DESCENT IN TRIANGULATED CATEGORIES

PAUL BALMER

Abstract. We establish effective descent for faithful ring objects in tensor triangulated categories. More generally, we discuss descent for monads in triangulated categories without tensor, where the answer is more subtle.

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Introduction

Descent is an important useful device and so are triangulated categories. Yet, there seems to be two misconceptions about their interaction. Put briefly, the first one is that descent could not hold for triangulated categories and the second one that descent always holds for them. Unsurprisingly, the truth lies in between.

The idea that descent could not hold for triangulated categories is based on the well-known failure of Zariski descent for derived categories of non-affine schemes; see Example 2.1. However, this failure has a rather bold reason, namely the existence of locally trivial morphisms which are not globally trivial, i.e. the *non-faithfulness* of the derived pull-back to a Zariski cover. We claim that this problem is irrelevant to descent theory since faithfulness really is the very first necessary condition one always requires. Even ordinary ring-theoretic descent trivially fails for non-faithful ring homomorphisms. And no one would say that descent could not hold for rings.

Here, we prove that this obvious necessary condition – faithfulness – is indeed already sufficient for descent to hold for ring objects in triangulated categories :

Theorem (Corollary 3.1). *Let A be a ring object in an idempotent-complete tensor triangulated category $\mathcal{C}$. Then A satisfies descent in $\mathcal{C}$ if and only if A is faithful.*

The second misconception is the expectation that this Theorem holds for monads as well. We shall return to this at the end of the Introduction. Let us first remind the reader of the theory of descent. Precise definitions can be found in Section 1.

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For commutative rings, $R \to A$, following Grothendieck's original insight [5], Knus and Ojanguren defined in their 1974 paper [8] a *descent datum* on an A-module M to be an $A \otimes A$-isomorphism $\gamma : A \otimes M \overset{\sim}{\to} M \otimes A$ satisfying some cocycle condition. Here, $\otimes$ means $\otimes_R$. Inspiration comes from Zariski covers of affine schemes, as recalled in Example 3.6 below. For a general homomorphism $R \to A$, the descent problem is to decide when extension of scalars $A \otimes -$ induces an equivalence between the category of R-modules and that of A-modules with descent data. The exact characterization, secretely due to Joyal-Tierney, is that $R \to A$ should be a so-called "pure monomorphism" of R-modules, see Mesablishvili [11]. This first notion of descent data can be extended verbatim to the framework of *symmetric* monoidal categories, as recalled in Remark 3.4 below. By comparison to the Joyal-Tierney result, our Theorem above shows how things become considerably simpler in the presence of a triangulated category structure: Essentially, descent *always* holds in that case. But let us be careful with this conclusion.

Actually, symmetry of the tensor is not necessary, as we now explain. In 1976, Cipolla [4] generalized descent to non-commutative rings, replacing the above γ by an A-linear morphism $M \to A \otimes M$ satisfying different conditions. His approach transposes equally well to triangulated categories with a non-necessarily symmetric monoidal structure (see Remark 3.3) and our Theorem holds in that generality.

So far, so good.

Nevertheless, descent is not specific to ring objects and it is usual nowadays to study descent in the generalized context of *monads*. The reader will find a 2-page vademecum on monads in Appendix A. In that world, there is no tensor and therefore no symmetry to worry about. Hence, we make our general discussion in the language of monads in Section 2 and specialize to ring objects in Section 3.

For a monad M on a category $\mathcal{C}$, the (effective) descent problem is to decide whether the free-M-module functor F_M, which replaces the above $A \otimes -$, yields an equivalence between $\mathcal{C}$ and the category $\mathrm{Desc}_{\mathcal{C}}(M)$ of M-modules equipped with a new notion of descent data, jazzed-up from Cipolla's; see Mesablishvili [12, § 2].

We prove in Corollary 2.15 that the functor $F_M : \mathcal{C} \to \mathrm{Desc}_{\mathcal{C}}(M)$ is always fully faithful (for M faithful, of course) and we give a description of its essential image in Main Lemma 2.18. In that respect, the program of faithful descent is fulfilled: For any faithful monad M one can uniquely reconstruct $\mathcal{C}$ from M-modules with descent data satisfying some additional property. We give a simple characterization of effective descent in Theorem 2.20 and further equivalent conditions in Theorem 2.23, maybe more user-friendly. For instance, we have:

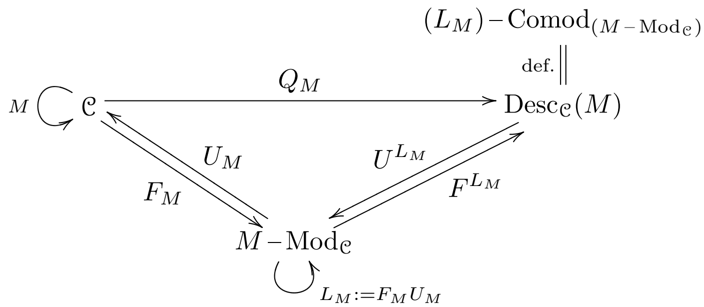
Theorem. *Let $M : \mathcal{C} \to \mathcal{C}$ be a monad on an idempotent-complete triangulated category $\mathcal{C}$, with M additive and faithful. Then M satisfies effective descent if and only if M reflects semi-simplicity, i.e. if a morphism g in $\mathcal{C}$ is such that $M(g)$ has a kernel in $\mathcal{C}$ then g itself has a kernel in $\mathcal{C}$.*

In view of those results, it is tempting to believe that every faithful monad on a triangulated category satisfies effective descent. This claim can even be found in the literature. It is however incorrect. In Theorem 4.6, we provide a hereditary abelian category $\mathcal{A}$ and a faithful monad M on $\mathrm{D^b}(\mathcal{A})$, even realized by an adjunction of derived categories, such that M does not satisfy effective descent. This counter-example is quite elementary. So, failure of effective descent for faithful monads should not be considered a pathological phenomenon.

1. The descent category

Let (M, μ, η) be a monad on a category $\mathcal{C}$ with multiplication $\mu : M^2 \to M$ and unit $\eta : \mathrm{Id}_{\mathcal{C}} \to M$. See details in Appendix A. Recall that the prototype of a monad is $M = A \otimes -$ for A a ring object in a monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$. And recall that the prototype of the prototype is to take for $\mathcal{C}$ the category of good old R-modules over a commutative ring R, for $\otimes$ the tensor product over R and for A any associative unital R-algebra. As in [12], one defines the *descent category,* $\mathrm{Desc}_{\mathcal{C}}(M)$*, for* M *in* $\mathcal{C}$ as a suitable category of comodules in the category of M-modules in the following way:

(1.1)



Let us read the above slightly intimidating picture step-by-step. We start in the upper-left corner with the given monad M on the category $\mathcal{C}$. We create the lower category $M - \mathrm{Mod}_{\mathcal{C}}$ of M-modules in $\mathcal{C}$ (a. k. a. M-algebras). It is related to the original category $\mathcal{C}$ by the Eilenberg-Moore adjunction $F_M : \mathcal{C} \leftrightarrows M - \mathrm{Mod}_{\mathcal{C}} : U_M$ with the usual free-module functor F_M and its right adjoing, the forgetful functor U_M. This adjunction "realizes M", that is, we have $U_M\, F_M = M$. The other composition $L_M := F_M\, U_M$ defines a *comonad* on the category $M - \mathrm{Mod}_{\mathcal{C}}$ of M-modules (see A.9). Now, we can build the category of comodules in $M - \mathrm{Mod}_{\mathcal{C}}$ with respect to that comonad L_M. This is by definition our descent category $\mathrm{Desc}_{\mathcal{C}}(M)$ which appears in the upper-right corner. We give an explicit description of $\mathrm{Desc}_{\mathcal{C}}(M)$, just in terms of M and $\mathcal{C}$, in Remark 1.4 below.

