PRESHEAVES OF TRIANGULATED CATEGORIES AND RECONSTRUCTION OF SCHEMES

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Abstract. To any triangulated category with tensor product $(K, \otimes)$, we associate a topological space $\mathrm{Spc}(K, \otimes)$, by means of thick subcategories of K, *à la* Hopkins-Neeman-Thomason. Moreover, to each open subset U of this space $\mathrm{Spc}(K, \otimes)$, we associate a triangulated category $\mathcal{K}(U)$, producing what could be thought of as a *presheaf of triangulated categories.* Applying this to the derived category $(K, \otimes) := (\mathrm{D}^{\mathrm{perf}}(X), \otimes^L)$ of perfect complexes on a noetherian scheme X, the topological space $\mathrm{Spc}(K, \otimes)$ turns out to be the underlying topological space of X; moreover, for each open $U \subset X$, the category $\mathcal{K}(U)$ is naturally equivalent to $\mathrm{D}^{\mathrm{perf}}(U)$.

As an application, we give a method to reconstruct any reduced noetherian scheme X from its derived category of perfect complexes $\mathrm{D}^{\mathrm{perf}}(X)$, considering the latter as a tensor triangulated category with $\otimes^L$.

1. Introduction

Triangulated categories are ubiquitous. They appear in Homological Algebra, Topology, Algebraic Geometry, Representation Theory, K-Theory, and so on. In most cases, these triangulated categories come along with a tensor product, in the very weak sense of Definition 2.3, which is so flexible as to admit the zero tensor product $\otimes = 0$ in any case. Let us then rephrase our anthem: *tensor* triangulated categories are ubiquitous.

In Algebraic Geometry, Grothendieck *et al.* [**2**] defined the notion of perfect complexes (see 2.1) and introduced the triangulated category $\mathrm{D}^{\mathrm{perf}}(X)$, which is very close to the "naive" derived category of bounded complexes of vector bundles. Actually, $\mathrm{D}^{\mathrm{perf}}(X)$ coincides with the latter on a big class of schemes, including affine ones, but is in general more suitable for geometry, as brightly demonstrated by Thomason in his state-of-the-art treatment of algebraic K-theory of schemes [**12**]. Observe that this triangulated category $\mathrm{D}^{\mathrm{perf}}(X)$ is equipped with a tensor product $\otimes = \otimes^L_{\mathcal{O}_X}$.

Our main result is:

Theorem. *The functor $X \mapsto (\mathrm{D}^{\mathrm{perf}}(X), \otimes^L_{\mathcal{O}_X})$, from reduced noetherian schemes to tensor triangulated categories, is faithful and reflects isomorphisms in the following sense: given two reduced noetherian schemes, if there exists a $\otimes$-equivalence of tensor triangulated categories between their derived categories of perfect complexes then the two schemes are isomorphic.*

This Theorem is to be found in Section 8. It follows from Theorem 8.4 and Corollary 8.6. We also give there partial results about non-necessarily reduced schemes.

Key words and phrases. $\otimes$-triangulated category, thick subcategories, perfect complexes, spectrum, triangular presheaf, atomic, geometric.

This result is interesting for various reasons. First of all, taken literally, it says that the scheme invariant $X \mapsto \mathrm{D}^{\mathrm{perf}}(X)$ taking values in tensor triangulated categories is a "complete invariant". It is a commonplace to say that schemes are supposed to be more subtle mathematical objects than tensor triangulated categories. Therefore, this result is rather surprising.

Secondly, we will actually define a left inverse functor to $X \mapsto \mathrm{D}^{\mathrm{perf}}(X)$ on a big class of $\otimes$-triangulated categories. Therefore, almost tautologically, some classical constructions which are performed on schemes, can be *extended* to those tensor triangulated categories as well. We have K-theory in mind. We do not claim that this is a clever way of defining K-theory of triangulated categories, but we observe that the K-theory of a scheme can be reconstructed from $\mathrm{D}^{\mathrm{perf}}(X)$ and $\otimes^L$, since the whole scheme can ! This goes against the general feeling among K-theorists that their invariant, or rather Quillen's, does not go through derived categories. Further comment about this can be found below in Remark 9.3.

Thirdly, the technique we develop in the proofs might perhaps be transposed to other frameworks. Our backbone is one of the last articles of Thomason's [**11**], where he classifies thick subcategories of $\mathrm{D}^{\mathrm{perf}}(X)$ in terms of X. Hopkins and Neeman (see [**6**]) did the case where X is affine and noetherian without using the tensor product. Thomason generalized it to non-affine, non-noetherian schemes, at the cost of having to remember the tensor product $\otimes^L$ on $\mathrm{D}^{\mathrm{perf}}(X)$. There is already a considerable literature about "classification of thick subcategories" in more than one area of mathematics and we will not lose ourselves in an overview of this material.

For our convenience, we introduce here the concept of *presheaf of triangulated categories* (see 5.6). This is again very general and might be useful in other contexts as well. It allows us to speak of things being true "locally" in a triangulated category. Together with some folklore observations about reconstructing reduced rings from their derived category, this "local" approach allows us to reconstruct reduced noetherian schemes as well.

Our reconstruction holds more precisely for any scheme whose underlying space is noetherian, see Remark 2.9. This is more general than being noetherian, but the reader unfamiliar with this subtlety should just think of noetherian schemes, like algebraic varieties for instance.

Reconstruction results were already obtained by Bondal and Orlov [**3**], without using the tensor structure, for smooth algebraic varieties with ample either canonical or anticanonical sheaf. In that case, $\mathrm{D}^{\mathrm{perf}}(X)$ is equivalent to $\mathrm{D}^{\mathrm{b}}_{coh}(X)$, the derived category of coherent $\mathcal{O}_X$-modules. In general, it is impossible to reconstruct a variety from $\mathrm{D}^{\mathrm{b}}_{coh}(X)$. There are known counterexamples to this due to Mukai, to Orlov and to Polishchuk (see [**5**], [**8**] and [**9**]).

After Section 2 which contains some prerequisites, the article is divided into two parts.

Part I (Sections 3 to 6) gives the generalities about $\otimes$-triangulated categories. We start with our definition of the *spectrum* $\mathrm{Spc}(K, \otimes)$ of a $\otimes$-triangulated category. We insist on "our" since it is not clear that our approach would be exactly the right one in other contexts, but it is good enough for our reconstruction purposes. In particular, we give this spectrum a topology, in order to recover the underlying space of X when applied to $\mathrm{D}^{\mathrm{perf}}(X)$. We also study the functoriality of this construction $\mathrm{Spc}(-)$. We are led to consider what we call *geometric* morphisms of $\otimes$-triangulated categories (Definition 4.1). These include equivalences and morphisms of the form $Lf^* : \mathrm{D}^{\mathrm{perf}}(X) \to \mathrm{D}^{\mathrm{perf}}(Y)$ for a morphism of noetherian schemes $f : Y \to X$. Then we also consider presheaves on $\mathrm{Spc}(K, \otimes)$. The basic construction being the *triangular presheaf* or presheaf of $\otimes$-triangulated categories, denoted $\mathcal{K}$, of Definition 5.1. This uses the general remark that the idempotent completion of a triangulated category can be naturally triangulated, which is proved in [**1**]. We then use rings of endomorphisms locally to produce a presheaf of rings whose sheafification is the wanted ringed space structure on $\mathrm{Spc}(K)$.

Part II (Sections 7 to 9) is mostly devoted to the scheme case. We unfold and apply the above definitions and constructions when K equals $\mathrm{D}^{\mathrm{perf}}(X)$. We show that the underlying space of X can be functorially recovered from $(\mathrm{D}^{\mathrm{perf}}(X), \otimes^L_X)$, which is tacitly in Thomason [**11**], and we identify the presheaf of triangulated categories $\mathcal{K}$ with $U \mapsto \mathrm{D}^{\mathrm{perf}}(U)$. We then check how the ringed space structure is linked to the one of X. This leads us to the result. We end the paper with an example and some further comments in Section 9.

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2. Preliminaries on triangulated categories

All categories under consideration are supposed to be essentially small.

An *abelian* or more generally an *exact* category is an additive category equipped with a collection of so-called *exact sequences* which encode the homological information. Similarly, a *triangulated* category is an additive category K where this homological information is encoded in *exact triangles*. More precisely, K comes equipped with a *translation*, *i.e.* an endo-equivalence $T : K \to K$ and a class of *exact triangles* of the form $P \xrightarrow{u} Q \xrightarrow{v} R \xrightarrow{w} T(P)$, a.k.a. *distinguished triangles*, satisfying some axioms (see [**14**, II.1.1.1] or [**15**, Chapter 10]). A functor $\varphi : K \to K'$ is called *exact*, a.k.a. *triangulated*, if it commutes with the translation up to isomorphism and preserves exact triangles. Basic examples of triangulated categories are derived categories of exact categories, stable categories of Frobenius categories, Spanier-Whitehead categories of model categories, and so on. In Algebraic Geometry and in Algebraic K-Theory, an important example is the derived category $\mathrm{D}^{\mathrm{perf}}(X)$ of perfect complexes on a scheme X (see Grothendieck *et al.* [**2**, Exposé I] or Thomason [**12**, Section 2] and [**11**, 3.1]) that we briefly recall here.

