The Gersten Conjecture for Witt Groups in the Equicharacteristic Case

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Abstract. We prove the Gersten conjecture for Witt groups in the equicharacteristic case, that is for regular local rings containing a field of characteristic not 2.

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0. Introduction

The *Witt group* is the classical invariant classifying symmetric spaces, up to isometry and modulo metabolic spaces, see for instance [12] for rings and [11] for schemes. The *Gersten conjecture for Witt groups*, stated by Pardon in 1982 [16], claims the existence and the exactness of a complex $\mathrm{GWC_a}(R)$:

$$0 \to \mathrm{W}(R) \to \mathrm{W}(K) \to \bigoplus_{x\in X^{(1)}} \mathrm{W}(\kappa(x)) \to \ldots \to \bigoplus_{x\in X^{(n-1)}} \mathrm{W}(\kappa(x)) \to \mathrm{W}(\kappa(\mathfrak{m})) \to 0$$

where $(R,\mathfrak{m})$ is an n-dimensional regular local ring in which 2 is invertible; we denote by $X = \mathrm{Spec}(R)$ the spectrum of R, by $X^{(p)}$ the primes of height p, by $\kappa(x)$ the residue field at a point $x \in X$, and by $K = \kappa(0)$ the field of fractions of R. We call $\mathrm{GWC_a}(R)$ *an augmented Gersten-Witt complex.* In [5] Balmer and Walter constructed a *Gersten-Witt complex*

$$\mathrm{GWC}(X) := \qquad \ldots \to 0 \longrightarrow \bigoplus_{x\in X^{(0)}} \mathrm{W}(\kappa(x)) \longrightarrow \ldots \longrightarrow \bigoplus_{x\in X^{(p)}} \mathrm{W}(\kappa(x)) \to \ldots$$

for general regular schemes X, not necessarily local or essentially of finite type, as part of the so-called *Gersten-Witt spectral sequence.* We will recall these constructions in Section 3. The augmented Gersten-Witt complex that we consider here is simply their complex $\mathrm{GWC}(R)$ augmented by the natural map $\mathrm{W}(R) \to \mathrm{W}(K)$. Our main result is Theorem 6.1 below, which says:

Theorem. *The augmented Gersten-Witt complex* $\mathrm{GWC_a}(R)$ *is exact for any equicharacteristic regular local ring* R*, i.e. for* R *regular local containing some field* k*.*

If the field k can be taken infinite with R essentially smooth over k, this has already been proven by Balmer [4] and independently by Pardon [17]. Here we extend this result first to essentially smooth local algebras over finite ground fields k. Then we extend it to regular local algebras which are not essentially of finite type to obtain the above Theorem, following a method introduced by Panin [15] to prove the equicharacteristic Gersten conjecture in K-theory.
Although these strategies have already been used for other theories, their application to Witt theory has not been rapid. For instance the Gersten conjecture for K-theory was proven by Quillen 30 years before the analogue for Witt groups. The most significant problem was that until recently [2] [3] [4] [7] it had not been established that Witt groups were part of a *cohomology theory with supports* in the sense of Colliot-Thélène, Hoobler and Kahn [6]. It is basically this observation which led to the proof of the conjecture for essentially smooth local algebras over infinite ground fields by means of a geometric proof whose roots reach back to Ojanguren's pioneering article [13].
Let us explain the general strategies to

(1) Go from infinite ground fields to any ground field.
(2) Go from essentially smooth local algebras to any regular local algebra.

The strategy for proving (1) is seemingly due to Colliot-Thélène, *cf.* [13], p. 115. One considers infinite towers of finite field extensions $k \subset F_1 \subset F_2 \subset \ldots$; the result holds "at the limit" by assumption, hence holds for some finite extension, and finally it holds for k itself by a transfer argument.
The strategy for proving (2) is due to Panin [15] and relies on results of Popescu [18] [19] which imply that any equicharacteristic regular local ring R is the filtered colimit of essentially smooth local algebras over some field $k \subset R$. There is usually no hope of getting this limit to commute with Gersten-type complexes because the morphisms in the colimit may be pretty wild. Panin's trick consists in finding a statement in terms of Zariski cohomology which is equivalent to the considered Gersten conjecture (he did it for K-theory) and then using a theorem of Grothendieck [1] asserting that the colimit and the cohomology commute.
We follow these strategies for Witt groups. The main difference between the usual cohomology theories (such as K-theory) and Witt groups is that the latter depend not only on a scheme or a category but also on a duality functor $E \mapsto E^*$ and biduality isomorphisms $\varpi_E : E \cong E^{**}$. Most schemes and categories which one studies this way come equipped with numerous choices for $({}^*, \varpi)$. For instance one can twist the duality functor for vector bundles by a line bundle, one can use shifted dualities for chain complexes, and one can change the sign of the biduality isomorphisms. When one wishes to apply a geometric argument with a pullback or a pushforward along a map $\pi : Y \to X$ one has to worry about which dualities on X and Y correspond for the construction in

question. Pushforwards (or transfers) in particular are not yet widely available (although some of the authors are working on it). Nevertheless, things have reached the point where one understands enough to construct the Gersten-Witt complex (Balmer-Walter [5]) and to treat pushforwards along a closed embedding $\operatorname{Spec}(R/fR) \hookrightarrow \operatorname{Spec} R$ of spectra of regular local rings (Gille [7]). This allows us to carry out (1) and (2).

Another reason we write this paper is that Panin's strategy for (2) is still quite new and has not yet been assimilated by the community. We hope that our exposition of this method will aid the process of digestion.

Note that our proof of the Gersten conjecture is independent of Ojanguren's and Panin's work [14] and hence we get a new proof of their main theorem, namely the purity theorem for equicharacteristic regular local rings. Nevertheless, the present work is more a generalization than a simplification of [14] since the various pieces of our proof (geometric presentation lemmas, transfers, and Panin's trick) are of similar complexity.

Apart from the ideas described in this introduction, our basic technical device is the recourse to *triangular Witt groups* [2] [3], namely Witt groups of suitable derived categories.