Of course, there also exists a "co"-Eilenberg-Moore adjunction as in the above picture, i.e. a free-comodule functor F^{L_M} and its left adjoint U^{L_M}, which forgets the L_M-coaction (only). This second adjunction *realizes the comonad* L_M, that is, $U^{L_M}\, F^{L_M} = L_M$. By finality of the Eilenberg-Moore adjunction among all adjunctions which realize a given comonad (here L_M), the left-hand adjunction of (1.1) maps uniquely into the right-hand one, that is, there is a unique functor $Q_M : \mathcal{C} \to \mathrm{Desc}_{\mathcal{C}}(M)$ such that $U^{L_M} Q_M = F_M$ and $Q_M U_M = F^{L_M}$. We also describe this comparison functor Q_M in simple terms in Remark 1.4.

We have now obtained the whole picture (1.1).

1.2. *Definition* ([12]). One says that the monad M *satisfies effective*[1] *descent in* $\mathcal{C}$ when the above functor $Q_M : \mathcal{C} \longrightarrow \mathrm{Desc}_{\mathcal{C}}(M)$ is an equivalence. In words, this property says that one can uniquely reconstruct objects and morphisms of $\mathcal{C}$ out of M-modules equipped with suitable "descent data", expressed via the L_M-coaction.

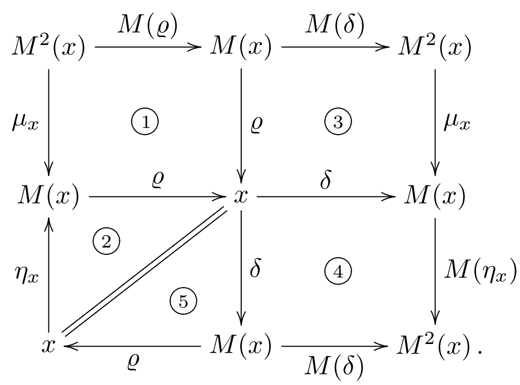
1.3. *Remark.* It is well-known that deciding when a comparison functor like Q_M is an equivalence can be done with Beck's Comonadicity Theorem, see Beck [2,

[1] Following [12], M "satisfies descent" just means that Q_M is fully faithful.

Thm. 1], or Mac Lane [10, Ex. VI.7.6, p. 155] applied to opposite categories, or Kashiwara-Schapira [7, Thm. 4.3.8]. Here, we shall not use Beck's Theorem because we do not have descent in general. Also, we want to give as self-contained a treatment as possible. Still, several arguments are reminiscent of Beck's techniques, with a triangulo-categorical twist and we shall refer to variants of Beck's result when possibly interesting for the reader.

1.4. *Remark.* Unfolding the definitions, the category $\mathrm{Desc}_{\mathcal{C}}(M)$ is the following. Its objects are triples (x, ϱ, δ) where x is an object in $\mathcal{C}$ and $\varrho : M(x) \to x$ and $\delta : x \to M(x)$ are morphisms in $\mathcal{C}$ such that the following five diagrams commute:

(1.5)



The meaning of (1.5) is the following. The morphism $\varrho : M(x) \to x$ defines a structure of M-module on x, which involves the commutativity of the square marked 1 and the triangle marked 2. The morphism $\delta : x \to M(x)$ is the *descent datum* on the M-module x. It is a morphism of M-modules from x to the free M-module $M(x)$, as expressed by the commutativity of the square marked 3. Moreover, δ is an L_M-coaction; this is expressed by the commutativity of the square marked 4 and the triangle marked 5. Indeed, the comonad $L_M = (L_M, \lambda, \epsilon)$ is defined by $L_M(x, \varrho) = (M(x), \mu_x)$ for every M-module (x, ϱ), and has comultiplication $\lambda_{(x,\varrho)} = M(\eta_x)$ and counit $\epsilon_{(x,\varrho)} = \varrho$. See details in Remark A.9.

A morphism $f : (x, \varrho, \delta) \to (x', \varrho', \delta')$ in $\mathrm{Desc}_{\mathcal{C}}(M)$ is a morphism $f : x \to x'$ in $\mathcal{C}$ which is compatible with the M-module structures and the descent data, that is, $\varrho' M(f) = f\,\varrho$ and $M(f)\,\delta = \delta' f$.

Finally, the comparison functor $Q_M : \mathcal{C} \longrightarrow \mathrm{Desc}_{\mathcal{C}}(M)$ is given by

$$Q_M(x) = \big(M(x)\,,\ \mu_x\,,\ M(\eta_x)\big) \tag{1.6}$$

for every object x in $\mathcal{C}$ and by $Q(f) = M(f)$ for every morphism f in $\mathcal{C}$.

We unfold all the above for the monad $M = A \otimes -$ in Section 3, where the case of a Zariski cover is also recalled in Example 3.6.

2. Faithful descent in triangulated categories

2.1. *Example.* As announced in the Introduction, here is an example of the non-faithfulness of Zariski localization in derived categories of schemes. Observe that if $\mathcal{E} \rightarrowtail \mathcal{F} \twoheadrightarrow \mathcal{G}$ is a non-split exact sequence of vector bundles over a scheme, the associated map $\mathcal{G} \to \mathcal{E}[1]$ in the derived category is non-zero but becomes zero over every affine open subscheme (where the sequence does split). This happens already with the Koszul sequence $\mathcal{O}_{\mathbb{P}^1} \rightarrowtail \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1}(2)$ over $\mathbb{P}^1$, so there is nothing exotic or pathological about this phenomenon.

2.2. *Remark.* Recall that an additive category $\mathcal{C}$ is *idempotent-complete* if every idempotent endomorphism splits. When $\mathcal{C}$ is triangulated, see [1].

From now on, $\mathcal{C}$ is a triangulated category in the sense of Grothendieck-Verdier, see Neeman [14]. We shall not use the Octahedron Axiom, so "triangulated" could as well mean "pre-triangulated" below. See also Remark 2.24.

2.3. *Remark.* Of course, if the comparison functor $Q_M : \mathcal{C} \longrightarrow \mathrm{Desc}_{\mathcal{C}}(M)$ of (1.1) is an equivalence, or simply if it is faithful, then so is $U_M \circ U^{L_M} \circ Q_M = M$ since both forgetful functors are faithful. So, faithfulness of the monad M is a necessary condition for descent. Conversely, let us see what we can prove by assuming only that. First, recall well-known facts about triangulated categories, see [14, § 1.2] :

2.4. *Remark.* For $f : x \to x'$ in a triangulated category, the following are equivalent :

(i) f is a monomorphism.

(ii) For some (hence every) distinguished triangle $x'' \overset{f_0}{\to} x \overset{f}{\to} x' \to \Sigma x''$, $f_0 = 0$.

(iii) f is a split monomorphism, i.e. admits a retraction $r : x' \to x$, $rf = \mathrm{id}_x$.

(iv) There exists an isomorphism $x' \simeq x \oplus y$ under which f becomes $\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$.

2.5. *Definition.* Let us say that a morphism g is *semi-simple* if it is the composition $g = g_2 \circ g_1$ of a split epimorphism g_1 followed by a split monomorphism g_2.