2.1. Definition. Let X be a scheme (below, X will always be assumed to have a noetherian underlying space). A *strict perfect complex* on X is a bounded complex of locally free $\mathcal{O}_X$-modules of finite type. A complex P of $\mathcal{O}_X$-modules is a *perfect complex* on X if any point has a neighborhood U such that the restriction of P to U is quasi-isomorphic to some strict perfect complex on U. We denote by $\mathrm{D}^{\mathrm{perf}}(X)$ the full subcategory of perfect complexes in $\mathrm{D}(X)$, the derived category of the abelian category of $\mathcal{O}_X$-modules.

Thomason [**11**] gives a description of some subcategories of $\mathrm{D}^{\mathrm{perf}}(X)$ in terms of X that we will use below. Let us recall what Verdier [**14**] called an "*épaisse*" subcategory, and is here called a "thick" subcategory. The following simplified version of the definition is due to Rickard [**10**].

2.2. Definition. Let K be a triangulated category. A subcategory $A \subset K$ is called *thick* if it is a full triangulated subcategory (stable by isomorphisms, translations, and taking cones) such that $P \oplus Q \in A$ with $P, Q \in K$ forces $P \in A$ and $Q \in A$.

2.3. Definition. We will call *tensor triangulated category* a triangulated category K equipped with a covariant functor $\otimes : K \times K \to K$ which is exact in each variable. A *morphism of $\otimes$-triangulated categories*, a.k.a. *$\otimes$-functor*, $\varphi : (K, \otimes) \to (L, \otimes)$ must commute with $\otimes$ up to isomorphism.

2.4. Remarks.

(1) $$\begin{tikzcd} T(P)\otimes T(Q) \arrow[r, "\cong"] \arrow[d, "\cong"'] & T(T(P)\otimes Q) \arrow[d, "\cong"] \\ T(P\otimes T(Q)) \arrow[r, "\cong"] & T^2(P\otimes Q) \end{tikzcd}$$
Quite often, the diagram on the left is assumed to be skew-commutative, as in the case of a derived tensor product. Here we will assume commutativity *up to a sign*, which includes strict commutativity as in Example 9.2.

(2) Observe that our definition of $\otimes$ is very flexible. We do not require associativity or commutativity or anything of the like. Moreover, any triangulated category can be equipped with a tensor structure: $\otimes = 0$ for instance.

2.5. Example. Let X be a scheme. The triangulated category $\mathrm{D}^{\mathrm{perf}}(X)$ is equipped with $\otimes^L$, the left derived functor of the usual tensor product $- \otimes_{\mathcal{O}_X} -$ (see if necessary [**11**, 3.1] and more references there).

2.6. Definition. Let $(K, \otimes)$ be a $\otimes$-triangulated category. We will say that a thick subcategory $A \subset K$ is *$\otimes$-thick* if $P \in A$ forces $P \otimes Q \in A$ and $Q \otimes P \in A$ for all $Q \in K$.

2.7. Notation. Let X be a scheme.

(1) Let $P \in \mathrm{D}^{\mathrm{perf}}(X)$, denote by $\mathrm{Supph}(P) := \{x \in X \,|\, P_x \not\simeq 0 \text{ in } \mathrm{D}^{\mathrm{perf}}(\mathcal{O}_{X,x})\}$ the support of the homology of P.

(2) Let $Y \subset X$, denote by $\mathrm{D}^{\mathrm{perf}}_Y(X) := \{P \in \mathrm{D}^{\mathrm{perf}}(X) \,|\, \mathrm{Supph}(P) \subset Y\}$ the full subcategory of $\mathrm{D}^{\mathrm{perf}}(X)$ whose objects are those acyclic outside Y.

2.8. Corollary (from Thomason's classification). *Let X be a scheme with noetherian underlying space. Let $\mathcal{C}$ be the collection of $\otimes$-thick subcategories of $\mathrm{D}^{\mathrm{perf}}(X)$ and let $\mathcal{S}$ be the collection of specialization closed subsets of X, i.e. subsets $Y \subset X$ such that $y \in Y$ implies $\overline{\{y\}} \subset Y$. Then there are inverse isomorphisms:*

$$\varphi : \mathcal{C} \to \mathcal{S} \qquad\qquad \psi : \mathcal{S} \to \mathcal{C}$$
$$A \mapsto \bigcup_{P \in A} \mathrm{Supph}(P) \qquad\qquad Y \mapsto \mathrm{D}^{\mathrm{perf}}_Y(X)$$

which preserve inclusions.

Proof. Thomason gives in [**11**, Theorem 3.15] a more general statement for quasi-compact and quasi-separated schemes, in which the specialization closed subsets of X are replaced by unions of closed subsets whose complement is quasi-compact. Under the noetherian assumption, this boils down to the above. Recall that a scheme with noetherian underlying space is quasi-separated, see [**4**, Section 6.1]. $\square$

2.9. Remark and Definition. Let us briefly comment on the hypothesis made throughout the article that our schemes have noetherian underlying spaces. First of all, we need Thomason's classification and thus we need our schemes to be quasi-compact and quasi-separated. But later on, we want to work with all their open subschemes and we will also need them to be quasi-compact and quasi-separated (actually for different reasons, also going back to Thomason, see Theorem 2.13 below). So we should deal with schemes X such that:

X is quasi-compact and quasi-separated, and any open subscheme of X also is $(\star)$.

This property is equivalent to the underlying topological space of X being noetherian. We will sometimes use the abbreviation *topologically noetherian* for this property. It is an easy exercise to check that our condition $(\star)$ forces X to be topologically noetherian. Conversely, if X is topologically noetherian, any open subscheme of X is quasi-compact and is moreover quasi-separated by [**4**, Corollary 6.1.13, p. 296].

2.10. Lemma. *Let X be a topologically noetherian scheme.*

(1) *Let $Z \subset X$ be a closed subset. Then there exists a perfect complex $P \in \mathrm{D}^{\mathrm{perf}}(X)$ such that* $\mathrm{Supph}(P) = Z$.
(2) *Let $f : Y \to X$ be a morphism of topologically noetherian schemes and let $P \in \mathrm{D}^{\mathrm{perf}}(X)$. Then* $\mathrm{Supph}(Lf^*(P)) = f^{-1}(\mathrm{Supph}(P))$.

Proof. These two facts come respectively from Lemma 3.4 and Lemma 3.3 part (b) of [**11**]. For more on Lf^* see [**11**, 3.1]. □

2.11. Localization. Let K be a triangulated category and $J \subset K$ be a thick subcategory (Definition 2.2). We can define the *localization* or the *quotient* K/J as the triangulated category being universal for the following property : there exists a functor $q : K \to L$ such that $q(P) \simeq 0$ for all $P \in J$. This category is obtained from K via calculus of fractions by inverting those morphisms of K whose cone is in J. See more on that in [**14**, Section II.2]. Below we shall use for K/J this model. The category K/J has the same objects as those of K and morphisms are classes of fractions

$$\xleftarrow[s]{} \longrightarrow$$

where the cone of s belongs to J. Clearly, when K is moreover a $\otimes$-triangulated category (Definition 2.3) and when J is a $\otimes$-thick subcategory (Definition 2.6), the localization inherits a tensor product as well.

2.12. Idempotent completion. A triangulated category K is in particular additive and we can consider its idempotent completion $\widetilde{K}$ as an additive category. It turns out that $\widetilde{K}$ has a unique and natural structure of triangulated category such that $K \to \widetilde{K}$ is exact. More about that can be found in [**1**]. Below we shall always consider $\widetilde{K}$ to be built as follows : objects are pairs (P, p) where $p = p^2 : P \to P$ is an idempotent and morphisms $f : (P, p) \to (Q, q)$ are morphisms in K such that $fp = qf = f$. Clearly, when K is moreover a $\otimes$-triangulated category then so is $\widetilde{K}$.

Localization and idempotent completion allow the following result.

2.13. Theorem. *Let X be a scheme with noetherian underlying space. Let $U \subset X$ be an open subscheme and let $Z = X - U$ its closed complement. Consider the $\otimes$-thick subcategory $\mathrm{D}^{\mathrm{perf}}_Z(X)$ of $\mathrm{D}^{\mathrm{perf}}(X)$. Consider the quotient $\mathrm{D}^{\mathrm{perf}}(X)/\mathrm{D}^{\mathrm{perf}}_Z(X)$ and its idempotent completion. Then there is a natural equivalence of $\otimes$-triangulated categories :*

$$\widetilde{\mathrm{D}^{\mathrm{perf}}(X)/\mathrm{D}^{\mathrm{perf}}_Z(X)} \longrightarrow \mathrm{D}^{\mathrm{perf}}(U)$$

which is compatible with the localization functors coming from $\mathrm{D}^{\mathrm{perf}}(X)$.