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1. Notations

Convention 1.1. Each time we consider the Witt group of a scheme X or of a category $\mathcal{A}$, we implicitly assume that 2 is invertible, *i.e.* that $1/2$ is in the ring of global sections $\Gamma(X, \mathcal{O}_X)$, respectively that $\mathcal{A}$ is $\mathbb{Z}[1/2]$-linear. Of course, this has nothing to do with "tensoring outside with $\mathbb{Z}[1/2]$" and our Witt groups might very well have non-trivial 2-torsion.

Let X be a noetherian scheme with structure sheaf $\mathcal{O}_X$, and let Z be a closed subset. For a complex P of quasi-coherent $\mathcal{O}_X$-modules we define the (*homological*) *support* of P to be

$$\operatorname{supph}(P) := \bigcup_{i \in \mathbb{Z}} \operatorname{supp}\big(H_i(P)\big).$$

We denote by $\mathcal{M}_X$ the category of quasi-coherent $\mathcal{O}_X$-modules and by $\mathcal{P}_X$ the category of locally free $\mathcal{O}_X$-modules of finite rank. We denote by $D^b(\mathcal{E})$ the bounded derived category of an exact category $\mathcal{E}$. Let $D^b_{\mathrm{coh}}(\mathcal{M}_X)$ be the full subcategory of $D^b(\mathcal{M}_X)$ of complexes whose homology modules are coherent, and let $D^b_{\mathrm{coh},\,Z}(\mathcal{M}_X)$ be the full subcategory of $D^b_{\mathrm{coh}}(\mathcal{M}_X)$ of those complexes whose homological support is contained in Z. The symbol $D^b_Z(\mathcal{P}_X)$ has an analogous self-explanatory meaning. For any positive integer $p \geq 0$ we set

$$D^b_Z(\mathcal{P}_X)^{(p)} := \bigcup_{\substack{\operatorname{codim} \mathrm{W} = p \\ \mathrm{W} \subset Z}} D^b_{\mathrm{W}}(\mathcal{P}_X).$$

The category $D^b_{\mathrm{coh},\, Z}(\mathcal{M}_X)^{(p)}$ is defined similarly.

Remark 1.2. We shall use here the following standard abbreviations:

(1) When $Z = X$, we drop its mention, as in $D^b(\mathcal{P}_X)$ to mean $D^b_X(\mathcal{P}_X)$.
(2) In the affine case, $X = \operatorname{Spec}(R)$, we drop "Spec", as in $D^b_{\mathrm{coh}}(\mathcal{M}_R)^{(p)}$ which stands for $D^b_{\mathrm{coh}}(\mathcal{M}_{\operatorname{Spec}(R)})^{(p)}$.
(3) If $X = \operatorname{Spec}(R)$ and $Z = V(I)$ is defined by an ideal $I \subset R$, we replace "Z" by "I", and even further we abbreviate $D^b_f(\mathcal{P}_R)$ instead of $D^b_{fR}(\mathcal{P}_R)$ where $f \in R$.

2. Triangulated categories with duality and their Witt groups

When not mentioned, the reference for this section is [2].

A *triangulated category with duality* is a triple $(\mathcal{K}, \sharp, \varpi)$, where $\mathcal{K}$ is a triangulated category, $\sharp : \mathcal{K} \to \mathcal{K}$ is a δ-exact contravariant functor ($\delta = \pm 1$) and $\varpi : \mathrm{id}_{\mathcal{K}} \to \sharp\sharp$ is an isomorphism of functors such that $\mathrm{id}_{M^\sharp} = (\varpi_M)^\sharp \cdot \varpi_{M^\sharp}$ and $\varpi_M[1] = \varpi_{M[1]}$.

Triangular Witt groups. We can associate to a triangulated category with duality a series of Witt groups $\mathrm{W}^n(\mathcal{K})$, for $n \in \mathbb{Z}$. The group $\mathrm{W}^n(\mathcal{K})$ classifies the n-symmetric spaces modulo Witt equivalence. Here an *n-symmetric space* is a pair (P, ϕ) with $P \in \mathcal{K}$ and with $\phi : P \xrightarrow{\sim} P^\sharp[n]$ an isomorphism such that $\phi^\sharp[n]\, \varpi_P = (-1)^{\frac{n(n+1)}{2}} \delta^n \phi$. The isometry classes of n-symmetric spaces form a monoid with the orthogonal sum as addition. Dividing this monoid by the submonoid of *neutral* n-symmetric spaces (see [2] Definition 2.12) gives the n-th Witt group of $\mathcal{K}$. These groups are 4-periodic, *i.e.* $\mathrm{W}^n(\mathcal{K}) = \mathrm{W}^{n+4}(\mathcal{K})$. The class of (P, ϕ) in the Witt group is written $[P, \phi]$.

Derived Witt groups of schemes. Let X be a scheme and $Z \subset X$ a closed subset. The derived functor of $\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X)$ is then a duality on $D^b_Z(\mathcal{P}_X)$ making it a triangulated category with 1-exact duality. We denote the corresponding triangular Witt groups by $\mathrm{W}^n_Z(X)$ and call them the *derived Witt groups of X with support in Z*. The abbreviations introduced in 1.2 also apply to this notation, like $\mathrm{W}^n(X)$ for $\mathrm{W}^n_X(X)$. The comparison with the classical Witt group of the scheme X defined by Knebusch [11] is given by the following fact ([3] Theorem 4.7): The natural functor $\mathcal{P}_X \to D^b(\mathcal{P}_X)$ induces an isomorphism $\mathrm{W}(X) \xrightarrow{\simeq} \mathrm{W}^0(X)$.