2.6. *Definition.* A complex is contractible if its identity is null-homotopic. We say that a (truncated) complex of the form $0 \to x \overset{f}{\to} y \overset{g}{\to} z$ is *contractible* if there exist morphisms $\ell : y \to x$ and $m : z \to y$ such that $\ell f = \mathrm{id}_x$ and $f \ell + mg = \mathrm{id}_y$:

$$0 \longrightarrow x \underset{\ell}{\overset{f}{\rightleftarrows}} y \underset{m}{\overset{g}{\rightleftarrows}} z\,. \tag{2.7}$$

Its is clear that in that case f is a kernel of g. Also, any additive functor preserves such contractible complexes.

2.8. **Lemma.** *For a morphism $g : y \to z$ in an idempotent-complete additive category, the following properties are equivalent :*

(i) The morphism g is semi-simple (Definition 2.5).

(ii) There exists a morphism $m : z \to y$ such that $g = gmg$.

(iii) There exists a contractible complex $0 \to x \overset{f}{\to} y \overset{g}{\to} z$ (Definition 2.6).

If moreover $\mathcal{C}$ is triangulated then these three properties are further equivalent to :

(iv) The morphism g has a kernel in $\mathcal{C}$.

Proof. All this is also a standard exercise. For (ii)⇒(iii), use the idempotent $e = mg$ on y. For (iv)⇒(iii), let $f : x \to y$ be a kernel of g. Since f is a monomorphism, Remark 2.4 gives us $y \cong x \oplus w$ and the following (solid) exact sequence

$$0 \longrightarrow x \underset{\ell := (\,1\;0\,)}{\overset{f = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)}{\rightleftarrows}} y \cong x \oplus w \underset{m := \left(\begin{smallmatrix}0\\r\end{smallmatrix}\right)}{\overset{g = (\,0\;\tilde{g}\,)}{\rightleftarrows}} z \tag{2.9}$$

for some object $w \in \mathcal{C}$ and some morphism $\tilde{g} : w \to z$. Then $\tilde{g}$ is a monomorphism since f is the kernel of g. Finally, set $m := \left(\begin{smallmatrix}0\\r\end{smallmatrix}\right)$ for any retraction r of $\tilde{g}$. □

2.10. **Lemma.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a faithful additive functor between triangulated categories. Let $0 \to x \xrightarrow{f} y \xrightarrow{g} z$ be a complex in $\mathcal{C}$. Then it is contractible (Definition 2.6) if and only if its image under F is contractible.*

Proof. Let us prove the non-trivial direction. Suppose $0 \to F(x) \xrightarrow{F(f)} F(y) \xrightarrow{F(g)} F(z)$ contractible in $\mathcal{D}$. Then $F(f)$ is a split monomorphism, hence by faithfulness of F, f is also a monomorphism. By Remark 2.4, f must be a split monomorphism. So, up to isomorphism, we can assume that $y = x \oplus w$ for some object $w \in \mathcal{C}$, so that $f = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $g = \begin{pmatrix}0 & \tilde{g}\end{pmatrix}$ for $\tilde{g} : w \to z$. By contractibility of

$$0 \longrightarrow F(x) \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} F(x) \oplus F(w) \xrightarrow{(0 \;\; F(\tilde{g}))} F(z)\,,$$

$F(\tilde{g})$ must be a monomorphism in $\mathcal{D}$. As for f, this implies that $\tilde{g}$ is already a split monomorphism in $\mathcal{C}$, say $r\,\tilde{g} = \mathrm{id}_w$. Hence we get the contractibility as in (2.9). □

2.11. *Remark.* The above argument can be repeated inductively to show that a complex in $\mathcal{C}$, which is bounded on one side, is contractible if and only if its image under F is contractible.

2.12. **Proposition.** *Let (M, μ, η) be a faithful additive monad on a triangulated category $\mathcal{C}$ and $x \in \mathcal{C}$ an object. Then the following complex is contractible in $\mathcal{C}$:*

$$0 \longrightarrow x \xrightarrow{\eta_x} M(x) \xrightarrow{\eta_{M(x)} - M(\eta_x)} M^2(x)\,. \tag{2.13}$$

In particular this complex is exact, i.e. $\eta_x : x \to M(x)$ is a kernel of $\eta_{M(x)} - M(\eta_x)$.

Proof. Note that (2.13) is a complex by naturality of $\eta : \mathrm{Id}_{\mathcal{C}} \to M$. By Lemma 2.10, it suffices to observe that the image of this complex under M is contractible with:

$$0 \longrightarrow M(x) \underset{\mu_x}{\overset{M(\eta_x)}{\rightleftarrows}} M^2(x) \underset{\mu_{M(x)}}{\overset{M(\eta_{M(x)}) - M^2(\eta_x)}{\rightleftarrows}} M^3(x)\,.$$

Here $\mu\, M(\eta) = \mathrm{id}_M$ by (A.2) and $\mu_{M(x)}\, M^2(\eta_x) = M(\eta_x)\,\mu_x$ by naturality of μ. □

2.14. *Remark.* The complex (2.13) is the beginning of the *Amitsur complex*, as in [8], whose contractibility (hence exactness everywhere) can be proven as above via Remark 2.11.

2.15. **Corollary.** *Let (M, μ, η) be a faithful additive monad on a triangulated category $\mathcal{C}$. Then the functor $Q_M : \mathcal{C} \longrightarrow \mathrm{Desc}_{\mathcal{C}}(M)$ of (1.1) is fully faithful.*

Proof. Proposition 2.12 implies that $x \xrightarrow{\eta_x} M(x) \underset{M(\eta_x)}{\overset{\eta_{M(x)}}{\rightrightarrows}} M^2(x)$ is an equalizer. Hence $\eta_x : x \to M(x)$ is a "regular monomorphism". It then follows from a variant of Beck's Comonadicity Theorem, see [12, Thm. 2.3 (i)], that Q_M is automatically fully faithful.

More directly, Q_M is faithful since M is, see (1.6) and it is full by following [8, *Démonstration* of Prop. 2.5]: A morphism $f : Q_M(x) \to Q_M(x')$ in $\mathrm{Desc}_{\mathcal{C}}(M)$

makes the right-hand square commute in the following diagram since f commutes both with ηM by naturality of η and with $M\eta$, which is the coaction on $Q_M(x)$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & x & \xrightarrow{\eta_x} & M(x) & \xrightarrow{\eta_{M(x)} - M(\eta_x)} & M^2(x) \\ & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle M(f)} \\ 0 & \longrightarrow & x' & \xrightarrow{\eta_{x'}} & M(x') & \xrightarrow{\eta_{M(x')} - M(\eta_{x'})} & M^2(x') . \end{array} \tag{2.16}$$

The two rows are exact by Proposition 2.12. So, there is a unique $g : x \to x'$ making the diagram commute. Then $f = f\,\mathrm{id}_{M(x)} = f\,\mu_x\,M(\eta_x) \overset{\star}{=} \mu_{x'}M(f)\,M(\eta_x) = \mu_{x'}M(f\,\eta_x) \overset{(2.16)}{=} \mu_{x'}M(\eta_{x'}g) = \mu_{x'}M(\eta_{x'})M(g) = \mathrm{id}_{M(x')}\,M(g) = M(g)$ where equality $\star$ holds by M-linearity of $f : M(x) \to M(y)$ between free M-modules. This means $f = Q_M(g)$. Hence Q_M is full, as wanted. □

We now analyze the essential image of our functor $Q_M : \mathcal{C} \to \mathrm{Desc}_{\mathcal{C}}(M)$.