Proof. This is basically in Thomason-Trobaugh [**12**] and we only make it explicit here for the convenience of the reader. It is clear that the restriction functor $\mathrm{D}^{\mathrm{perf}}(X) \to \mathrm{D}^{\mathrm{perf}}(U)$ induces a functor : $\mathrm{D}^{\mathrm{perf}}(X)/\mathrm{D}^{\mathrm{perf}}_Z(X) \longrightarrow \mathrm{D}^{\mathrm{perf}}(U)$. This functor is faithful by [**12**, Proposition 5.2.4 (a)] and full by [**12**, Proposition 5.2.3 (a)]. Observe that $\mathrm{D}^{\mathrm{perf}}(U)$ is idempotent complete (exercise, see also [**1**]). It follows immediately that the idempotent completion of $\mathrm{D}^{\mathrm{perf}}(X)/\mathrm{D}^{\mathrm{perf}}_Z(X)$ is also a full subcategory of $\mathrm{D}^{\mathrm{perf}}(U)$. Now it suffices to see that any object of $\mathrm{D}^{\mathrm{perf}}(U)$ is a direct summand of the restriction of an object of X. Let $F \in \mathrm{D}^{\mathrm{perf}}(U)$, then $[F \oplus T(F)] = 0$ in $K_0(U)$. The result now follows from [**12**, Proposition 5.2.2 (a)]. □

Part I. Triangular presheaves

3. Our definition of the spectrum

3.1. Notation. Let $(K, \otimes)$ be a $\otimes$-triangulated category and D be any collection of objects in K. We will denote by $\langle\, D \,\rangle$ the smallest $\otimes$-thick subcategory which contains D, that is the intersection of all $\otimes$-thick subcategories which contain D. (See Definition 2.6 for $\otimes$-thick.) As usual, when $D = \{d\}$ is reduced to a singleton, we shall write $\langle d \rangle$ instead of $\langle \{d\} \rangle$.

3.2. Exercise. Let $(K, \otimes)$ be a $\otimes$-triangulated category and D be any collection of objects in K. Denote by $\Theta(D)$ the following subset of objects of K: those $a \in K$ such that there exist $b, c \in D$, $e, e', f \in K$, an exact triangle $b \longrightarrow c \longrightarrow e \oplus e' \longrightarrow T(b)$ and an isomorphism $a \simeq e$, $a \simeq e \otimes f$ or $a \simeq f \otimes e$. Let $\Theta^0(D) = D \cup \{0\}$ and for each $n \geq 1$, let $\Theta^n(D) = \Theta(\Theta^{n-1}(D))$. Show that $\langle D \rangle$ is the full subcategory of K on the following objects: $\cup_{n \in \mathbb{N}} \Theta^n(D)$.

3.3. Definition. Let $(K, \otimes)$ be a $\otimes$-triangulated category. A $\otimes$-thick subcategory $A \subset K$ will be called *atomic* if the following condition holds: whenever we have a collection of objects $D \subset K$ such that $A \subset \langle D \rangle$ then there exists an element $d \in D$ such that $A \subset \langle d \rangle$.

3.4. Remarks. Let $(K, \otimes)$ be a $\otimes$-triangulated category.

(1) An atomic $\otimes$-thick subcategory A of K is in particular *principal* in the sense that there is an element $a \in A$ such that $A = \langle a \rangle$. The converse is false.
(2) It is an easy exercise to check that the condition of Definition 3.3 is equivalent to: whenever we have a collection of $\otimes$-thick subcategories $\{B_i \subset K \,|\, i \in I\}$ such that $A \subset \langle \cup_{i \in I} B_i \rangle$ then there exists an index $i \in I$ for which $A \subset B_i$.

3.5. Definition. Let $(K, \otimes)$ be a $\otimes$-triangulated category. We denote by $\mathrm{Spc}(K, \otimes)$ the set of all non-zero atomic subcategories of K. We will sometimes abbreviate this set by $\mathrm{Spc}(K)$ when the tensor structure is understood. We now give it a topology.

3.6. Notation. Let $(K, \otimes)$ be a $\otimes$-triangulated category. For any object $a \in K$, let

$$U(a) := \{B \in \mathrm{Spc}(K) \,\big|\, B \not\subset \langle a \rangle \}.$$

We shall also make use later of the complement $F(a) := \{B \in \mathrm{Spc}(K) \,\big|\, B \subset \langle a \rangle \}$.

3.7. Lemma. *With the above notations, for any $a, b \in K$ we have $U(a) \cap U(b) = U(a \oplus b)$. Moreover, $U(0) = \mathrm{Spc}(K)$.*

Proof. Observe that $\langle a \oplus b \rangle = \langle\, \{a, b\} \,\rangle$. Thus by definition, any atomic subcategory $B \subset K$ is contained in $\langle a \oplus b \rangle$ if and only if it is contained in $\langle a \rangle$ or in $\langle b \rangle$. $\square$

3.8. Definition. Taking the collection $\{U(a) \,\big|\, a \in K\}$ as a basis of a topology, we regard $\mathrm{Spc}(K, \otimes)$ as a topological space. We call it *the spectrum* of the $\otimes$-triangulated category $(K, \otimes)$.

3.9. Example. Let X be a topologically noetherian scheme. Then $\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(X), \otimes^L)$ is homeomorphic to the underlying topological space of X. This is Theorem 7.3 below.

3.10. Lemma. *Let $(K,\otimes)$ be a $\otimes$-triangulated category. Let $F\subset \mathrm{Spc}(K)$ be a closed subset and $A\in F$. For any $A'\in\mathrm{Spc}(K)$ such that $A'\subset A$, we have $A'\in F$.*

Proof. For an elementary closed subset of the form $F=F(a)$ with $a\in K$ (see 3.6), it is immediate. Now any closed subset of $\mathrm{Spc}(K)$ is an intersection of such subsets. □

4. The geometric functors

In order to make the construction $\mathrm{Spc}(-)$ functorial, we have to restrict ourselves to some morphisms, that we call *geometric morphisms* of $\otimes$-triangulated categories.

4.1. Definition. Let $\varphi:(K,\otimes)\to(L,\otimes)$ be a morphism of $\otimes$-triangulated categories. (We denote both tensor products by $\otimes$ since no confusion can occur from that.) We say that φ is *geometric* if the following condition is satisfied: For any collection of $\otimes$-thick subcategories $\{C_i\,\big|\,i\in I\}$ in K one has
$$\langle\varphi(\bigcap_{i\in I}C_i)\rangle=\bigcap_{i\in I}\langle\varphi(C_i)\rangle\,.$$
Observe that the inclusion $\subset$ is always true. The above condition roughly says that $\langle\varphi(-)\rangle$ commutes with arbitrary intersections.

4.2. Definition. Let $\varphi:(K,\otimes)\to(L,\otimes)$ be a morphism of $\otimes$-triangulated categories. We say that φ is *dense* if $\langle\varphi(K)\rangle=L$. See Remark 4.7.

4.3. Example. Let $f:Y\to X$ be a morphism of topologically noetherian schemes. Then $Lf^*:\mathrm{D}^{\mathrm{perf}}(X)\to\mathrm{D}^{\mathrm{perf}}(Y)$ is geometric and dense. This is Proposition 7.4 below.

4.4. Example. Let $\varphi:(K,\otimes)\to(L,\otimes)$ be a $\otimes$-equivalence of $\otimes$-triangulated categories. Then φ is geometric and dense.

4.5. Lemma. *Let $\varphi:K\to L$ be a morphism of $\otimes$-triangulated categories. Let $D\subset K$ be a collection of objects. Then $\langle\varphi(D)\rangle=\langle\,\varphi(\langle D\rangle)\,\rangle$.*

Proof. The left-hand side is clearly a subcategory of the right-hand side. Conversely, the subcategory $\varphi^{-1}(\langle\varphi(D)\rangle)$ of K is $\otimes$-thick and contains D, so it contains $\langle D\rangle$. □

4.6. Proposition and Definition. *Let $\varphi:K\to L$ be a morphism of $\otimes$-triangulated categories. Assume that φ is geometric and dense. Let $C\in\mathrm{Spc}(L)$. Define*
$$\Phi(C):=\bigcap_{\substack{H\subset K\ \otimes\text{-thick subcategory}\\ \text{s.t. } C\subset\langle\varphi(H)\rangle}}H.$$
Then $\Phi(C)$ is a non-zero atomic $\otimes$-thick subcategory of K. Moreover, we have $C\subset\langle\,\varphi(\Phi(C))\,\rangle$.

Proof. From φ being geometric, we commute $\langle\varphi(-)\rangle$ with the above intersection and get immediately
$$\langle\varphi(\Phi(C))\rangle=\bigcap_{\substack{H\subset K\ \otimes\text{-thick subcategory}\\ \text{s.t. } C\subset\langle\varphi(H)\rangle}}\langle\varphi(H)\rangle\qquad\supset C$$
which is the "moreover part". Thus it is clear that $C\neq 0$ implies $\Phi(C)\neq 0$.

Let $D\subset K$ be a collection of objects of K such that $\Phi(C)\subset\langle D\rangle$. Applying $\langle\varphi(-)\rangle$ to this relation, using the previous Lemma and the above observation, we have that: $C\subset\langle\varphi(D)\rangle$. Since C is atomic, there is an element $d\in D$ with $C\subset\langle\varphi(d)\rangle=\langle\varphi(\langle d\rangle)\rangle$. By the very definition of $\Phi(C)$, this forces $\Phi(C)\subset\langle d\rangle$ (think of $H=\langle d\rangle$). This gives the result. □

4.7. Remark. We have only used φ dense to be sure that there exists at least one $\otimes$-thick subcategory H of K such that $\langle\varphi(H)\rangle \supset C$. Pedantically speaking, requiring φ to be geometric implies already that φ is dense. To see this, apply the defining condition to the empty family: $\{C_i \,|\, i \in I\} = \emptyset$ in Definition 4.1. Actually, we could even weaken the condition $\langle\varphi(K)\rangle = L$ into: $\langle\varphi(K)\rangle \supset C$ for all $C \in \mathrm{Spc}(L)$.

