!/H}_{P/H} \times \mathrm{Res}^G_P} \\ \mathrm{StPerm}(P; k) & \underset{\cong}{\xrightarrow{(\Psi^H, \bar{\Upsilon})}} & \mathrm{StPerm}(P/H; k) \times \mathrm{StMod}(k(P))). \end{array} \tag{7.7}$$

The bottom horizontal functor is an equivalence by Lemma 7.4 applied to the p-group P, and to $N = Z(P) = H$. The vertical functors satisfy descent (Remark 4.6), identifying $\mathcal{T} = \mathrm{StPerm}(G; k)$ with $\mathrm{Desc}_{\mathcal{T}}(A)$ for $A = k(G/P)$ and $\mathcal{S} = \mathrm{StPerm}(G/\!\!/H; k) \times \mathrm{StMod}(kG)$ with $\mathrm{Desc}_{\mathcal{S}}(B)$ for $B = \big(k((G/\!\!/H)/(P/H)), k(G/P)\big)$ in $\mathcal{S}$. The latter tt-ring B is obtained by applying the adjoint $\mathrm{Ind}^{G/\!\!/H}_{P/H} \times \mathrm{Ind}^G_P$ to the unit (k, k). The first component of B is $k(N_G(H)/P)$ since $(G/\!\!/H)/(P/H) \cong N_G(H)/P$. Now the functor $\Psi^H\colon \mathrm{perm}(G; k) \to \mathrm{perm}(G/\!\!/H; k)$ maps $k(G/P)$ to $k\big((G/P)^H\big) = k(N_G(H, P)/P)$ and crucially $N_G(H, P) = \{\, g \in G \,|\, {}^gH \leq P \,\}$ is equal to $N_G(H)$ since H is the *unique* cyclic subgroup of order 2 in the generalized quaternion group P. It follows that the top functor in (7.7) maps $A = k(G/P)$ to $(\Psi^H(k(G/P)), \bar{\Upsilon}(k(G/P))) = (k(N_G(H)/P), k(G/P)) = B$ and therefore, by general nonsense, it is also an equivalence. (If a tensor functor $F\colon \mathcal{T} \to \mathcal{S}$ sends a ring A to $F(A) = B$ and if the induced functor $F\colon \mathrm{Mod}_{\mathcal{T}}(A) \to \mathrm{Mod}_{\mathcal{S}}(B)$ is an equivalence then F induces an equivalence $\mathrm{Desc}_{\mathcal{T}}(A) \xrightarrow{\sim} \mathrm{Desc}_{\mathcal{S}}(B)$; therefore, if A and B satisfy descent then the original $F\colon \mathcal{T} \to \mathcal{S}$ was an equivalence.) □

7.8. *Example.* Let $p = 2$ and $G = Q_8$ be the quaternion group, $N = Z(G) \cong C_2$ and $G/Z(G) = D_4 = V_4$ the Klein-four group. As displayed in [BG25a, (2.13)], the spectrum $\mathrm{Spc}(\mathcal{K}(Q_8))$ of the derived permutation category looks as follows:

(7.9)



This space is barely connected, thanks to the one closed point $\mathcal{M}(N)$. To obtain the spectrum $\mathrm{Spc}(\mathrm{stperm}(Q_8; k))$ of the stable permutation category, we remove all the closed points. The resulting open is disconnected: One piece is $\mathrm{Spc}(\mathrm{stperm}(V_4; k))$ (as pictured in Figure 1) and the other is a single point $* = \mathrm{Spc}(\mathrm{stmod}(kQ_8))$. As we saw in Lemma 7.4, the disconnection of the spectrum reflects the decomposition of the category $\mathrm{StPerm}(Q_8; k) \cong \mathrm{StPerm}(V_4; k) \times \mathrm{StMod}(kQ_8)$.

7.10. **Theorem.** *Let G be a finite group whose p-Sylow subgroup is cyclic of order p^n for $n \geq 1$. Choose a tower $1 = H_n < H_{n-1} < \ldots < H_1 < H_0 \leq G$ of cyclic subgroups H_i of order p^{n-i}. Then the stable permutation category* $\mathrm{StPerm}(G; k)$ *is equivalent to the product of the stable* module *categories* $\mathrm{StMod}(k(G/\!\!/H_i))$ *for all $i = 1, \ldots, n-1$. The components* $\mathrm{StPerm}(G; k) \to \mathrm{StMod}(k(G/\!\!/H_i))$ *of this equivalence are given by the (stable) modular H_i-fixed points functors Ψ^{H_i} composed with the canonical localization $\bar{\Upsilon}\colon$* $\mathrm{StPerm}(G/\!\!/H_i; k) \twoheadrightarrow \mathrm{StMod}(k(G/\!\!/H_i))$.

Proof. The argument is similar to the proof of Theorem 7.6: One proves it for the p-Sylow and one reduces to that case by descent. For the p-group case (the Sylow) one proceeds by an easy induction on n using Lemma 7.4, since in this case G/N is cyclic of order p^{n-1}. For the descent argument one uses once again the little miracle about the uniqueness of the subgroup H of a certain order in the p-Sylow, which gives us $N_G(H, P) = N_G(H)$ for every $H = H_1, \ldots, H_{n-1}$. Details are left to the interested reader. □

Theorems 7.6 and 7.10 specialize to Theorems 1.3 and 1.4 in the introduction in the case of p-groups.

7.11. *Remark.* In the above two proofs, we use central cyclic subgroups $N \lhd P$ of the p-Sylow of G, typically their center $N = Z(P)$. Note that we cannot apply Lemma 7.4 directly to N in G as there is no guarantee that N remains normal in G. For instance, for $P = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ the usual quaternion group, we can find a finite abelian group A of odd order on which Q_8 acts in such a way that the action of $N = \{\pm 1\}$ is non-trivial, for instance the $\mathbb{F}_3$-vectorspace $A = \mathbb{F}_3^2$ with the action of Q_8 via the usual embedding $Q_8 \hookrightarrow \mathrm{GL}_2(\mathbb{F}_3)$ given by $i \mapsto \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $j \mapsto \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right)$. In such a case, if we let $G = Q_8 \ltimes A$ then the 2-Sylow $P = Q_8 \times \{1\}$ is not normal and the element $(-1, a)$ conjugates N to a different cyclic subgroup, for any $a \in A$ not fixed by -1.

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