2.17. **Lemma.** *Let (M, μ, η) be an additive monad on $\mathcal{C}$ and let (x, ϱ, δ) be an object in the descent category $\mathrm{Desc}_{\mathcal{C}}(M)$. Then the complex*

$$0 \longrightarrow x \underset{\varrho}{\overset{\delta}{\rightleftarrows}} M(x) \underset{\mu_x}{\overset{M(\eta_x-\delta)}{\rightleftarrows}} M^2(x)$$

is contractible in $\mathcal{C}$, with the contraction given by ϱ and μ_x as above.

Proof. This is a direct verification, using $\mu_x\,M(\eta_x) = \mathrm{id}_{M(x)}$ from (A.2), as well as $M(\eta_x - \delta)\,\delta = 0$, $\varrho\,\delta = \mathrm{id}_x$ and $\mu_x\,M(\delta) = \delta\,\varrho$ from the relations marked 4, 5 and 3 in (1.5), respectively. □

2.18. **Main Lemma.** *Let (M, μ, η) be a faithful additive monad on an idempotent-complete triangulated category $\mathcal{C}$. Let (x, ϱ, δ) be an object in $\mathrm{Desc}_{\mathcal{C}}(M)$. Recall $Q_M : \mathcal{C} \to \mathrm{Desc}_{\mathcal{C}}(M)$ from (1.6). Then the following conditions are equivalent:*

(i) (x, ϱ, δ) belongs to the essential image of the functor Q_M.
(ii) The morphism $\eta_x - \delta : x \to M(x)$ has a kernel in $\mathcal{C}$ (see Lemma 2.8).

Proof. (i)⇒(ii) : One easily checks that property (ii) is stable under isomorphism in $\mathrm{Desc}_{\mathcal{C}}(M)$, so we can assume that $(x, \varrho, \delta) = Q_M(y) = (M(y), \mu_y, M(\eta_y))$ for some $y \in \mathcal{C}$. In that case, $\eta_x - \delta = \eta_{M(y)} - M(\eta_y)$ and this morphism admits $\eta_y : y \to M(y) = x$ as kernel by Proposition 2.12.

For (ii)⇒(i), Lemma 2.8 applied to $g = \eta_x - \delta$ gives a contractible exact complex $0 \longrightarrow y \xrightarrow{f} x \xrightarrow{\eta_x - \delta} M(x)$ for some morphism $f : y \to x$ in $\mathcal{C}$. Let us show that $Q_M(y) \simeq x$ in $\mathrm{Desc}_{\mathcal{C}}(M)$. This is a standard argument. First, apply the additive functor M to the above contractible exact complex to get the contractible exact complex in the upper row of the following commutative diagram in $\mathcal{C}$

$$\begin{array}{ccccccc} 0 & \longrightarrow & M(y) & \xrightarrow{M(f)} & M(x) & \xrightarrow{M(\eta_x-\delta)} & M^2(x) \\ & & {\scriptstyle \varphi}\downarrow{\scriptstyle \simeq} & & \| & & \| \\ 0 & \longrightarrow & x & \xrightarrow{\delta} & M(x) & \xrightarrow{M(\eta_x-\delta)} & M^2(x) \end{array} \tag{2.19}$$

whose second row is a contractible exact complex by Lemma 2.17. Therefore there exists an isomorphism $\varphi : M(y) \overset{\sim}{\to} x$ as in the above diagram satisfying $\delta\varphi = M(f)$, hence $\varphi = \varrho\, M(f)$. This φ is the wanted isomorphism between $Q_M(y) = (M(y), \mu_y, M(\eta_y))$ and (x, ϱ, δ) in $\mathrm{Desc}_{\mathcal{C}}(M)$. Indeed, by the square marked 1 in (1.5) and naturality of μ, we have $\varrho\, M(\varphi) = \varrho\, M(\varrho)\, M^2(f) = \varrho\, \mu_x\, M^2(f) = \varrho\, M(f)\, \mu_y = \varphi\, \mu_y$ which shows that φ is M-linear. Finally φ respects the L_M-coactions since $M(\varphi)\, M(\eta_y) = M(\varrho)\, M^2(f)\, M(\eta_y) = M(\varrho)\, M(\eta_x)\, M(f) = M(f) = \delta\, \varphi$ in which we used naturality of η and the relation marked 2 in (1.5). □

2.20. **Theorem.** *Let (M, μ, η) be a faithful additive monad on an idempotent-complete triangulated category $\mathcal{C}$. Then M satisfies effective descent, i.e. the natural functor $Q_M : \mathcal{C} \longrightarrow \mathrm{Desc}_{\mathcal{C}}(M)$ of (1.1) is an equivalence, if and only if the following condition holds: For every object $(x, \varrho, \delta) \in \mathrm{Desc}_{\mathcal{C}}(M)$, the morphism $\eta_x - \delta : x \to M(x)$ has a kernel in $\mathcal{C}$ (that is, $\eta_x - \delta$ is semi-simple, see Definition 2.5 and Lemma 2.8).*

Proof. Immediate from Corollary 2.15 and Main Lemma 2.18. □

2.21. *Remark.* The proof of Lemma 2.18 describes the inverse of $Q_M : \mathcal{C} \to \mathrm{Desc}_{\mathcal{C}}(M)$ relatively explicitly. In other words, it tells us how to *do descent.* For every M-module with descent datum (x, ϱ, δ), we have $Q_M^{-1}(x, \varrho, \delta) := \ker(\eta_x - \delta)$.

The following is standard, see for instance Borceux [3, Prop. 4.3.2] :

2.22. **Lemma.** *Let $\mathcal{D}$ be an idempotent-complete additive category and let (L, λ, ϵ) be a comonad on $\mathcal{D}$ (Definition A.4) with $L : \mathcal{D} \to \mathcal{D}$ additive. Let g be a morphism in $L\text{–}\mathrm{Comod}_{\mathcal{D}}$ such that $U^L(g) = g$ is semi-simple in $\mathcal{D}$ (Definition 2.5). Then, the kernel of g in $\mathcal{D}$ can be equipped with an L-coaction which makes it a kernel of g in $L\text{–}\mathrm{Comod}_{\mathcal{D}}$. In particular, if $U^L(g)$ is semi-simple then g has a kernel in $L\text{–}\mathrm{Comod}_{\mathcal{D}}$.*

Proof. We sketch the proof for the reader's convenience: Say that $g : y \to y'$ for L-comodules (y, δ) and (y', δ') in $L\text{–}\mathrm{Comod}_{\mathcal{D}}$. By Lemma 2.8, there exists a contractible exact sequence $0 \longrightarrow y_0 \xrightarrow{f} y \xrightarrow{g} y'$, for a morphism $f : y_0 \to y$ in $\mathcal{D}$. We need to prove that y_0 can be made into an L-comodule in such a way that f is L-colinear. Since $L : \mathcal{D} \to \mathcal{D}$ is additive it preserves the contractible exact sequence above (and so does L^2 of course). Hence the rows of the following commutative diagram are exact:

$$\begin{array}{ccccccc} 0 & \longrightarrow & y_0 & \xrightarrow{f} & y & \xrightarrow{g} & y' \\ & & \downarrow{\scriptstyle \delta_0} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta'} \\ 0 & \longrightarrow & L(y_0) & \xrightarrow{L(f)} & L(y) & \xrightarrow{L(g)} & L(y')\,. \end{array}$$

As the right-hand square commutes there exists a morphism $\delta_0 : y_0 \to L(y_0)$ in $\mathcal{D}$ making the diagram commute. One easily proves that δ_0 is the wanted L-coaction. Indeed, the proof of $\lambda\, \delta_0 = L(\delta_0)\, \delta_0$ uses that $L^2(f)$ is a monomorphism; the proof of $\epsilon_{y_0} \delta_0 = \mathrm{id}_{y_0}$ uses that f is a monomorphism; finally the proof that $f : (y_0, \delta_0) \to (y, \delta)$ is the kernel of g in $L\text{–}\mathrm{Comod}_{\mathcal{D}}$ uses that $L(f)$ is a monomorphism. □

2.23. **Theorem.** *Let (M, μ, η) be a faithful additive monad on an idempotent-complete triangulated category $\mathcal{C}$. Then the following properties are equivalent (see Definition 2.5 and Lemma 2.8 for semi-simplicity and existence of kernels):*

(A) The functor $M : \mathcal{C} \to \mathcal{C}$ detects semi-simplicity: If $g : y \to z$ in $\mathcal{C}$ is such that $M(g)$ has a kernel in $\mathcal{C}$ then g has a kernel in $\mathcal{C}$.