The cone construction and the localization long exact sequence. The main theorem of triangular Witt theory is the localization theorem. Let

$$0 \to \mathcal{J} \longrightarrow \mathcal{K} \xrightarrow{q} \mathcal{K}/\mathcal{J} \to 0 \tag{2.1}$$

be an exact sequence of triangulated categories with duality, *i.e.* a localization with $\mathcal{J}^\sharp \subset \mathcal{J}$. Let z be an element of $\mathrm{W}^0(\mathcal{K}/\mathcal{J})$. Then there exists a symmetric morphism $\psi : P \to P^\sharp$ (*i.e.* $\psi^\sharp \varpi_P = \psi$) such that $z = [q(P), q(\psi)]$. In particular $C := \operatorname{cone} \psi$ belongs to $\mathcal{J}$. By [2] Theorem 2.6, there is a commutative diagram

$$\begin{tikzcd}
P \arrow[r, "\psi"] \arrow[d, "\delta\varpi_P"'] & P^\sharp \arrow[r] \arrow[d, "="'] & C \arrow[r] \arrow[d, "\simeq"', "\phi"] & P[1] \arrow[d, "\delta\varpi_P[1]"] \\
P^{\sharp\sharp} \arrow[r, "\delta\psi^\sharp"'] & P^\sharp \arrow[r] & C^\sharp[1] \arrow[r] & P^{\sharp\sharp}[1]
\end{tikzcd}$$

such that the upper and the lower rows are exact triangles dual to each other and such that $\phi^\sharp[1]\varpi_C = -\delta\phi$. The last property means that (C, ϕ) is a 1-symmetric space, *i.e.* represents an element of $\mathrm{W}^1(\mathcal{J})$. The isometry class of (C, ϕ) is uniquely determined by the isometry class of (P, ψ). We get a morphism $\mathrm{W}^0(\mathcal{K}) \longrightarrow \mathrm{W}^1(\mathcal{J})$ sending $z = [q(P), q(\psi)]$ to $[C, \phi]$. In the same manner we can define morphisms $\partial : \mathrm{W}^n(\mathcal{K}) \longrightarrow \mathrm{W}^{n+1}(\mathcal{J})$ fitting in a long exact sequence, the *localization sequence* associated to the exact sequence (2.1) of triangulated categories with duality:

$$\ldots \longrightarrow \mathrm{W}^n(\mathcal{K}) \longrightarrow \mathrm{W}^n(\mathcal{K}/\mathcal{J}) \xrightarrow{\partial} \mathrm{W}^{n+1}(\mathcal{J}) \longrightarrow \mathrm{W}^{n+1}(\mathcal{K}) \longrightarrow \ldots$$

3. Gersten-Witt spectral sequences and complexes

We review the Gersten-Witt spectral sequence, which was introduced by Balmer and Walter [5] for regular schemes, and generalized by Gille [7] to Gorenstein schemes of finite Krull dimension.

The construction: Let X be a regular scheme of finite Krull dimension and $Z \subset X$ a closed subset. Then $D^b_Z(\mathcal{P}_X)$ has a filtration

$$D^b_Z(\mathcal{P}_X) = D^0_Z \supset D^1_Z \supset \cdots \supset D^{\dim X}_Z \supset D^{\dim X+1}_Z \simeq 0$$

where we have written $D^p_Z := D^b_Z(\mathcal{P}_X)^{(p)}$. The localization exact sequences

$$\ldots \longrightarrow \mathrm{W}^i(D^{p+1}_Z) \longrightarrow \mathrm{W}^i(D^p_Z) \longrightarrow \mathrm{W}^i(D^p_Z/D^{p+1}_Z) \longrightarrow \mathrm{W}^{i+1}(D^{p+1}_Z) \longrightarrow \ldots,$$

can be organized into an exact couple, giving rise to a convergent spectral sequence:

$$E_1^{p,q}(X, Z) = \mathrm{W}^{p+q}(D^p_Z/D^{p+1}_Z) \quad \Longrightarrow \quad \mathrm{W}^{p+q}_Z(X).$$

This is the *Gersten-Witt spectral sequence* for X with supports in Z.
Using the 4-periodicity of Witt groups, as well as [5] Theorem 7.2 and [7] Theorem 3.14, one sees that the E_1-page is zero everywhere except for the

lines with $q \equiv 0 \pmod 4$, which are all the same and which vanish outside the interval $\mathrm{codim}_X Z \leq p \leq \dim X$. So the information of the E_1-page is essentially given by the complex

$$0 \to E_1^{\mathrm{codim} Z,0}(X,Z) \to E_1^{\mathrm{codim} Z+1,0}(X,Z) \to \cdots \to E_1^{\dim X,0}(X,Z) \to 0.$$

Definition 3.1. Let X be a regular scheme of finite Krull dimension and $Z \subset X$ a closed subset. Then we define the complex $\mathrm{GWC}^\bullet(X,Z) := E_1^{\bullet,0}(X,Z)$. In other words, we have

$$\mathrm{GWC}^p(X,Z) = \mathrm{W}^p\left(D_Z^p / D_Z^{p+1}\right)$$

where $D_Z^p = D_Z^b(\mathcal{P}_X)^{(p)}$, and the differential $d^p = d_1^{p,0}$ is the composition

$$\mathrm{W}^p\left(D_Z^p / D_Z^{p+1}\right) \xrightarrow{\partial} \mathrm{W}^{p+1}\left(D_Z^{p+1}\right) \longrightarrow \mathrm{W}^{p+1}\left(D_Z^{p+1} / D_Z^{p+2}\right)$$

where ∂ is the connecting homomorphism of the localization long exact sequence and where the second homomorphism is the natural one. When $X = Z$ we write $E_1^{p,q}(X)$ instead of $E_1^{p,q}(X,X)$, and similarly for $\mathrm{GWC}^\bullet(X)$. We adopt the notation $\mathrm{GWC}^\bullet(X)$ to avoid confusion with the Grothendieck-Witt group $\mathrm{GW}(X)$.

Adding in the edge morphism $E^0(X) \to E_1^{0,0}(X)$ of the spectral sequence gives the *augmented Gersten-Witt complex* of X

$$\mathrm{GWC_a}(X): \quad 0 \longrightarrow \mathrm{W}(X) \longrightarrow \mathrm{GWC}^0(X) \longrightarrow \mathrm{GWC}^1(X) \longrightarrow \cdots$$

The E_2-page of the spectral sequence has $E_2^{p,0}(X) = H^p(\mathrm{GWC}^\bullet(X))$. From this we deduce the following result which will be used in the proofs of Theorems 4.4 and 6.1.