Hereafter, we shall drop the mention "dense" for geometric morphisms.

4.8. Proposition. *Let $\varphi : K \to L$ be a geometric morphism of $\otimes$-triangulated categories. Let $\Phi : \mathrm{Spc}(L) \to \mathrm{Spc}(K)$ be defined as in 4.6. Then Φ is continuous.*

Proof. Let $a \in K$. Recall the notation for closed subsets: $F(a) = \{B \in \mathrm{Spc}(K) \,|\, B \subset \langle a\rangle\}$ (see 3.6). It is enough to prove the following Lemma:

4.9. Lemma. *With the above notations, $\Phi^{-1}(F(a)) = F(\varphi(a))$.*

Proof. Let $C \in \mathrm{Spc}(L)$ be an atomic subcategory of L.

Assume that $C \in \Phi^{-1}(F(a))$. This means that $\Phi(C) \subset \langle a\rangle$. Applying $\langle\varphi(-)\rangle$ to both sides, using the "moreover part" of Proposition 4.6 on the left, and Lemma 4.5 on the right, we obtain $C \subset \langle\varphi(a)\rangle$. This means $C \in F(\varphi(a))$.

Conversely assume that $C \in F(\varphi(a))$. This means $C \subset \langle\varphi(a)\rangle$. Using Lemma 4.5 again, this implies that $H_0 := \langle a\rangle$ is one of the $\otimes$-thick subcategories H of K satisfying $C \subset \langle\varphi(H)\rangle$. By the very definition of $\Phi(C)$, we have $\Phi(C) \subset H_0 = \langle a\rangle$. This means $C \in \Phi^{-1}(F(a))$.

This finishes the proof of the Lemma and of the Proposition. □

4.10. Definition. Let $\varphi : K \to L$ be a geometric morphism of $\otimes$-triangulated categories. Define $\mathrm{Spc}(\varphi) : \mathrm{Spc}(L) \to \mathrm{Spc}(K)$ to be the continuous map Φ defined in 4.6.

4.11. Proposition. *We have a contravariant functor* $\mathrm{Spc}(-)$ *from $\otimes$-triangulated categories with geometric morphisms to the category of topological spaces. Two isomorphic geometric morphisms of $\otimes$-triangulated categories induce the same map.*

Proof. Left as an easy familiarization exercise. □

4.12. Example. Let $f : Y \to X$ be a morphism of topologically noetherian schemes. Via the identifications announced in Example 3.9, the morphism $\mathrm{Spc}(Lf^*)$ coincides with the underlying morphism of spaces $f : Y \to X$. This is Theorem 7.7 below.

5. A presheaf of tensor triangulated categories

5.1. Definition. Let $(K, \otimes)$ be a $\otimes$-triangulated category. Let $V \subset \mathrm{Spc}(K, \otimes)$ be an open subset. Denote by $J(V) := \Big\langle \bigcup_{A \in \mathrm{Spc}(K)-V} A \Big\rangle$. Since this is a $\otimes$-thick subcategory, the idempotent completion of the localization

$$\mathcal{K}(V) := \big(\widetilde{K/J(V)}\big)$$

inherits a structure of $\otimes$-triangulated category (see 2.11 and 2.12). Below, we will denote by $q_V : K \to \mathcal{K}(V)$ the natural morphism.

5.2. Example. Let $(K, \otimes)$ be a $\otimes$-triangulated category. Let $V = \mathrm{Spc}(K)$. Then $J(V) = 0$ and $\mathcal{K}(V) = \widetilde{K}$ is the idempotent completion of K.

5.3. Proposition. *Let $(K, \otimes)$ be a $\otimes$-triangulated category. Let $V_1 \subset V_2$ be open subsets of* $\mathrm{Spc}(K)$*. Adopt the notations of 5.1.*

(1) *We have $J(V_2) \subset J(V_1)$.*
(2) *There is a unique morphism of $\otimes$-triangulated categories $\rho_{V_1V_2} : \mathcal{K}(V_2) \to \mathcal{K}(V_1)$ such that $q_{V_1} = \rho_{V_1V_2} \circ q_{V_2}$.*

Moreover, we have $\rho_{V_1V_3} = \rho_{V_1V_2} \circ \rho_{V_2V_3}$ for any triple of open subsets $V_1 \subset V_2 \subset V_3$ in $\mathrm{Spc}(K)$*.*

Proof. The first assertion is immediate. The second follows from the universal property of localization and idempotent completion. Similarly for the "moreover part". Actually, the above equalities are only isomorphisms of functors. If we choose an explicit construction of all the localizations and idempotent completions in terms of K, calculus of fractions and idempotents, then these equalities hold strictly (see 2.11 and 2.12). □

5.4. Remark. Heuristically, we should consider $V \mapsto \mathcal{K}(V)$ as a presheaf of $\otimes$-triangulated categories on $\mathrm{Spc}(K, \otimes)$. We will not develop here a whole theory of presheaves of triangulated categories, dealing with all equalities which are in fact isomorphisms and the like. We leave these improvements to the conscientious reader.

5.5. Example. On a topologically noetherian scheme X, the above presheaf $\mathcal{K}$ associated to the tensor triangulated category $(K, \otimes) = (\mathrm{D}^{\mathrm{perf}}(X), \otimes^L)$ produces $\mathrm{D}^{\mathrm{perf}}(U)$ on each open $U \subset X$. See Theorem 7.8 below.

5.6. Definition. We shall call here *triangular presheaf* a pair $(X, \mathcal{K})$ consisting of a topological space X and a presheaf $\mathcal{K}$ of triangulated categories on X. Given a continuous map $f : X' \to X$ and a triangular presheaf $(X', \mathcal{K}')$, we denote as usual by $f_*\mathcal{K}'$ the presheaf of triangulated categories on X defined by $f_*\mathcal{K}'(V) = \mathcal{K}'(f^{-1}(V))$. A *morphism of triangular presheaves* $(f, \mathcal{F}) : (X', \mathcal{K}') \to (X, \mathcal{K})$ consists of a continuous map $f : X' \to X$ and of a morphism of presheaves of triangulated categories $\mathcal{F} : \mathcal{K} \to f_*\mathcal{K}'$ on X.

5.7. Remark. We now want to briefly study the functoriality of the above described construction $K \mapsto (\mathrm{Spc}(K), \mathcal{K})$ with respect to K, although, strictly speaking, this is not necessary for the reconstruction of schemes. It turns out that this functoriality is more tricky that one could expect and that restricting ourselves to geometric morphisms is not enough. Therefore, we restrict our study to a smaller class of triangulated categories, still cautiously keeping in our basket all derived categories of perfect complexes on topologically noetherian schemes.

5.8. Definition. Let $(K, \otimes)$ be a $\otimes$-triangulated category. We say that K is *molecular* if any $\otimes$-thick subcategory $B \subset K$ is generated by the atomic subcategories it contains:

$$B = \Big\langle \bigcup_{\substack{A \in \mathrm{Spc}(K) \\ \text{s.t. } A \subset B}} A \Big\rangle.$$

5.9. Example. Let X be a topologically noetherian scheme. The $\otimes$-triangulated category $\mathrm{D}^{\mathrm{perf}}(X)$ is molecular. This is an easy exercise once we have Theorem 7.3 below.

5.10. Lemma. *Let $\varphi : K \to K'$ be a geometric morphism of $\otimes$-triangulated categories. Assume that K' is molecular. Let $\Phi := \mathrm{Spc}(\varphi) : \mathrm{Spc}(K') \to \mathrm{Spc}(K)$ be the induced map of 4.6. Let $V \subset \mathrm{Spc}(K)$. Recall the notations of 5.1. Then we have the following inclusion:*

$$\varphi(J(V)) \subset J\big(\Phi^{-1}(V)\big).$$

of subcategories of K'.

Proof. Let $C \in \mathrm{Spc}(K')$ such that $C \subset \langle\, \varphi(J(V))\,\rangle$. We claim that $\Phi(C) \in \mathrm{Spc}(K) - V$. It is obvious from the definition of Φ given in 4.6 that the inclusion $C \subset \langle\, \varphi(J(V))\,\rangle$ implies $\Phi(C) \subset J(V)$. By definition 5.1, we have $J(V) = \big\langle \cup_{A \in \mathrm{Spc}(K)-V} A \big\rangle$. Now $\Phi(C)$ is atomic and is contained in $J(V)$, which forces the existence of an $A \in \mathrm{Spc}(K) - V$ with $\Phi(C) \subset A$. It follows from Lemma 3.10 that $\Phi(C)$ also belongs to the closed subset $\mathrm{Spc}(K) - V$ of $\mathrm{Spc}(K)$. This proves the claim.

We have proved above that any $C \in \mathrm{Spc}(K')$ such that $C \subset \langle\, \varphi(J(V))\,\rangle$ is contained in the $\otimes$-thick subcategory $J\big(\Phi^{-1}(V)\big)$. We now use the hypothesis that K' is molecular to conclude that $\langle\, \varphi(J(V))\,\rangle$ is itself contained in $J\big(\Phi^{-1}(V)\big)$. □

5.11. Proposition. *Let $\varphi : K \to K'$ be a geometric morphism of $\otimes$-triangulated categories. Assume that K' is molecular. Let $\Phi := \mathrm{Spc}(\varphi) : \mathrm{Spc}(K') \to \mathrm{Spc}(K)$ be the induced map and let $\mathcal{K}$ and $\mathcal{K}'$ be the respective presheaves of $\otimes$-triangulated categories (see 5.1). Then there is a morphism of presheaves of $\otimes$-triangulated categories on* $\mathrm{Spc}(K)$ *:*

$$\mathcal{F} : \mathcal{K} \longrightarrow \Phi_* \mathcal{K}'$$

whose global section $\mathcal{F}(\mathrm{Spc}(K)) : \widetilde{K} \to \widetilde{K}'$ coincides with the idempotent completion of φ.