(B) The free M-module functor $F_M : \mathcal{C} \to M-\mathrm{Mod}_{\mathcal{C}}$ detects semi-simplicity: If $g : y \to z$ in $\mathcal{C}$ is such that $F_M(g)$ is semi-simple in the additive category $M-\mathrm{Mod}_{\mathcal{C}}$ then g is semi-simple in $\mathcal{C}$ (i.e. it has a kernel in $\mathcal{C}$).

(C) The monad M satisfies effective descent, i.e. $Q_M : \mathcal{C} \xrightarrow{\sim} \mathrm{Desc}_{\mathcal{C}}(M)$.

Proof. (A)⇒(C): By Theorem 2.20, we need to show that for every $(x, \varrho, \delta) \in \mathrm{Desc}_{\mathcal{C}}(M)$, the morphism $\eta_x - \delta : x \to M(x)$ has a kernel. By (A), it suffices to check that $M(\eta_x - \delta) : M(x) \to M^2(x)$ is semi-simple in $\mathcal{C}$, which follows from Lemma 2.17. (Apply Lemma 2.8 (ii) with $m := \mu_x : M^2(x) \to M(x)$.)

(C)⇒(B): Let g in $\mathcal{C}$ such that $F_M(g)$ is semi-simple in $M-\mathrm{Mod}_{\mathcal{C}}$. Since $F_M = U^{L_M} Q_M$, it follows from Lemma 2.22 that $Q_M(g)$ has a kernel in $\mathrm{Desc}_{\mathcal{C}}(M)$. Since we assume that Q_M is an equivalence, g has a kernel in $\mathcal{C}$, which is triangulated. Hence g is semi-simple by the part of Lemma 2.8 which uses "triangulated".

(B)⇒(A): It suffices to show that if $g : y \to z$ in $\mathcal{C}$ is such that $M(g)$ is semi-simple in $\mathcal{C}$ then $F_M(g)$ is semi-simple in $M-\mathrm{Mod}_{\mathcal{C}}$. Let $m : M(z) \to M(y)$ in $\mathcal{C}$ such that $M(g)\, m\, M(g) = M(g)$. Of course, $F_M(g) = M(g) : M(y) \to M(z)$ but, a priori, m is not M-linear. However, $\tilde{m} := \mu_y\, M(m)\, M(\eta_z) : M(z) \to M(y)$ is M-linear (using associativity and naturality of μ) and satisfies $F_M(g)\, \tilde{m}\, F_M(g) = F_M(g)$ by naturality of μ and η and the above relation $M(g)\, m\, M(g) = M(g)$. These straightforward verifications are left to the reader. □

2.24. *Remark.* Mesablishvili pointed out that the above proofs rely only on the following property of triangulated categories: Monomorphisms are split. Hence, the above results hold in a broader setting, see [13]. He also pointed out to me that exactness of the monad, which we required (out of habit) in a previous version of the article, was indeed superfluous. Only additivity is necessary, as stated above.

2.25. *Remark.* We have seen in Proposition 2.12 that when M is faithful then $\eta_x : x \to M(x)$ is a split monomorphism. However, the retraction might not be natural in x. When a natural retraction $\pi : M \to \mathrm{Id}_{\mathcal{C}}$ of η exists, then effective descent follows by (A) of Theorem 2.23. Indeed, suppose that $g : y \to z$ is such that $M(g)$ is semi-simple in $\mathcal{C}$. By Lemma 2.8, there exists $m : M(z) \to M(y)$ with $M(g)\, m\, M(g) = M(g)$ in $\mathcal{C}$. Let then $\bar{m} := \pi_y\, m\, \eta_z : z \to y$. A direct verification (using naturality of π) shows that $g\, \bar{m}\, g = g$. This recovers, in the special case of $\mathcal{C}$ triangulated, the following general result for additive categories:

2.26. **Corollary** (Mesablishvili [12, Cor. 3.17])**.** *Let (M, μ, η) be an additive monad on an idempotent-complete additive category $\mathcal{C}$, admitting a natural transformation $\pi : M \to \mathrm{Id}_{\mathcal{C}}$ such that $\pi \circ \eta = \mathrm{id}$. Then M satisfies effective descent.*

3. The case of ring objects

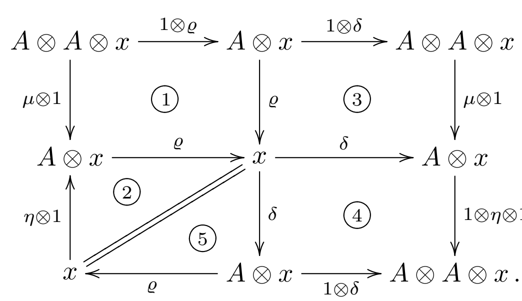
We consider a triangulated category $\mathcal{C}$ with a monoidal functor, the *tensor*, $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ with unit object $\mathbb{1} \in \mathcal{C}$. We just need $- \otimes -$ additive in each variable. Recall Remark 2.24 for generalization beyond triangulated categories.

3.1. **Corollary.** *Let (A, μ_A, η_A) be a ring object in an idempotent-complete tensor triangulated category $\mathcal{C}$. Define $\mathrm{Desc}_{\mathcal{C}}(A) = \mathrm{Desc}_{\mathcal{C}}(M)$ to be the descent category* *for the associated monad $(M, \mu, \eta) = (A \otimes -, \mu_A \otimes -, \eta_A \otimes -)$ in $\mathcal{C}$. (See details in Remark 3.3 below.) Then the natural functor $Q_A : \mathcal{C} \longrightarrow \mathrm{Desc}_{\mathcal{C}}(A)$ is an equivalence, i.e. A satisfies effective descent, if and only if A is faithful.*

Proof. As in Remark 2.3, A must be faithful to satisfy descent. Conversely, suppose that A, hence $M = A \otimes -$, is faithful. We want to apply Corollary 2.26. By Proposition 2.12, $\eta_A : \mathbb{1} \to A$ is a split monomorphism. Choose a retraction $\pi_A : A \to \mathbb{1}$ of η_A in $\mathcal{C}$ and define the requested natural retraction π of $\eta = \eta_A \otimes - : \mathrm{Id}_{\mathcal{C}} \to A \otimes -$ to be $\pi_x := \pi_A \otimes \mathrm{id}_x : A \otimes x \to x$ for every $x \in \mathcal{C}$. □

3.2. *Remark.* In the above statement, we do not need the tensor $\otimes$ to be *symmetric* monoidal. This illustrates the clarity of the monadic approach. Similarly, if $\mathcal{C}$ acts on some other category $\mathcal{D}$ via a bi-additive $\boxtimes : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{D}$, then $M = A \boxtimes -$ is an additive monad on $\mathcal{D}$ to which we can apply the results of Section 2.