Lemma 3.2. *The following hold true:*

1. *Let X be a regular scheme. Assume that $H^i(\mathrm{GWC}^\bullet(X)) = 0$ for all $i \geq 1$. Then the augmented Gersten-Witt complex for X is exact.*
2. *Let R be a regular local ring. Assume only that $H^i(\mathrm{GWC}^\bullet(R)) = 0$ for all $i \geq 4$. Then the Gersten conjecture for Witt groups holds for R.*

Proof. We start with (1). The hypothesis implies that $E_2^{p,q}(X) = 0$ for all $p \neq 0$. It follows that the spectral sequence degenerates at E_2, and so the edge morphisms give isomorphisms $E^q(X) \xrightarrow{\sim} E_2^{0,q}(X)$ for all q. For $q = 0$ this means that the natural map $\mathrm{W}(X) \to H^0(\mathrm{GWC}^\bullet(X))$ is an isomorphism. This is what we needed to show.

For (2), recall from above that E_1 is concentrated in the lines $q \equiv 0 \bmod 4$. Therefore the spectral sequence degenerates again at $E_2 = E_5$ because no nonzero higher differentials can occur. Observe that for $p = 1, 2, 3$, the homology $H^p(\mathrm{GWC}^\bullet(R))$ is simply $E_2^{p,0}$ and the latter is isomorphic to $\mathrm{W}^p(R)$ by the convergence of the spectral sequence. Now, when R is local, we have $\mathrm{W}^p(R) = 0$ for $p = 1, 2, 3$ by [4] Theorem 5.6. So we can apply (1). ∎

Dévissage: The localization morphisms $\operatorname{Spec}\mathcal{O}_{X,x} \longrightarrow X$ induce an isomorphism (see [5] Proposition 7.1 or [7] Theorem 3.12)

$$\mathrm{GWC}^p(X,Z) \xrightarrow{\simeq} \bigoplus_{x\in X^{(p)}\cap Z} \mathrm{W}^p_{\mathfrak{m}_x}(\mathcal{O}_{X,x}), \tag{3.1}$$

where $X^{(p)}$ is the set of points of codimension p and $(\mathcal{O}_{X,x},\mathfrak{m}_x)$ is the local ring at $x\in X$.
Let $(S,\mathfrak{n})$ be a regular local ring and $\ell = S/\mathfrak{n}$. Then we have an isomorphism $\mathrm{W}(\ell) \xrightarrow{\sim} \mathrm{W}^{\dim S}_{\mathfrak{n}}(S)$ which depends on the choice of local parameters (see [5] Theorem 6.1) and hence

$$\mathrm{GWC}^p(X,Z) \simeq \bigoplus_{x\in X^{(p)}\cap Z} \mathrm{W}(\kappa(x))$$

for any regular scheme X with closed subset Z. It follows that our Gersten-Witt complex has the form announced in the Introduction.
There are two ways of having a complex independent of choices. Either work as we do here in Definition 3.1 with the underlying complex before *dévissage*, or twist the dualities on the residue fields, see [5]. The latter means that one can consider for each $x\in X^{(p)}$ the Witt group $\mathrm{W}(\kappa(x),\omega_{x_p/X})$ with twisted coefficients in the one-dimensional $\kappa(x)$-vector space $\omega_{x_p/X} := \mathrm{Ext}^p(\kappa(x),\mathcal{O}_{X,x}) = \Lambda^p(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$, and we have then a canonical isomorphism $\mathrm{W}(\kappa(x),\omega_{x_p/X}) \cong \mathrm{W}^p_{\mathfrak{m}_x}(\mathcal{O}_{X,x})$ and thus a canonical Gersten-Witt complex with

$$\mathrm{GWC}^p(X,Z) \cong \bigoplus_{x\in X^{(p)}\cap Z} \mathrm{W}(\kappa(x),\omega_{x_p/X}).$$

After all, this *dévissage* is only relevant for cognitive reasons since it relates the terms of the Gersten-Witt complex with quadratic forms over the residue fields. But we will see in the sequel that our initial canonical definition $\mathrm{GWC}^p(X,Z) = \mathrm{W}^p\left(D^p_Z/D^{p+1}_Z\right)$ is more convenient to handle.

Another construction of a Gersten Witt spectral sequence has been given by Gille [7], Section 3. Let Y be a Gorenstein scheme of finite Krull dimension and $Z\subset Y$ a closed subset. Then the derived functor of $\mathcal{H}\mathrm{om}_{\mathcal{O}_Y}(-,\mathcal{O}_Y)$ is a duality on $D^b_{\mathrm{coh},\,Z}(\mathcal{M}_Y)$ making it a triangulated category with 1-exact duality. Following [7] we denote the associated so called *coherent Witt groups* by $\tilde{\mathrm{W}}^i_Z(Y)$. On the triangulated category $D^b_{\mathrm{coh}}(\mathcal{M}_Y)$ we have also a finite filtration

$$D^b_{\mathrm{coh},\,Z}(\mathcal{M}_Y) = D^0_Z \supset D^1_Z \supset D^2_Z \supset \ldots \supset D^{\dim Y}_Z\,,$$

where $D^p_Z := D^b_{\mathrm{coh},\,Z}(\mathcal{M}_Y)^{(p)}$. As above this gives us long exact sequences

$$\ldots \longrightarrow \mathrm{W}^i(D^{p+1}_Z) \longrightarrow \mathrm{W}^i(D^p_Z) \longrightarrow \mathrm{W}^i(D^p_Z/D^{p+1}_Z) \longrightarrow \mathrm{W}^{i+1}(D^{p+1}_Z) \longrightarrow \ldots,$$

and hence by Massey's method of exact couples a convergent spectral sequence

$$\tilde{E}^{p,q}(Y,Z) := \mathrm{W}^{p+q}(D^{p+q}_Z/D^{p+q+1}_Z) \implies \tilde{\mathrm{W}}^{p+q}_Z(Y)\,.$$