Proof. Let $V \subset \mathrm{Spc}(K)$ be an open subset. Recall the notations of 5.1. From the above Lemma, we know that φ induces a commutative (solid) diagram:

$$\begin{tikzcd}
0 \arrow[r] & J(V) \arrow[r] \arrow[d] & K \arrow[r] \arrow[d, "\varphi"'] & K/J(V) \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & J(\Phi^{-1}(V)) \arrow[r] & K' \arrow[r] & K'/J(\Phi^{-1}(V)) \arrow[r] & 0.
\end{tikzcd}$$

Therefore φ induces a morphism of $\otimes$-triangulated categories: $K/J(V) \to K'/J(\Phi^{-1}(V))$. The idempotent completion of this morphism is the wanted morphism $\mathcal{K}(V) \to \Phi_*\mathcal{K}'(V)$. The details are left to the reader. □

5.12. Corollary. *The construction $(K, \otimes) \mapsto (\mathrm{Spc}(K, \otimes), \mathcal{K})$ described in 5.1 defines a contravariant functor from molecular $\otimes$-triangulated categories (see 5.8) and geometric morphisms (see 4.1) towards the category of triangular presheaves (see 5.6).*

Proof. Obvious from the proof of the above Proposition. □

6. Presheaves of rings

6.1. Definition. Let K be a triangulated category. We denote by $\mathrm{End}(K)$ the ring of those endomorphisms of the identity functor on K which commute with translation. This is the same as the set of collections $(\alpha_a)_{a \in K}$ of endomorphisms $\alpha_a : a \to a$ such that for any morphism $f : a \to b$ in K one has $\alpha_b\, f = f\, \alpha_a$ and such that $\alpha_{T(a)} = T(\alpha_a)$. Observe that $\mathrm{End}(K)$ is commutative.

6.2. Definition. An endomorphism of the identity $\alpha \in \mathrm{End}(K)$ is called *pointwise nilpotent* if for any $a \in K$, α_a is nilpotent. We shall denote by $\mathrm{PNil}(K)$ the ideal of pointwise nilpotent elements of $\mathrm{End}(K)$.

6.3. Proposition. *Let K be a triangulated category and D be a set of generators. (That is the classical notion, i.e. $K = \langle D \rangle$ for $\otimes = 0$.) Assume that an element $\alpha \in \mathrm{End}(K)$ is nilpotent on each object of D. Then α is pointwise nilpotent.*

Proof. Clearly, if $\alpha_a^n = 0$ then the same holds for a direct summand of a. So it is enough to prove the following. Assume that

$$a \xrightarrow{u} b \xrightarrow{v} c \xrightarrow{w} T(a)$$

is an exact triangle and that α_a and α_b are nilpotent, then so is α_c. Replacing α by some power of itself, we may as well assume that $\alpha_a = 0$ and $\alpha_b = 0$. But then from $\alpha_c\, v = v\, \alpha_b = 0$ we deduce the existence of $\beta : T(a) \to c$ such that $\alpha_c = \beta\, w$. It follows that $\alpha_c^2 = \beta\, w\, \alpha_c = \beta\, T(\alpha_a)\, w = 0$. More generally, in an exact triangle of endomorphisms (obvious sense), if two of them are nilpotent then so is the third. $\square$

6.4. Lemma. *Let $\varphi : K \simeq K'$ be an equivalence of triangulated categories. Then φ induces an isomorphism of rings $\mathrm{End}(K) \simeq \mathrm{End}(K')$, which preserves pointwise nilpotent elements.*

Proof. Let $\psi : K' \to K$ be an inverse equivalence and let $\zeta : \mathrm{Id}_{K'} \xrightarrow{\sim} \varphi\psi$ be an isomorphism of functors. Define $f : \mathrm{End}(K) \to \mathrm{End}(K')$ by the following formula: $f(\alpha)_b = \zeta_b^{-1}\varphi(\alpha_{\psi(b)})\zeta_b$. Then the rest of the proof is straightforward. $\square$

6.5. Lemma. *Let K be a triangulated category and $J \subset K$ be a thick subcategory. Consider the idempotent completion of the quotient $L = \widetilde{K/J}$ and $q : K \to L$ the natural functor. Then there is a ring homomorphism $\rho : \mathrm{End}(K) \to \mathrm{End}(L)$ such that $\rho(\alpha)_{q(a)} = q(\alpha_a)$ for any $\alpha \in \mathrm{End}(K)$ and $a \in K$. Moreover, this homomorphism preserves the pointwise nilpotent elements.*

Proof. Observe first of all that the previous Lemma allows us to choose our favorite model for localization and idempotent completion, namely those of 2.11 and 2.12. The localization step is very easy. The idempotent completion step is only easy: define $\rho(\alpha)$ on an object (a, p), where $p = p^2 : a \to a$ to be $\rho(\alpha)_{(a,p)} := p\,\alpha_a = \alpha_a\, p$. The "moreover part" is obvious. $\square$

6.6. Definition. Let $(K, \otimes)$ be a $\otimes$-triangulated category. Consider the triangular presheaf $(\mathrm{Spc}(K), \mathcal{K})$ of Section 5. We consider the presheaf of rings $V \mapsto \mathrm{End}(\mathcal{K}(V))$ on $\mathrm{Spc}(K, \otimes)$, whose sheafification will be denoted by $\mathcal{O}'_K$. We have defined a *ringed space*

$$\mathrm{Space}(K, \otimes) := (\mathrm{Spc}(K), \mathcal{O}'_K).$$

Similarly, we consider the presheaf of rings $V \mapsto \mathrm{End}(\mathcal{K}(V))/\,\mathrm{PNil}(\mathcal{K}(V))$ on $\mathrm{Spc}(K, \otimes)$, whose sheafification will be denoted by $\mathcal{O}''_K$. We have defined another *ringed space*

$$\mathrm{Space}_{\mathrm{pt.red}}(K, \otimes) := (\mathrm{Spc}(K), \mathcal{O}''_K).$$

(We avoid using the tempting notation $\mathcal{O}_K$ because other sheaves should also be consider: see Section 9. For the time being, we see no reason to designate one of them as the "right" one.)

6.7. Example. Let X be a topologically noetherian scheme. We will see that the ringed space $\mathrm{Space}_{\mathrm{pt.red}}\big(\mathrm{D}^{\mathrm{perf}}(X)\big)$ is the reduced ringed space X_{red}. In general, $\mathrm{Space}\big(\mathrm{D}^{\mathrm{perf}}(X)\big)$ will be a bigger ringed space than X. See more in Theorem 8.5.

6.8. Remark. It is unclear for what nice class of morphisms of $\otimes$-triangulated categories the above constructions $\mathrm{Space}_{(\mathrm{pt.red})}(K, \otimes)$ could be made functorial. At the very least, two equivalent categories have isomorphic ringed spaces, by Lemma 6.4, which is enough for the reconstruction purposes. For general morphisms $\varphi : K \to K'$, it seems rather unpredictable when an endomorphism of the identity on K extends to K'.

6.9. Proposition. *Let $(K, \otimes)$ be a $\otimes$-triangulated category. There is a surjective morphism $r_K : \mathcal{O}'_K \longrightarrow \mathcal{O}''_K$ of sheaves of rings on $\mathrm{Spc}(K, \otimes)$.*

Proof. The morphism is the sheafification of the obvious epimorphism $\mathrm{End} \to \mathrm{End}/\,\mathrm{PNil}$. $\square$

Part II. Reconstruction of schemes

7. The underlying topological space

7.1. Lemma. *Let X be a topologically noetherian scheme. Let $\{Y_i \,|\, i \in I\}$ be a collection of specialization closed subsets of X. Then their union $Y = \cup_{i\in I} Y_i$ is also specialization closed and*

$$\mathrm{D}^{\mathrm{perf}}_Y(X) = \Big\langle \bigcup_{i\in I} \mathrm{D}^{\mathrm{perf}}_{Y_i}(X) \Big\rangle.$$

Proof. By Corollary 2.8, the $\otimes$-thick subcategory $\big\langle \cup_{i\in I} \mathrm{D}^{\mathrm{perf}}_{Y_i}(X) \big\rangle$ of $\mathrm{D}^{\mathrm{perf}}(X)$ has to be of the form $\mathrm{D}^{\mathrm{perf}}_Z(X)$ for some specialization closed subset $Z \subset X$. Since Thomason's correspondence preserves inclusions, it is immediate that Z must be the smallest specialization closed subset containing all the Y_i's, that is their union. □

7.2. Proposition. *Let X be a topologically noetherian scheme and $Y \subset X$ be a specialization closed subset. Then $\mathrm{D}^{\mathrm{perf}}_Y(X)$ is a non-zero atomic subcategory of $\mathrm{D}^{\mathrm{perf}}(X)$ in the sense of Definition 3.3 if and only if Y is non-empty, closed and irreducible.*

Proof. Assume that Y is non-empty, closed and irreducible. There is a point $y \in Y$ such that $Y = \overline{\{y\}}$. Let $A := \mathrm{D}^{\mathrm{perf}}_Y(X)$ in $K := \mathrm{D}^{\mathrm{perf}}(X)$. First of all $A \neq 0$ by Lemma 2.10 (1). We have to see that A is atomic. We choose to verify Definition 3.3 via the equivalent condition of Remark 3.4 (2). Choose a collection of $\otimes$-thick subcategories $\{B_i \subset K \,|\, i \in I\}$ such that $A \subset \langle \cup_{i\in I} B_i \rangle$. For each B_i take the unique specialization closed subset $Y_i \subset X$ such that $B_i = \mathrm{D}^{\mathrm{perf}}_{Y_i}(X)$. The assumption that $A \subset \langle \cup_{i\in I} B_i \rangle$ and the previous Lemma imply that $Y \subset \cup_{i\in I} Y_i$. Since $y \in Y$, this forces $y \in Y_i$ for some $i \in I$; therefore $Y = \overline{\{y\}} \subset Y_i$ and $A \subset B_i$.