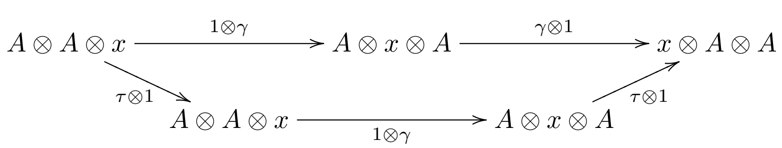
3.3. *Remark.* For convenience of quotation, let us unfold the definition of the descent category, $\mathrm{Desc}_{\mathcal{C}}(A)$, for a ring object (A, μ, η) in $\mathcal{C}$, first without assuming symmetry of $\otimes$. An object (x, ϱ, δ) in $\mathrm{Desc}_{\mathcal{C}}(A)$ is a triple with $x \in \mathcal{C}$ an object, $\varrho : A \otimes x \to x$ the A-module structure on x and $\delta : x \to A \otimes x$ the comodule structure (a. k. a. the *descent datum*); this triple must satisfy the five conditions of (1.5), which are here:



In Cipolla's words [4], the relations marked 4 and 5 express conditions (1) and (2) of [4, *Definizione* p. 45]. (Note that there is a misprint in the definition of d^3 in [4], which should read $d^3(s \otimes m) = s \otimes f(m)$.) A morphism of A-modules with descent data $f : (x, \varrho, \delta) \to (x', \varrho', \delta')$ is simply a morphism $f : x \to x'$ in $\mathcal{C}$ which commutes with action and descent data, as before: $f\,\varrho = \varrho'\,(1 \otimes f) : A \otimes x \to x'$ and $\delta' f = (1 \otimes f)\,\delta : x \to A \otimes x'$. The functor $Q_A : \mathcal{C} \longrightarrow \mathrm{Desc}_{\mathcal{C}}(A)$ maps an object x to $(A \otimes x,\, \mu \otimes 1,\, 1 \otimes \eta \otimes 1)$ and a morphism $f : x \to x'$ to $1 \otimes f : A \otimes x \to A \otimes x'$.

3.4. *Remark.* Continuing Remark 3.3, let us now assume that $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ is *symmetric* monoidal. Let us denote the switch of factors by $\tau_{x,y} : x \otimes y \xrightarrow{\sim} y \otimes x$. In that case, one can verify that the category $\mathrm{Desc}_{\mathcal{C}}(A)$ is isomorphic to the category whose objects are triples (x, ϱ, γ) with (x, ϱ) an A-module and $\gamma : A \otimes x \xrightarrow{\sim} x \otimes A$ an isomorphism of $A \otimes A$-modules such that the following diagram commutes:

(3.5)



Morphisms of such triples are defined in the obvious way. The isomorphism of categories with $\mathrm{Desc}_{\mathcal{C}}(A)$ relates (x,ϱ,δ) and (x,ϱ,γ) via the following dictionary:

$$\begin{array}{ccc} \delta : x \to A\otimes x & \longmapsto & \gamma(\delta) := (\varrho\otimes 1)\,(1\otimes\tau)\,(1\otimes\delta) \\ \delta(\gamma) := \tau\,\gamma\,(\eta\otimes 1) & \longleftarrow & \gamma : A\otimes x \overset{\sim}{\to} x\otimes A\,. \end{array}$$

This verification was an exercise thirty-five years ago in [4] and remains so today.

The formulation with the isomorphism $\gamma : A\otimes x \overset{\sim}{\to} x\otimes A$ instead of the coaction $\delta : x \to A\otimes x$ is essentially the one of [8, § II.3]. Equation (3.5) is the usual cocycle condition, which would read "$\gamma_2 = \gamma_3\gamma_1$" in the notation of [8].

3.6. *Example.* Let R be a commutative ring and $\mathrm{Spec}(R) = D(s_1)\cup\ldots\cup D(s_n)$ a Zariski cover by principal open subsets and set $A = R[1/s_1]\times\cdots\times R[1/s_n]$. Then an A-module in $\mathcal{C} = R-\mathrm{Mod}$ is the same thing as a collection of $R[1/s_i]$-modules M_i and the isomorphism γ provides isomorphisms between the restrictions of M_i and M_j to $D(s_i)\cap D(s_j)$. Here, the cocycle condition (3.5) yields the usual compatibility over triple intersections and descent just means gluing of modules.

4. A faithful triangular monad without descent

Let $\Bbbk$ be a field and $R = \left(\begin{smallmatrix}\Bbbk & 0\\ \Bbbk & \Bbbk\end{smallmatrix}\right) \subset \mathrm{M}_2(\Bbbk)$ the $\Bbbk$-algebra of lower-triangular (2×2)-matrices in $\Bbbk$. It is the path algebra of the A_2-quiver $\bullet \to \bullet$. Let $\mathcal{A} = R-\mathrm{mod}$ be the category of finitely generated left R-modules. The ring R is hereditary and satisfies Krull-Schmidt. Thinking of R in columns, we have

$$R = P_2 \oplus P_1$$

where $P_2 = \left(\begin{smallmatrix}\Bbbk\\ \Bbbk\end{smallmatrix}\right)$ and $P_1 = \left(\begin{smallmatrix}0\\ \Bbbk\end{smallmatrix}\right)$ have left R-action by matrix multiplication. These are indecomposable projective R-modules and there is only one other isomorphism class of indecomposable R-module, given by the non-projective module $M = \Bbbk$ with left R-action $\left(\begin{smallmatrix}a & 0\\ b & c\end{smallmatrix}\right)\cdot x = ax$. We have an exact sequence:

$$0 \to P_1 \xrightarrow{\alpha} P_2 \xrightarrow{\beta} M \to 0\,, \tag{4.1}$$

where $\alpha : P_1 \rightarrowtail P_2$ is the obvious inclusion $\alpha\left(\begin{smallmatrix}0\\ y\end{smallmatrix}\right) = \left(\begin{smallmatrix}0\\ y\end{smallmatrix}\right)$ and where $\beta\left(\begin{smallmatrix}x\\ y\end{smallmatrix}\right) = x$.

Dually, we have *right* R-modules $Q_1 = (\Bbbk\ 0)$ and $Q_2 = (\Bbbk\ \Bbbk)$, with right R-action by matrix multiplication again. They are projective: $R = Q_1 \oplus Q_2$ (in rows) and there is also a right R-module $N := \Bbbk$ given by $y\cdot\left(\begin{smallmatrix}a & 0\\ b & c\end{smallmatrix}\right) = yc$. We now have an exact sequence of right R-modules, with $\gamma(x\ 0) = (x\ 0)$ and $\delta(x\ y) = y$:

$$0 \to Q_1 \xrightarrow{\gamma} Q_2 \xrightarrow{\delta} N \to 0\,. \tag{4.2}$$

We have interesting isomorphisms of right R-modules

$$\mathrm{Hom}_R(P_2,R) \simeq Q_1 \quad\text{and}\quad \mathrm{Hom}_R(P_1,R) \simeq Q_2 \tag{4.3}$$

under which $\mathrm{Hom}_R(\alpha,R)$ is γ. Note that $\mathrm{Hom}_R(-,R)$ swaps dimension two and dimension one (over $\Bbbk$). The isomorphisms of (4.3) are easy to establish, for instance using that $P_1 = \mathrm{coker}\left(R \xrightarrow{\cdot d} R\right)$ for $d = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$ with dual $\ker\left(R \xrightarrow{d\cdot} R\right) = Q_2$.