If now Y is regular we have equivalences $D^b_Z(\mathcal{P}_Y)^{(p)} \xrightarrow{\simeq} D^b_{\mathrm{coh},\,Z}(\mathcal{M}_Y)^{(p)}$ which are duality preserving and hence give isomorphisms $\mathrm{W}^i(D^b_Z(\mathcal{P}_Y)^{(p)}) \simeq \mathrm{W}^i(D^b_{\mathrm{coh},\,Z}(\mathcal{M}_Y)^{(p)})$. We get then from the functorial properties of the localization sequence an isomorphism of spectral sequences $E_1^{p,q}(Y,Z) \xrightarrow{\simeq} \tilde{E}_1^{p,q}(Y,Z)$. Hence in the regular case both constructions lead to the same result.

One advantage of this "coherent" approach is the following. Let $Y = \operatorname{Spec} R$ with R a Gorenstein ring of finite Krull dimension and let the closed subset Z be defined by a regular element f of R, *i.e.* $Z = \operatorname{Spec} R/Rf$. We set $\bar{D}^p := D^b_{\mathrm{coh}}(\mathcal{M}_Z)^{(p)}$ and as before $D^p_Z := D^b_{\mathrm{coh},\,Z}(\mathcal{M}_Y)^{(p)}$. The natural morphism $\alpha : Z \hookrightarrow Y$ induces a pushforward functor $\alpha_* : \bar{D}^p \to D^{p+1}_Z$ for any $p \in \mathbb{N}$. This functor shifts the duality structure by 1 (*cf.* [7], Theorem 4.2), *i.e.* it induces morphisms $\mathrm{W}^i(\bar{D}^p) \to \mathrm{W}^{i+1}(D^{p+1}_Z)$ for all $i \in \mathbb{Z}$ and $p \in \mathbb{N}$. From the functoriality of the localization sequence (*cf.* [7] Theorem 2.9) we get commutative diagrams with exact rows

$$\begin{array}{ccccccc}
\cdots\ \mathrm{W}^i(\bar{D}^{p+1}) & \longrightarrow & \mathrm{W}^i(\bar{D}^p) & \longrightarrow & \mathrm{W}^i(\bar{D}^p/\bar{D}^{p+1}) & \longrightarrow & \mathrm{W}^{i+1}(\bar{D}^{p+1})\ \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots\ \mathrm{W}^{i+1}(D^{p+2}_Z) & \longrightarrow & \mathrm{W}^{i+1}(D^{p+1}_Z) & \longrightarrow & \mathrm{W}^{i+1}(D^{p+1}_Z/D^{p+2}_Z) & \longrightarrow & \mathrm{W}^{i+2}(D^{p+2}_Z)\ \cdots
\end{array}$$

(*cf.* [7], diagram on the bottom of p. 130). In particular we have a morphism of spectral sequences $\tilde{\alpha}_* : \tilde{E}_1^{p,q}(Z) \longrightarrow \tilde{E}_1^{p+1,q}(Y,Z)$ which is an isomorphism as shown in [7], Section 4.2.3.

If now R is regular local and f a regular parameter, *i.e.* R/Rf is regular too, the identification above gives the following

Lemma 3.3. *Let R be a regular local ring and f a regular parameter. Then we have an isomorphism of spectral sequences*

$$E_r^{p,q}(R/fR) \xrightarrow{\simeq} E_r^{p+1,q}(R,fR).$$

In particular, we have isomorphisms of complexes

$$\mathrm{GWC}^\bullet(R/fR) \xrightarrow{\simeq} \mathrm{GWC}^{\bullet+1}(R,fR)\,.$$

4. A reformulation of the conjecture

Let X be a regular scheme and Z a closed subset. From Definition 3.1 and from the dévissage formula (3.1), we immediately obtain a degree-wise split short exact sequence

$$0 \to \mathrm{GWC}^\bullet(X,Z) \longrightarrow \mathrm{GWC}^\bullet(X) \longrightarrow \mathrm{GWC}^\bullet(X \setminus Z) \to 0.$$

We will consider below the long exact cohomology sequence of this short exact sequence of complexes in the case $X = \operatorname{Spec} R$, for R a regular local ring, and Z is defined by a regular parameter f.

DEFINITION 4.1. Recall from the introduction that the Gersten conjecture asserts that for a regular local ring R the Gersten complex $\mathrm{GWC}^\bullet(R)$ is an exact resolution of $\mathrm{W}(R)$.

We denote by $\mathcal{W}$ the *Witt sheaf*, *i.e.* the sheafification of the presheaf on the Zariski site

$$U \longmapsto \mathrm{W}(U) \qquad U \subset X \qquad \text{open}.$$

We have also a Gersten-Witt complex $\mathcal{GWC}^\bullet$ of sheaves on any regular scheme of finite Krull dimension X. The definition of this complex in degree $p \geq 0$ is:

$$U \longmapsto \mathrm{GWC}^p(U) \qquad U \subset X \qquad \text{open}.$$

LEMMA 4.2. *Let X be a regular scheme of finite Krull dimension and assume that the Gersten conjecture holds for all local rings $\mathcal{O}_{X,x}$ of X. Then for all i we have*

$$\mathrm{H}^i_{\mathrm{Zar}}(X, \mathcal{W}) \simeq H^i(\Gamma(X, \mathcal{GWC}^\bullet)) = H^i(\mathrm{GWC}^\bullet(X)).$$

Proof. Note that $\mathcal{GWC}^p$ is a flabby sheaf and that $(\mathcal{GWC}^p)_x \simeq \mathrm{GWC}^p(\mathcal{O}_{X,x})$ for all points x of the scheme X. Since the natural morphism $\mathrm{W}(\mathcal{O}_{X,x}) \longrightarrow \mathcal{W}_x$ is an isomorphism for all $x \in X$ it follows that if the Gersten conjecture is true for every local ring of the regular scheme X, then $\mathcal{GWC}^\bullet$ is a flabby resolution of $\mathcal{W}$ on X. □

DEFINITION 4.3. Let $\mathcal{C}$ be a class of regular local rings. We say that $\mathcal{C}$ is *nepotistic* if the following holds: whenever R belongs to $\mathcal{C}$, so do $R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$ and R/fR for all regular parameters $f \in R$.