Conversely assume that $A := \mathrm{D}^{\mathrm{perf}}_Y(X)$ is non-zero and atomic. From A non-zero one has Y non-empty. Since $Y \subset \cup_{y\in Y} \overline{\{y\}}$, the previous Lemma forces $A \subset \big\langle \cup_{y\in Y} \mathrm{D}^{\mathrm{perf}}_{\overline{\{y\}}}(X) \big\rangle$. Using that A is atomic, there exists a $y \in Y$ such that $A \subset \mathrm{D}^{\mathrm{perf}}_{\overline{\{y\}}}(X)$. This in turn implies that $Y \subset \overline{\{y\}}$. That is $Y = \overline{\{y\}}$ is an irreducible closed subset of X. □

7.3. Theorem. *Let X be a scheme with noetherian underlying space. Consider the map*

$$\begin{aligned} E:\ X &\longrightarrow \mathrm{Spc}\big(\mathrm{D}^{\mathrm{perf}}(X)\big) \\ x &\mapsto \mathrm{D}^{\mathrm{perf}}_{\overline{\{x\}}}(X) = \big\{ P \in \mathrm{D}^{\mathrm{perf}}(X) \,\big|\, \mathrm{Supph}(P) \subset \overline{\{x\}} \,\big\}. \end{aligned}$$

This is a well-defined homeomorphism between the underlying space of X and $\mathrm{Spc}\big(\mathrm{D}^{\mathrm{perf}}(X)\big)$ with the topology of Definition 3.8.

Proof. The map $E : X \to \mathrm{Spc}(\mathrm{D}^{\mathrm{perf}} X)$ is well-defined and bijective by Proposition 7.2 and Thomason's classification (Corollary 2.8). We want to prove that E respects the topology. Recall from Definition 3.8, that the topology on $\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}} X)$ is given by the following basis $\mathcal{B} := \{U(a) \,|\, a \in \mathrm{D}^{\mathrm{perf}}(X)\}$, where $U(a) = \{B \in \mathrm{Spc}(\mathrm{D}^{\mathrm{perf}} X) \,|\, B \not\subset \langle a \rangle \,\}$. Let $a \in \mathrm{D}^{\mathrm{perf}}(X)$ and consider the closed subset $Y := \mathrm{Supph}(a)$ of X. We have $\langle a \rangle = \mathrm{D}^{\mathrm{perf}}_Y(X)$ because Y is the smallest specialization closed $Z \subset X$ such that $a \in \mathrm{D}^{\mathrm{perf}}_Z(X)$, using Corollary 2.8 again. Therefore $U(a) = \{E(x) \,|\, x \in X, \text{ s.t. } \mathrm{D}^{\mathrm{perf}}_{\overline{\{x\}}}(X) \not\subset \mathrm{D}^{\mathrm{perf}}_Y(X)\} = E(X - Y) = E(X - \mathrm{Supph}(a))$.

In other words, under our bijection E, the basis $\mathcal{B}$ of $\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}} X)$ corresponds to the following collection of open subsets of X: $\mathcal{B}' = \{X - \mathrm{Supph}(a) \,|\, a \in \mathrm{D}^{\mathrm{perf}}(X)\}$. Conversely, since X is topologically noetherian, any open in X is an element of $\mathcal{B}'$, by Lemma 2.10 (1). □

7.4. Proposition. *Let $f : Y \to X$ be a morphism of topologically noetherian schemes. Let $\varphi := Lf^* : \mathrm{D}^{\mathrm{perf}}(X) \to \mathrm{D}^{\mathrm{perf}}(Y)$. Then φ is geometric and dense in the sense of Definitions 4.1 and 4.2.*

7.5. Lemma. *With the notations of the above Proposition, let $Z \subset X$ be specialization closed. Observe that $f^{-1}(Z)$ is specialization closed in Y. We have*

$$\Big\langle \varphi(\mathrm{D}^{\mathrm{perf}}_Z(X)) \Big\rangle = \mathrm{D}^{\mathrm{perf}}_{f^{-1}(Z)}(Y).$$

Proof. Let $y \in f^{-1}(Z)$. By continuity of f, we have $f(\overline{\{y\}}) \subset \overline{\{f(y)\}} \subset Z$ since Z is specialization closed. This means that $\overline{\{y\}} \subset f^{-1}(Z)$. Thus $f^{-1}(Z)$ is specialization closed.

Therefore, the right-hand category is a $\otimes$-thick subcategory of $\mathrm{D}^{\mathrm{perf}}(Y)$. The following inclusion: $\langle \varphi(\mathrm{D}^{\mathrm{perf}}_Z(X)) \rangle \subset \mathrm{D}^{\mathrm{perf}}_{f^{-1}(Z)}(Y)$ is a direct consequence of Lemma 2.10 (2). We want to prove the converse inclusion. By Thomason's classification, there exists W specialization closed in Y such that $\langle \varphi(\mathrm{D}^{\mathrm{perf}}_Z(X)) \rangle = \mathrm{D}^{\mathrm{perf}}_W(Y)$. We are left to prove $W \supset f^{-1}(Z)$.

Let $x \in Z$. Then there is a complex $e_x \in \mathrm{D}^{\mathrm{perf}}(X)$ such that $\mathrm{Supph}(e_x) = \overline{\{x\}}$ by Lemma 2.10 (1). Since $\varphi(e_x) \in \big\langle \varphi(\mathrm{D}^{\mathrm{perf}}_Z(X)) \big\rangle$ we have $W \supset \mathrm{Supph}(\varphi(e_x)) = f^{-1}(\mathrm{Supph}(e_x))$ by Lemma 2.10 (2). This means that $W \supset f^{-1}(\overline{\{x\}}) \supset f^{-1}(\{x\})$ and this holds for arbitrary $x \in Z$. In other words, we have $W \supset f^{-1}(Z)$ which is the claim. $\square$

7.6. Proof of Proposition 7.4. Let $\{C_i \,|\, i \in I\}$ be a collection of $\otimes$-thick subcategories in $\mathrm{D}^{\mathrm{perf}}(X)$ and, for each $i \in I$, let $Z_i \subset X$ be the specialization closed subset such that $C_i = \mathrm{D}^{\mathrm{perf}}_{Z_i}(X)$. Let also $Z := \cap_{i\in I} Z_i$ and observe that $f^{-1}(Z) = \cap_{i \in I} f^{-1}(Z_i)$. We then compute directly, using the above Lemma:

$$\Big\langle \varphi(\bigcap_{i\in I} C_i) \Big\rangle = \Big\langle \varphi(\bigcap_{i\in I} \mathrm{D}^{\mathrm{perf}}_{Z_i}(X)) \Big\rangle = \big\langle \varphi(\mathrm{D}^{\mathrm{perf}}_Z(X)) \big\rangle = \mathrm{D}^{\mathrm{perf}}_{f^{-1}(Z)}(Y) =$$
$$= \bigcap_{i\in I} \mathrm{D}^{\mathrm{perf}}_{f^{-1}(Z_i)}(Y) = \bigcap_{i\in I} \big\langle \varphi(\mathrm{D}^{\mathrm{perf}}_{Z_i}(X)) \big\rangle = \bigcap_{i\in I} \big\langle \varphi(C_i) \big\rangle.$$

This proves that φ is geometric. It is dense because $\mathcal{O}_Y \simeq \varphi(\mathcal{O}_X) \in \big\langle \varphi(\mathrm{D}^{\mathrm{perf}}(X)) \big\rangle$. $\square$

7.7. Theorem. *Let $f : Y \to X$ be a morphism of schemes with noetherian underlying spaces. Consider the morphism of $\otimes$-triangulated categories $Lf^* : \mathrm{D}^{\mathrm{perf}}(X) \to \mathrm{D}^{\mathrm{perf}}(Y)$, that we know to be geometric. Then the following diagram of topological spaces:*

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "E"', "\simeq"] & X \arrow[d, "E"', "\simeq"] \\
\mathrm{Spc}\big(\mathrm{D}^{\mathrm{perf}}(Y)\big) \arrow[r, "\mathrm{Spc}(Lf^*)"'] & \mathrm{Spc}\big(\mathrm{D}^{\mathrm{perf}}(X)\big)
\end{tikzcd}$$

is commutative. The vertical homeomorphisms are those of Theorem 7.3, the lower line involves the functor $\mathrm{Spc}(-)$ *of Section 4.*