On the other hand, duals over $\Bbbk$ do respect dimension. We have isomorphisms

$$\mathrm{Hom}_{\Bbbk}(Q_1,\Bbbk) \simeq M \quad,\quad \mathrm{Hom}_{\Bbbk}(Q_2,\Bbbk) \simeq P_2 \quad\text{and}\quad \mathrm{Hom}_{\Bbbk}(N,\Bbbk) \simeq P_1 \tag{4.4}$$

of left R-modules (R acts on $\mathrm{Hom}_{\Bbbk}(X,\Bbbk)$ via $(af)(x) = f(xa)$ as usual). These are easy to establish on $\Bbbk$-bases and are left to the reader.

4.5. **Lemma.** *There are three adjunctions of exact functors as follows:*

$$\begin{array}{c} \mathrm{D^b}(\mathcal{A}) \\ \mathrm{RHom}_R(P_2\,,-)\Big\downarrow\Big\uparrow M\underset{\Bbbk}{\otimes}- \\ \mathrm{D^b}(\Bbbk) \end{array} \qquad \begin{array}{c} \mathrm{D^b}(\mathcal{A}) \\ \mathrm{RHom}_R(P_1\,,-)\Big\downarrow\Big\uparrow P_2\underset{\Bbbk}{\otimes}- \\ \mathrm{D^b}(\Bbbk) \end{array} \qquad \begin{array}{c} \mathrm{D^b}(\mathcal{A}) \\ \mathrm{RHom}_R(M,-)\Big\downarrow\Big\uparrow P_1[1]\underset{\Bbbk}{\otimes}- \\ \mathrm{D^b}(\Bbbk)\,. \end{array}$$

Proof. Recall $\mathcal{A}=R-\mathrm{mod}$. For every $X\in\mathrm{D^b}(\mathcal{A})$ and every $Y\in\mathrm{D^b}(\Bbbk)$, we have

$$\mathrm{RHom}_R(X,-)\cong\mathrm{RHom}_R(X,R)\otimes^{\mathrm{L}}_R- \quad\text{and}\quad \mathrm{Hom}_{\Bbbk}(Y,-)\cong\mathrm{Hom}_{\Bbbk}(Y,\Bbbk)\otimes_{\Bbbk}-\,.$$

Combining this with the usual $\otimes^{\mathrm{L}}$–RHom adjunction, yields an adjunction

$$\begin{array}{c} \mathrm{D^b}(\mathcal{A}) \\ \mathrm{RHom}_R(X,-)\cong Y\overset{\mathrm{L}}{\underset{R}{\otimes}}- \quad\Big\downarrow\Big\uparrow\quad \mathrm{Hom}_{\Bbbk}(Y\,,-)\cong Z\underset{\Bbbk}{\otimes}- \\ \mathrm{D^b}(\Bbbk) \end{array}$$

where $Y:=\mathrm{RHom}_R(X,R)$ is seen as a complex of right R-modules and where $Z:=\mathrm{Hom}_{\Bbbk}(Y,\Bbbk)$ is again a complex of left R-modules.

A direct computation with the notation introduced before the Lemma yields for the three indecomposable values of X, namely P_2, P_1 and M, the following result:

$$\begin{array}{llllll} \text{If} & X=P_2 & \text{then} & Y\simeq Q_1 & \text{and} & Z\simeq M\,. \\ \text{If} & X=P_1 & \text{then} & Y\simeq Q_2 & \text{and} & Z\simeq P_2\,. \\ \text{If} & X=M & \text{then} & Y\simeq N[-1] & \text{and} & Z\simeq P_1[1]\,. \end{array}$$

To check the last row, for instance, use the projective resolution (4.1) of M, apply $\mathrm{Hom}_R(-,R)$ and use (4.3) and (4.2) to see that $\mathrm{RHom}_R(M,R)=N[-1]$. Then $Z=\mathrm{Hom}_{\Bbbk}(N[-1],\Bbbk)\cong\mathrm{Hom}_{\Bbbk}(N,\Bbbk)[1]\simeq P_1[1]$ by (4.4). $\square$

4.6. **Theorem.** *Let $\Bbbk$ be a field, $R=\left(\begin{smallmatrix}\Bbbk & 0\\ \Bbbk & \Bbbk\end{smallmatrix}\right)$ and $\mathcal{A}=R-\mathrm{mod}$ the category of left R-modules as above. (Equivalently, $\mathcal{A}$ is the category $\mathrm{Arr}(\Bbbk-\mathrm{vect})$ of arrows of finite-dimensional $\Bbbk$-vector spaces.) Consider the adjunction obtained by adding up the three adjunctions of Lemma 4.5:*

$$\begin{array}{c} \mathrm{D^b}(\mathcal{A}) \\ F:=\mathrm{RHom}_R\big(P_2\oplus P_1\oplus M\,,-\big)\Big\downarrow\Big\uparrow G:=\big(M\oplus P_2\oplus P_1[1]\big)\underset{\Bbbk}{\otimes}- \\ \mathrm{D^b}(\Bbbk) \end{array}$$

Then the associated monad $\mathbb{M}:=G\,F$ on $\mathcal{C}:=\mathrm{D^b}(\mathcal{A})$, see (A.7)*, is faithful (and exact) but does not satisfy effective descent in the triangulated category $\mathcal{C}$.*

Proof. Both functors F and G are faithful. For the latter it follows obviously from $G\neq 0$. For F, note that $L:=P_2\oplus P_1\oplus M$ is the direct sum of all indecomposable objects, up to isomorphisms. Moreover, since R is hereditary, every object X in $\mathrm{D^b}(\mathcal{A})$ is a sum of translates of these three objects. Faithfulness of F then easily follows from $\mathrm{H}_i(\mathrm{RHom}_R(L,X))\cong\mathrm{Hom}_{\mathrm{D^b}(\mathcal{A})}(L,X[-i])$ for every $i\in\mathbb{Z}$.

Let us discuss descent. In $\mathcal{C}=\mathrm{D^b}(\mathcal{A})$, consider the morphism $\alpha:P_1\to P_2$ of (4.1). As every other morphism of $\mathrm{D^b}(\Bbbk)$, the morphism $F(\alpha)$ is semi-simple. Hence so is $G(F(\alpha))=\mathbb{M}(\alpha)$. However, α is not semi-simple in $\mathcal{C}=\mathrm{D^b}(\mathcal{A})$ (if it satisfied Lemma 2.8 (ii), it would do so in $\mathcal{A}$ since $\mathcal{A} \hookrightarrow \mathrm{D}^{\mathrm{b}}(\mathcal{A})$ is fully faithful, but this is impossible since $\mathrm{Hom}_{\mathcal{A}}(P_2, P_1) = 0$). So, M does not satisfy property (A) of Theorem 2.23, hence does not satisfy descent. □

4.7. *Remark.* Unfolding the above counter-example, one can check that the faithful monad $M : \mathcal{C} \to \mathcal{C}$ is indeed given by $A \otimes^{\mathrm{L}}_R -$ for some ring object A. This might puzzle the reader, in view of Corollary 3.1. Explicitly, $A = \mathrm{Hom}_{\Bbbk}(Y, Y)$ where $Y = \mathrm{RHom}_R(L, R)$ is a complex of right R-modules. However, this A is not a ring object in $\mathcal{C}$ itself (which has no obvious tensor structure) but in the bigger derived category of R-bimodules, which only *acts on* $\mathcal{C}$. What happens is that although $A \otimes^{\mathrm{L}}_R - : \mathcal{C} \to \mathcal{C}$ is faithful, the ring A itself is not faithful in the derived category of R-bimodules. Therefore Theorem 4.6 does not contradict Corollary 3.1.