The main result of this section is the following:

THEOREM 4.4. *If $\mathcal{C}$ is a nepotistic class of regular local rings* (*see 4.3*)*, then the following conditions are equivalent*:

(i) *The Gersten conjecture for Witt groups is true for any $R \in \mathcal{C}$.*
(ii) *For any $R \in \mathcal{C}$ and for any regular parameter $f \in R$, we have for all $i \geq 1$ that $\mathrm{H}^i_{\mathrm{Zar}}(\operatorname{Spec} R_f, \mathcal{W}) = 0$. (When R is a field, this condition is empty and thus always true.)*

Proof. The short exact sequence $0 \to \mathrm{GWC}^\bullet(R, fR) \to \mathrm{GWC}^\bullet(R) \to \mathrm{GWC}^\bullet(R_f) \to 0$ of Gersten-Witt complexes gives rise to a long exact sequence of cohomology

$$0 \to H^0(\mathrm{GWC}^\bullet(R)) \longrightarrow H^0(\mathrm{GWC}^\bullet(R_f)) \xrightarrow{\delta} H^1(\mathrm{GWC}^\bullet(R, fR)) \to \cdots$$

(i) ⇒ (ii). Let $R \in \mathcal{C}$, and let $f \in R$ be a regular parameter. Then the Gersten conjecture for Witt groups holds for R and R/fR, and so we have $H^i(\mathrm{GWC}^\bullet(R)) = 0$ and $H^i(\mathrm{GWC}^\bullet(R/fR)) = 0$ for all $i \geq 1$. Because of the isomorphism of Lemma 3.3, we get $H^i(\mathrm{GWC}^\bullet(R, fR)) = 0$ for all $i \geq 2$. It now follows from the long exact sequence that $H^i(\mathrm{GWC}^\bullet(R_f)) = 0$ for all $i \geq 1$. The local rings of $\operatorname{Spec} R_f$ are the $R_\mathfrak{p}$ with $f \notin \mathfrak{p}$, so they are all in $\mathcal{C}$. So we also have $\mathrm{H}^i_{\mathrm{Zar}}(\operatorname{Spec} R_f, \mathcal{W}) = 0$ for all $i \geq 1$ by Lemma 4.2.

(ii) ⇒ (i). We will prove the Gersten conjecture for Witt groups for $R \in \mathcal{C}$ by induction on $n = \dim R$. For $n = 0$ the ring R is a field, and this is trivial.
So suppose $n \geq 1$ and that the Gersten conjecture for Witt groups is true for all $S \in \mathcal{C}$ with $\dim S < n$. Let $f \in \mathfrak{m} \setminus \mathfrak{m}^2$ be a regular parameter. The local rings of R_f are the local rings $R_\mathfrak{p}$ for primes with $f \notin \mathfrak{p}$, and they satisfy $R_\mathfrak{p} \in \mathcal{C}$ and $\dim R_\mathfrak{p} < n$. So the Gersten conjecture holds for all local rings of $\operatorname{Spec} R_f$, and so by (ii) and Lemma 4.2 we have $H^i(\mathrm{GWC}^\bullet(R_f)) = 0$ for all $i \geq 1$. We also have $R/fR \in \mathcal{C}$ with $\dim R/fR = n-1$, so we get $H^i(\mathrm{GWC}^\bullet(R/fR)) = 0$ for all $i \geq 1$. The identification of Lemma 3.3 now gives us $H^i(\mathrm{GWC}^\bullet(R, fR)) = 0$ for all $i \geq 2$. So from the long exact sequence we get $H^i(\mathrm{GWC}^\bullet(R)) = 0$ for all $i \geq 2$ and the Gersten conjecture for Witt groups holds for R by Lemma 3.2, part (ii). □

5. A pairing, a trace map and a projection formula

We recall here some techniques of Gille and Nenashev [8] that we shall use below. For more details and a more general point of view see [8].

The pairing $\mathrm{W}(k) \times \mathrm{W}^i(D^b(\mathcal{P}_R)) \longrightarrow \mathrm{W}^i(D^b(\mathcal{P}_R))$. Let k be a field of characteristic not 2, and let R be a regular k-algebra of Krull dimension n. Denote the duality on k-mod by $V^* = \operatorname{Hom}_k(V, k)$ and that on $D^b(\mathcal{P}_R)$ by $F^\sharp := \operatorname{Hom}_R(F, R)$. Let (V, φ) be a nondegenerate symmetric bilinear space over k. Then $V \otimes_k - : D^b(\mathcal{P}_R) \longrightarrow D^b(\mathcal{P}_R)$ is an exact functor, and the system of isomorphisms between $V \otimes_k F^\sharp$ and $(V \otimes_k F)^\sharp \cong V^* \otimes_k F^\sharp$ given by $\varphi \otimes 1_{F^\sharp}$ makes $V \otimes_k -$ duality-preserving. Actually, the duality-preserving functor is formally the pair $(V \otimes_k -, \varphi \otimes 1)$, and we will abbreviate it as $(V, \varphi) \otimes_k -$. Moreover, if we let $D_R^p = D^b(\mathcal{P}_R)^{(p)}$ be the subcategory of complexes of homological support of codimension at least p, then $(V, \varphi) \otimes_k -$ is compatible with the filtration

$$D^b(\mathcal{P}_R) = D_R^0 \supset D_R^1 \supset \cdots \supset D_R^{n+1} \simeq 0.$$

Hence the maps $\mathrm{W}^i(D_R^p) \longrightarrow \mathrm{W}^i(D_R^p)$ and $\mathrm{W}^i(D_R^p/D_R^{p+1}) \longrightarrow \mathrm{W}^i(D_R^p/D_R^{p+1})$ induced by $(V, \varphi) \otimes_k -$ are compatible with the localization exact sequences and induce endomorphisms of the Gersten-Witt exact couple and spectral sequence for R. These endomorphisms depend only on the Witt class of (V, φ), and they are compatible with the orthogonal direct sum $(V, \varphi) \perp (W, \psi)$ and tensor product $(V \otimes W, \varphi \otimes \psi)$ of symmetric bilinear spaces over k. This gives us the following result.