Proof. Let $y \in Y$. Let us use the following notations: $K := \mathrm{D}^{\mathrm{perf}}(X)$, $L := \mathrm{D}^{\mathrm{perf}}(Y)$, $\varphi := Lf^* : K \to L$ and $\Phi := \mathrm{Spc}(\varphi) : \mathrm{Spc}(L) \to \mathrm{Spc}(K)$. We want to compute $\Phi(E(y))$. By Definition 4.6,

$$\Phi(E(y)) = \bigcap_{\substack{H \subset K\ \otimes\text{-thick subcategory} \\ \text{s.t. } E(y) \subset \langle \varphi(H) \rangle}} H.$$

Using Thomason's classification to replace the H's and using Lemma 7.5, it is easy to see that $\Phi(E(y)) = \mathrm{D}^{\mathrm{perf}}_W(X)$ where $W \subset X$ is given by

$$W = \bigcap_{\substack{Z \subset X \text{ specialization closed} \\ \text{s.t. } y \in f^{-1}(Z)}} Z.$$

In other words, W is the smallest specialization closed subset of X which contains $f(y)$. This is $\overline{\{f(y)\}}$. This means that $\Phi(E(y)) = \mathrm{D}^{\mathrm{perf}}_{\overline{\{f(y)\}}}(X) = E(f(y))$, which is the claim. □

7.8. Theorem. *Let X be a scheme with noetherian underlying space. Let U be any open subscheme and $Z = X - U$ its closed complement. Let $V = E(U)$ be the corresponding open in $\mathrm{Spc}\big(\mathrm{D}^{\mathrm{perf}}(X)\big)$. Then, in the notations of Definition 5.1, we have $J(V) = \mathrm{D}^{\mathrm{perf}}_Z(X)$ and there is a natural equivalence of tensor triangulated categories with unit:*

$$\mathcal{K}(V) \simeq \mathrm{D}^{\mathrm{perf}}(U)$$

in a coherent way with respect to inclusions $U_1 \subset U_2$ – see Proposition 5.3.

Proof. By Definition 5.1 and Theorem 7.3, $J(V) = \big\langle \cup_{x \in Z} \mathrm{D}^{\mathrm{perf}}_{\overline{\{x\}}}(X) \big\rangle$. From Lemma 7.1, we have then $J(V) = \mathrm{D}^{\mathrm{perf}}_Z(X)$ since $\cup_{x \in Z} \overline{\{x\}} = Z$. The second assertion follows immediately from Theorem 2.13. □

8. The main result

We will start with a folklore observation. Let us first of all recall that over a commutative ring R the derived category $\mathrm{D}^{\mathrm{perf}}(R)$ is simply the homotopy category $\mathrm{K}^{\mathrm{b}}(\mathcal{P}(R))$ of bounded complexes of finitely generated projective R-modules.

8.1. Proposition. *Let R be a commutative ring. Consider the ring homomorphism $\lambda : R \to \mathrm{End}(\mathrm{D}^{\mathrm{perf}}(R))$ sending $r \in R$ to the multiplication by r everywhere (see 6.1).*

1. *This homomorphism admits a left inverse $\sigma : \mathrm{End}(\mathrm{D}^{\mathrm{perf}}(R)) \to R$ given by $\alpha \mapsto \alpha_{\mathbb{1}}$, where $\mathbb{1} = (\cdots \to 0 \to 0 \to R \to 0 \to 0 \to \cdots)$ is concentrated in degree zero.*
2. *The kernel of σ is made of pointwise nilpotent elements (see 6.2).*
3. *These homomorphisms induce isomorphisms $R_{\mathrm{red}} \simeq \mathrm{End}(\mathrm{D}^{\mathrm{perf}}(R))/\,\mathrm{PNil}(\mathrm{D}^{\mathrm{perf}}(R))$.*

Proof. The first assertion is obvious. The second follows from the fact that $\mathbb{1}$ is a generator of $\mathrm{D}^{\mathrm{perf}}(R)$ and from Proposition 6.3. Now λ and σ clearly induce homomorphisms on the level of R_{red} and $\mathrm{End}(\mathrm{D}^{\mathrm{perf}}(R))/\,\mathrm{PNil}(\mathrm{D}^{\mathrm{perf}}(R))$. The third assertion follows from (1) and (2). □

8.2. Remark. The above can be generalized to non-commutative rings, reconstructing in this way the reduced ring of the center of R.

8.3. Corollary. *Let R be a commutative ring. The topological space* $\mathrm{Spec}(R)$ *with its Zariski topology is an invariant of the derived category* $\mathrm{D}^{\mathrm{perf}}(R)$*, even for R non-noetherian.*

Proof. This is immediate from $\mathrm{Spec}(R_{\mathrm{red}}) = \mathrm{Spec}(R)$ and the above Proposition. The construction $\mathrm{End}(-)/\,\mathrm{PNil}(-)$ was entirely made on the level of triangulated categories. □

8.4. Theorem. *Let $f, g : Y \to X$ be two morphisms of schemes and suppose that the two induced morphisms of triangulated categories, Lf^* and $Lg^* : \mathrm{D}^{\mathrm{perf}}(X) \to \mathrm{D}^{\mathrm{perf}}(Y)$ are isomorphic. Then $f = g$.*

Proof. We know already from Theorem 7.7 and Proposition 4.11 that the underlying morphisms of topological spaces f and g are equal. The end goes easily, for instance as follows.

Assume first that $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ are affine and that $f, g : A \to B$ are ring homomorphisms such that $Lf^* \simeq Lg^*$. Evaluating this isomorphism $Lf^* \simeq Lg^*$ on the object A, viewed in $\mathrm{D}^{\mathrm{perf}}(A)$ as a complex concentrated in degree zero, we get an isomorphism of B-modules $B \simeq B$. This is just multiplication by an invertible element $s \in B$. Now, we have two commutative diagrams:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\mu"', "\simeq"] & B \arrow[d, "\simeq"', "\mu"] \\ \mathrm{End}_{\mathrm{D}^{\mathrm{perf}}(A)}(A) \arrow[r, "Lf^*"'] & \mathrm{End}_{\mathrm{D}^{\mathrm{perf}}(B)}(B) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "g"] \arrow[d, "\mu"', "\simeq"] & B \arrow[d, "\simeq"', "\mu"] \\ \mathrm{End}_{\mathrm{D}^{\mathrm{perf}}(A)}(A) \arrow[r, "Lg^*"'] & \mathrm{End}_{\mathrm{D}^{\mathrm{perf}}(B)}(B) \end{tikzcd}$$

where the isomorphisms μ just send an element to the multiplication by it. Our assumption that $Lf^* \simeq Lg^*$ boils down to $s \cdot f(a) \cdot s^{-1} = g(a)$ for any $a \in A$. This means $f = g$.

Consider now X and Y not necessarily affine. Given $U \subset X$ and $V \subset f^{-1}(U) = g^{-1}(U)$, with U and V both affine, we want to prove that $Lf^* \simeq Lg^*$ implies $L\tilde{f}^* \simeq L\tilde{g}^*$, where $\tilde{f}$ and $\tilde{g}$ are the restrictions of f and g to morphisms $V \to U$, as in the left-hand diagram below:

$$\begin{tikzcd} V \arrow[r] \arrow[dr, "\tilde{f}"'] & f^{-1}(U) \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ & U \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} \mathrm{D}^{\mathrm{perf}}(X) \arrow[r, "Lf^*"] \arrow[d, "q"'] & \mathrm{D}^{\mathrm{perf}}(Y) \arrow[d, "q"] \\ \mathrm{D}^{\mathrm{perf}}(U) \arrow[r, "L\tilde{f}^*"'] & \mathrm{D}^{\mathrm{perf}}(V) \end{tikzcd}$$

We also have the above right-hand commutative diagram, where we denote by q the restrictions and similarly for g instead of f, *mutatis mutandis*. Now recall from Theorem 2.13, that $\mathrm{D}^{\mathrm{perf}}(U)$ is just the idempotent completion of a localization of $\mathrm{D}^{\mathrm{perf}}(X)$. Thus $L\tilde{f}^*$ is characterized up to isomorphism by $L\tilde{f}^* \circ q$. Therefore, since $Lf^* \simeq Lg^*$ forces $L\tilde{f}^* \circ q \simeq q \circ Lf^* \simeq q \circ Lg^* \simeq L\tilde{g}^* \circ q$, we have $L\tilde{f}^* \simeq L\tilde{g}^*$ as wanted. The first part of the proof implies that $\tilde{f} = \tilde{g}$. This being true for arbitrary affine open $U \subset X$ and $V \subset f^{-1}(U) = g^{-1}(U)$, it forces $f = g$. □

8.5. Theorem. *Let X be a scheme with noetherian underlying space. Recall the notations of Definition 6.6. There is a natural commutative diagram of sheaves of rings on the space X :*

$$\begin{tikzcd} \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{O}'_{\mathrm{D}^{\mathrm{perf}}(X)} \arrow[d, "r_{\mathrm{D}^{\mathrm{perf}}(X)}"] \\ \mathcal{O}_X^{\mathrm{red}} \arrow[r] & \mathcal{O}''_{\mathrm{D}^{\mathrm{perf}}(X)} \end{tikzcd}$$

where the vertical homomorphisms are the natural epimorphisms (see 6.9 for the right one). Moreover, the upper morphism is split injective and the lower morphism is an isomorphism. That is, there is a natural ringed space isomorphism $X_{\mathrm{red}} \simeq \mathrm{Space}_{\mathrm{pt.red}}(\mathrm{D}^{\mathrm{perf}}(X), \otimes^L)$.