Lemma 5.1. *If R is a regular k-algebra of finite Krull dimension, then the pairing makes the Gersten-Witt spectral sequence $E_r^{p,q}(R)$ into a spectral sequence of $\mathrm{W}(k)$-modules.* □

Base change. Let ℓ/k be a separable algebraic field extension. Denote by π the projection $\pi : \operatorname{Spec}(\ell \otimes_k R) \to \operatorname{Spec} R$. Then $\ell \otimes_k R$ is a regular ℓ-algebra of Krull dimension n (see [9], and in particular Prop. 6.7.4, p. 146 for regularity). Moreover, $D^b(\mathcal{P}_{\ell\otimes_k R})$ has a duality $(-)^\flat$ given by $G^\flat := \operatorname{Hom}_{\ell\otimes_k R}(G, \ell \otimes_k R)$, and the exact functor $\ell \otimes_k - : D^b(\mathcal{P}_R) \longrightarrow D^b(\mathcal{P}_{\ell\otimes_k R})$ is naturally duality-preserving. It is compatible with the filtration of $D^b(\mathcal{P}_R)$ and the corresponding filtration

$$D^b(\mathcal{P}_{\ell\otimes_k R}) = D^0_{\ell\otimes_k R} \supset D^1_{\ell\otimes_k R} \supset \cdots \supset D^{n+1}_{\ell\otimes_k R} = 0.$$

of $D^b(\mathcal{P}_{\ell\otimes_k R})$, so it induces a morphism of spectral sequences $\pi^* : E_r^{p,q}(R) \longrightarrow E_r^{p,q}(\ell \otimes_k R)$.

A trace map. If ℓ/k is finite and separable, then choose $0 \neq \tau \in \operatorname{Hom}_k(\ell, k)$ and extend it to an R-linear map $\tau_R : \ell \otimes_k R \longrightarrow R$ by setting $\tau_R(l \otimes a) = \tau(l)a$. Let $\pi_* : \mathcal{P}_{\ell\otimes_k R} \to \mathcal{P}_R$ be the restriction-of-scalars functor. The natural homomorphism of R-modules

$$\pi_*(G^\flat) = \operatorname{Hom}_{\ell\otimes_k R}(G, \ell \otimes_k R) \xrightarrow{\tau_{R*}} \operatorname{Hom}_R(G, R) = (\pi_* G)^\sharp$$

sending $f \mapsto \tau_R \circ f$ is an isomorphism for any G in $\mathcal{P}_{\ell\otimes_k R}$, and it makes π_* into a duality-preserving exact functor $\pi_* : D^b(\mathcal{P}_{\ell\otimes_k R}) \to D^b(\mathcal{P}_R)$. Actually, the duality-preserving functor is formally the pair (π_*, τ_{R*}), and we will abbreviate it as $\operatorname{Tr}^\tau_{\ell\otimes_k R/R}$. Since $\ell \otimes_k R$ is flat and finite over R, the restriction-of-scalars functor preserves the codimension of the support of the homology modules, and so π_* is compatible with the filtrations on the two derived categories. So we again get a morphism of spectral sequences $\operatorname{Tr}^\tau_{\ell\otimes_k R/R} : E_r^{p,q}(\ell \otimes_k R) \longrightarrow E_r^{p,q}(R)$.

Remark 5.2. For $R = k$ and $i = 0$ our $\operatorname{Tr}^\tau_{\ell/k}$ is just the Scharlau transfer $\tau_* : \mathrm{W}(\ell) \longrightarrow \mathrm{W}(k)$ (*cf.* [20] Section 2.5).

Let (U, ψ) be a nondegenerate symmetric bilinear space over ℓ. The following diagram of duality-preserving functors commutes up to isomorphism of duality-preserving functors (see [5] §4 for the definition):

$$\begin{tikzcd}
D^b(\mathcal{P}_R) \arrow[r, "\ell \otimes_k -"] \arrow[d, "\operatorname{Tr}^\tau_{\ell/k}(U,\psi) \otimes_k -"'] & D^b(\mathcal{P}_{\ell\otimes_k R}) \arrow[d, "(U,\psi) \otimes_\ell -"] \\
D^b(\mathcal{P}_R) & D^b(\mathcal{P}_{\ell\otimes_k R}) \, . \arrow[l, "\operatorname{Tr}^\tau_{\ell\otimes_k R/R}"']
\end{tikzcd}$$

The induced maps on derived Witt groups, exact couples, and spectral sequences are then the same (*cf.* [5] Lemma 4.1(b)). This gives us a projection formula (*cf.* [8] Theorem 4.1):

Theorem 5.3. *Let R be a regular k-algebra of finite Krull dimension, let ℓ/k be a finite separable extension of fields, and let $\pi : \operatorname{Spec}(\ell \otimes_k R) \to \operatorname{Spec} R$ be* *the projection. Let $E_r^{p,q}(R)$ and $E_r^{p,q}(\ell \otimes_k R)$ be the two Gersten-Witt spectral sequences. Then*

$$\mathrm{Tr}^{\tau}_{\ell\otimes_k R/R}(u \cdot \pi^*(x)) = \mathrm{Tr}^{\tau}_{\ell/k}(u) \cdot x,$$

for all $u \in \mathrm{W}(\ell)$ and all $x \in E_r^{p,q}(R)$. $\square$

Odd-degree extensions. If ℓ/k is of odd degree, then there exists a $\tau \in \mathrm{Hom}_k(\ell, k)$ such that $\mathrm{Tr}^{\tau}_{\ell/k}(1_{\mathrm{W}(\ell)}) = 1_{\mathrm{W}(k)}$ ([20] Lemma 2.5.8). We then have $\mathrm{Tr}^{\tau}_{\ell\otimes_k R/R}(\pi^* x) = x$ for all $x \in E_r^{p,q}(R)$. In other words:

Corollary 5.4. *If ℓ/k is separable of odd degree, then $\pi^* : E_r^{p,q}(R) \to E_r^{p,q}(\ell \otimes_k R)$ is a split monomorphism of spectral sequences. In particular $H^i(\mathrm{GWC}^\bullet(R)) \to H^i(\mathrm{GWC}^\bullet(\ell\otimes_k R))$ is a split monomorphism for every i.* $\square$

The base change maps for separable algebraic extensions commute with filtered colimits.

Corollary 5.5. *If ℓ/k is a filtered colimit of separable finite extensions of odd degree, then $\pi^* : E_r^{p,q}(R) \to E_r^{p,q}(\ell \otimes_k R)$ is a filtered colimit of split monomorphisms of spectral sequences. In particular $H^i(\mathrm{GWC}^\bullet(R)) \to H^i(\mathrm{GWC}^\bullet(\ell \otimes_k R))$ is a monomorphism for every i.* $\square$

6. The equicharacteristic case of the Gersten conjecture for Witt groups

We are now ready to prove the main result of the paper.

Theorem 6.1. *Let R be an equicharacteristic regular local ring, i.e. R contains some field* (*of characteristic not* 2). *Then the Gersten conjecture for Witt groups is true for R.*

Fix the following notation: $\mathfrak{m}$ is the maximal ideal of R. We prove the theorem in two steps.

Step 1. Assume that R is essentially smooth over some field k.

When k is an infinite field, this is a special case of [4] Theorem 4.3 which states the following. If S is a semilocal ring essentially smooth over a field ℓ (*i.e.* S is the semi-localization of a smooth scheme over ℓ), and if the field ℓ is infinite, then the Gersten conjecture for Witt groups is true for S, *i.e.* $H^i(\mathrm{GWC}(S)) = 0$ for all $i \geq 1$.

Assume now k is a finite field and hence perfect. Fix an odd prime s. For any $n \geq 0$ let ℓ_n be the unique extension of the finite field k of degree s^n, and let $\ell = \bigcup_{n=0}^{\infty} \ell_n$. The ℓ-algebra $\ell \otimes_k R$ is integral over R and hence semilocal. It is further essentially smooth over the infinite field ℓ ([10] Prop. 10.1.b) and hence by the result above the Gersten conjecture is true for $\ell \otimes_k R$. Using Corollary 5.5, we see that the same is true for R.

Step 2. General equicharacteristic R.

For this case we use the following result:

Theorem 6.2. *Let R be an equicharacteristic regular local ring, and let $f \in \mathfrak{m} \setminus \mathfrak{m}^2$ be a regular parameter. Then:*

(1) *There exist a perfect field k contained in R and a filtered system of pairs (R_j, f_j) such that each R_j is an essentially smooth local k-algebra, each f_j is a regular parameter in R_j, and such that $R = \operatorname{colim} R_j$ and $R_f = \operatorname{colim}(R_j)_{f_j}$, and the morphisms $R_j \to R$ are local.*

(2) *In addition the natural maps*

$$\operatorname{colim}_j \ \mathrm{H}^i_{\mathrm{Zar}}(\operatorname{Spec}(R_j)_{f_j}, \mathcal{W}) \longrightarrow \mathrm{H}^i_{\mathrm{Zar}}(\operatorname{Spec} R_f, \mathcal{W})$$

are isomorphisms for all $i \geq 0$.

Proof. The first part is a consequence of Popescu's Theorem [18] [19] (see also [15] §3), while the second part follows from [15] Theorem 6.6, which was inspired by the étale analogue of this result : [1] Exposé VII, Théorème 5.7. □

Let $\mathcal{C}_{\mathrm{eq}}$ be the class of all equicharacteristic regular local rings, and let $\mathcal{C}_{\mathrm{sm}}$ be the subclass of regular local rings that are essentially smooth over a field. Both $\mathcal{C}_{\mathrm{eq}}$ and $\mathcal{C}_{\mathrm{sm}}$ are nepotistic (Definition 4.3). The class $\mathcal{C}_{\mathrm{sm}}$ satisfies condition (i) of Theorem 4.4 by the first step of our proof, and we wish to show that $\mathcal{C}_{\mathrm{eq}}$ satisfies the same condition. But conditions (i) and (ii) of Theorem 4.4 are equivalent, so it is enough to show that $\mathcal{C}_{\mathrm{eq}}$ satisfies condition (ii) of Theorem 4.4, knowing that $\mathcal{C}_{\mathrm{sm}}$ satisfies the same condition.

Let R be in $\mathcal{C}_{\mathrm{eq}}$, and let f be a regular parameter of R. By Theorem 6.2 there exist a perfect subfield $k \subset R$ and a filtered system (R_j, f_j) of essentially smooth local k-algebras R_j plus regular parameters $f_j \in R_j$ such that $R_f = \operatorname{colim}(R_j)_{f_j}$. Since the R_j are in $\mathcal{C}_{\mathrm{sm}}$, we have $\mathrm{H}^i_{\mathrm{Zar}}(\operatorname{Spec}(R_j)_{f_j}, \mathcal{W}) = 0$ for all $i \geq 1$ and all j because $\mathcal{C}_{\mathrm{sm}}$ satisfies condition (ii) of Theorem 4.4. Then by Theorem 6.2 (2) we also have $\mathrm{H}^i_{\mathrm{Zar}}(\operatorname{Spec} R_f, \mathcal{W}) = 0$ for $i \geq 1$. This is condition (ii) of Theorem 4.4 for the class $\mathcal{C}_{\mathrm{eq}}$, so we have completed the proof.

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