Proof. Observe first of all that those four sheaves of rings are defined on the same topological space $\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(X), \otimes^L) = X$, via the identifications of Theorem 7.3.

For each open $U \subset X$ there is a ring homomorphism $\lambda : \mathcal{O}_X(U) \to \mathrm{End}(\mathrm{D}^{\mathrm{perf}}(U))$ given by the multiplication. This is compatible with localization and idempotent completion and induces a morphism of presheaves of rings $\lambda : \mathcal{O}_X(-) \to \mathrm{End}(\mathcal{K}(-))$. Its sheafification is the wanted morphism $\mathcal{O}_X \to \mathcal{O}'_{\mathrm{D}^{\mathrm{perf}}(X)}$.

Conversely, there is a morphism $\sigma : \mathrm{End}(\mathrm{D}^{\mathrm{perf}}(U)) \to \mathcal{O}_X(U)$ given by evaluation at $\mathcal{O}_U$ when U is affine. This is again compatible with restriction between affine open subsets. Since the latter form a sub-basis of X, we have the wanted section $\sigma : \mathcal{O}'_{\mathrm{D}^{\mathrm{perf}}(X)} \to \mathcal{O}_X$.

Of course, nilpotent elements in $\mathcal{O}_X(U)$ are sent by λ to pointwise nilpotent elements of $\mathrm{End}(\mathrm{D}^{\mathrm{perf}}(U))$ and we have therefore the announced commutative diagram. To see that the lower morphism $\lambda : \mathcal{O}_{X_{\mathrm{red}}}(-) \to \mathrm{End}(\mathcal{K}(-))/\,\mathrm{PNil}(\mathcal{K}(-))$ is an isomorphism, it suffices to do it locally. This is Proposition 8.1 (3). $\square$

8.6. Corollary. *Let X and Y be topologically noetherian schemes. Assume that there is a $\otimes$-equivalence of $\otimes$-triangulated categories* $(\mathrm{D}^{\mathrm{perf}}(X), \otimes^L) \simeq (\mathrm{D}^{\mathrm{perf}}(Y), \otimes^L)$*, not necessarily coming from a morphism between X and Y. Then the reduced schemes X_{red} and Y_{red} are isomorphic.* $\square$

8.7. Corollary. *Let X and Y be noetherian schemes. Assume that there is a $\otimes$-equivalence of $\otimes$-triangulated categories* $(\mathrm{D}^{\mathrm{perf}}(X), \otimes^L) \simeq (\mathrm{D}^{\mathrm{perf}}(Y), \otimes^L)$*. Then X and Y have the same G-theory (K-theory of coherent $\mathcal{O}_X$-modules).*

Proof. We know by *dévissage* that G-theory does not distinguish X from X_{red}. $\square$

9. Comments

9.1. Remark. We do not claim that our definition of $\mathrm{Spc}(-)$ will be useful as it stands for triangulated categories not necessarily coming from schemes via $\mathrm{D}^{\mathrm{perf}}(-)$. The first dissuasion comes from non-noetherian schemes, even in the affine case. See for instance Neeman's comments in [**6**, Section 4] and Bökstedt's appendix. Anyway, this question is interesting.

9.2. Example. Consider one of the simplest examples of tensor triangulated category. Let k be a field. Let $K = k - mod$ be the category of finite dimensional vector spaces, let $T = \mathrm{Id}$ (!) and call a triangle $V_1 \to V_2 \to V_3 \to V_1$ exact when the (long) sequence of vector spaces is exact at V_1, V_2 and V_3. This is a triangulated category. It is equipped with a tensor $\otimes = \otimes_k$. Now let A be a $\otimes$-thick subcategory of K. Either $A = 0$ or it contains some sum of copies of k and therefore it contains k since it is thick, and thus $A = K$. In particular, $\mathrm{Spc}(K) = \{K\}$ is a point. Similarly, $\mathrm{Spc}(\mathrm{D}^{\mathrm{perf}}(k)) = \mathrm{Spec}(k)$ is reduced to a point. In both cases, the sheaf of rings is simply k. This shows that the above ringed space constructions lose a lot of information about triangulated categories, which is of course not a surprise.

9.3. Remark. Corollary 8.7 suggests that K-theory might be defined on the level of *tensor* triangulated categories. Note that a good K-theory of triangulated categories is still to be found. (If "good" supposes agreement with Thomason's K-theory on schemes and some long exact sequences on the triangular level.) Of course, there is Thomason-Trobaugh [**12**], but the second part of the title "Higher algebraic K-theory of schemes *and of derived categories*" is slightly misleading. The K-theory "of $\mathrm{D}^{\mathrm{perf}}(X)$" requires actually more data than just the category $\mathrm{D}^{\mathrm{perf}}(X)$, namely it requires the underlying bi-Waldhausen category of perfect complexes and quasi-isomorphisms. Similarly, more recent attempts by Neeman [**7**] to define K-theory of triangulated categories still faces some drawbacks, including the recourse to underlying models, leading in particular to his construction not being functorial. Vaknin [**13**] has an improved functorial version, but still uses models. The hope resulting from the above reconstruction result is that models could be replaced, for K-theory purposes, by tensor products. In the author's personal opinion this would be a conceptual and esthetic improvement.

9.4. Remark. There is another way of reconstructing the structure sheaf if we do not want to assume that the scheme is reduced. Namely, we can keep track of the *unit* for the tensor product $\mathbb{1} = \mathcal{O}_X$. In other words, we keep more information and consider the functor $X \mapsto (\mathrm{D}^{\mathrm{perf}}(X), \otimes^L, \mathcal{O}_X)$ as going from topologically noetherian schemes to tensor triangulated categories with unit $(K, \otimes, \mathbb{1})$. For this, we take the following definition of a unit.

9.5. Definition. Given a $\otimes$-triangulated category $(K, \otimes)$, we will call an element $\mathbb{1} \in K$ a *unit for* $\otimes$ if there are natural isomorphisms $\eta_P : P \otimes \mathbb{1} \simeq P$ and $\mu_Q : \mathbb{1} \otimes Q \simeq Q$ for all $P, Q \in K$. Here, we shall also assume that the two isomorphisms $\eta_{\mathbb{1}}$ and $\mu_{\mathbb{1}} : \mathbb{1} \otimes \mathbb{1} \to \mathbb{1}$ coincide. A morphism of $\otimes$-triangulated categories with unit must preserve the unit up to isomorphism, in a coherent way with respect to the η's and the μ's.

9.6. Lemma. *Let $(K, \otimes, \mathbb{1})$ be a tensor triangulated category with unit. Then the endomorphism ring $\mathrm{End}_K(\mathbb{1})$ is commutative.*

Proof. Let $f, g \in \mathrm{End}_K(\mathbb{1})$. Observe that the following diagram is commutative in $K \times K$:

$$\begin{tikzcd}
(\mathbb{1}, \mathbb{1}) \arrow[r, "{(f, \mathrm{Id})}"] \arrow[d, "{(\mathrm{Id}, g)}"'] \arrow[dr, "{(f, g)}"] & (\mathbb{1}, \mathbb{1}) \arrow[d, "{(\mathrm{Id}, g)}"] \\
(\mathbb{1}, \mathbb{1}) \arrow[r, "{(f, \mathrm{Id})}"'] & (\mathbb{1}, \mathbb{1})
\end{tikzcd}$$

Apply the functor $\otimes : K \times K \to K$ and replace $\mathbb{1} \otimes \mathbb{1}$ by $\mathbb{1}$ using the fact that both isomorphisms $\mathbb{1} \otimes \mathbb{1} \to \mathbb{1}$ coincide. The result follows. $\square$

Let $(K, \otimes, \mathbb{1})$ be a tensor triangulated category with unit. One can define the spectrum $\mathrm{Spc}(K, \otimes)$ and the presheaf of triangulated categories $\mathcal{K}$ exactly as before, observing that the latter is equipped with a unit on each open by localizing the unit of K. On this topological space $\mathrm{Spc}(K)$, one can then replace the presheaf of rings $\mathrm{End}(\mathcal{K}) / \mathrm{PNil}(\mathcal{K})$ which we used above by the presheaf of rings $\mathrm{End}_{\mathcal{K}}(\mathbb{1})$. Exactly the same kind of arguments as above give us the following result.

9.7. Theorem. *Consider the functor $X \mapsto (\mathrm{D}^{\mathrm{perf}}(X), \otimes^L, \mathcal{O}_X)$ from topologically noetherian schemes to tensor triangulated categories with unit. Then this functor is faithful and reflects isomorphisms.* $\square$

9.8. Remark. A more precise statement is true. Let $\mathcal{T}$ be the category of molecular tensor triangulated categories with unit, and geometric morphisms, as defined in Definitions 4.1 and 5.8. Then there exists a functor from $\mathcal{T}$ into ringed spaces such that its pre-composition with the functor $X \mapsto (\mathrm{D}^{\mathrm{perf}}(X), \otimes^L, \mathcal{O}_X)$ just gives back the natural inclusion from topologically noetherian schemes into ringed spaces.

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