4.8. *Remark.* When the triangulated category $\mathcal{C}$ comes with a model (and if triangular descent fails), the reader might want to recourse to the interesting approaches proposed in Hess [6] or Lurie [9], via enriched structures.

Appendix A. Monads, modules, and co

A.1. *Definition.* Recall that a *monad* (M, μ, η) on a category $\mathcal{C}$ is an endofunctor $M : \mathcal{C} \to \mathcal{C}$, together with two natural transformations, $\mu : M^2 \to M$ (*multiplication*) and $\eta : \mathrm{Id}_{\mathcal{C}} \to M$ (*two-sided unit*), satisfying the usual commutativities:

(A.2)
$$\begin{array}{ccc} M^3 & \xrightarrow{M\mu} & M^2 \\ {\scriptstyle \mu M}\downarrow & & \downarrow{\scriptstyle \mu} \\ M^2 & \xrightarrow[\mu]{} & M \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} M & \xrightarrow{M\eta} & M^2 & \xleftarrow{\eta M} & M \\ & \searrow\!\!\!\!= & \downarrow{\scriptstyle \mu} & =\!\!\!\!\swarrow & \\ & & M & & \end{array}$$

which express associativity of μ and the two-sided unit, respectively. Inspiration comes from the monad $M(-) = A \otimes -$ for some ring object $A = (A, \mu, \eta)$ in a monoidal category $\mathcal{C}$, with multiplication $\mu : A \otimes A \to A$ and unit $\eta : \mathbb{1} \to A$.

An M-*module* ([2]) *in* $\mathcal{C}$ is a pair (x, ϱ) where x is an object in $\mathcal{C}$ and $\varrho : M(x) \to x$ (the M-*action*) is a morphism such that the following diagrams commute:

$$\begin{array}{ccc} M^2(x) & \xrightarrow{M(\varrho)} & M(x) \\ {\scriptstyle \mu_x}\downarrow & & \downarrow{\scriptstyle \varrho} \\ M(x) & \xrightarrow{\varrho} & x \end{array} \qquad \text{and} \qquad \begin{array}{ccc} x & \xrightarrow{\eta_x} & M(x) \\ & \searrow\!\!\!\!= & \downarrow{\scriptstyle \varrho} \\ & & x\,. \end{array}$$

Morphisms $f : (x, \varrho) \to (x', \varrho')$ of M-modules are morphisms $f : x \to x'$ in $\mathcal{C}$ which are M-linear, i.e. $\varrho' \circ M(f) = f \circ \varrho$. We denote the category of M-modules in $\mathcal{C}$ by $M\text{–}\mathrm{Mod}_{\mathcal{C}}$. The *free-module functor* $F_M : \mathcal{C} \longrightarrow M\text{–}\mathrm{Mod}_{\mathcal{C}}$ is given by $F_M(x) = (M(x), \mu_x)$. It is left adjoint to the *forgetful functor* $U_M : M\text{–}\mathrm{Mod}_{\mathcal{C}} \longrightarrow \mathcal{C}$, which forgets the action. This yields the so-called *Eilenberg-Moore adjunction*:

(A.3)
$$\begin{array}{c} \mathcal{C} \\ {\scriptstyle F_M}\downarrow\uparrow{\scriptstyle U_M} \\ M\text{–}\mathrm{Mod}_{\mathcal{C}}\,. \end{array}$$

[2] In [10], "M-modules" are called "M-algebras". We prefer "modules" since this concept coincides with that of A-modules, not of A-algebras, in the motivating example $M = A \otimes -$.

Its unit $\mathrm{Id}_{\mathcal{C}} \to U_M\, F_M = M$ is simply the given unit η of M whereas its counit $\epsilon_M : F_M\, U_M \to \mathrm{Id}_{M-\mathrm{Mod}_{\mathcal{C}}}$ is given for every M-module (x, ϱ) by $(\epsilon_M)_{(x,\varrho)} = \varrho$.

Let us quickly recall the dual notions, to fix notations.

A.4. *Definition.* A *comonad in a category* $\mathcal{D}$ is a triple (L, λ, ϵ) where $L : \mathcal{D} \to \mathcal{D}$ is a functor with *comultiplication* $\lambda : L \to L^2$ and *counit* $\epsilon : L \to \mathrm{Id}_{\mathcal{D}}$ such that $(\lambda L) \circ \lambda = (L\lambda) \circ \lambda$ and $(L\epsilon) \circ \lambda = (\epsilon L) \circ \lambda = \mathrm{id}_L$. The category $L-\mathrm{Comod}_{\mathcal{D}}$ of *L-comodules in* $\mathcal{D}$ is the following: Its objects are pairs (x, δ) where $\delta : x \to L(x)$ is a morphism in $\mathcal{D}$ (the *L-coaction* on x) such that $L(\delta) \circ \delta = \lambda \circ \delta$ and $\epsilon_x \circ \delta = \mathrm{id}_x$. A morphism of L-comodules $f : (x, \delta) \to (x', \delta')$ is an $f : x \to x'$ in $\mathcal{D}$ such that $L(f) \circ \delta = \delta' \circ f$. There is a *free-comodule functor* $F^L : \mathcal{D} \longrightarrow L-\mathrm{Comod}_{\mathcal{D}}$, $x \mapsto \big(L(x), \lambda_x\big)$, which is right (!) adjoint to the obvious forgetful functor $U^L : L-\mathrm{Comod}_{\mathcal{D}} \longrightarrow \mathcal{D}$. This is the "co"-Eilenberg-Moore adjunction:

$$\begin{array}{c} L-\mathrm{Comod}_{\mathcal{D}} \\ {\scriptstyle U^L}\big\downarrow\big\uparrow{\scriptstyle F^L} \\ \mathcal{D}\,. \end{array} \tag{A.5}$$

Its counit is the given $\epsilon : U^L\, F^L = L \to \mathrm{Id}_{\mathcal{D}}$ and the unit $\eta^L : \mathrm{Id}_{L-\mathrm{Comod}_{\mathcal{D}}} \to F^L\, U^L$ is defined for every L-comodule (x, δ) by $\delta : x \to L(x) = F^L\, U^L(x, \delta)$.

A.6. *Remark.* It is well-known that every adjunction $F : \ \mathcal{C} \leftrightarrows \mathcal{D} \ : G$ with unit $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ and counit $\epsilon : FG \to \mathrm{Id}_{\mathcal{D}}$, induces both

$$\begin{array}{lll} \text{a monad} & \big(\, M := GF\,,\ \mu := G\epsilon F\,,\ \eta\,\big) & \text{on } \mathcal{C}\,, \quad \text{and} \\ \text{a comonad} & \big(\, L := FG\,,\ \lambda := F\eta G\,,\ \epsilon\,\big) & \text{on } \mathcal{D}\,. \end{array} \tag{A.7}$$

Adjunctions (A.3) and (A.5) show that any (co)monad can be realized in this way. In fact, the Eilenberg-Moore adjunctions are the final such realizations, in the following sense (see [10, Thm. VI.3.1]): Given an adjunction $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ and setting $M = GF : \mathcal{C} \longrightarrow \mathcal{C}$ and $L = FG : \mathcal{D} \longrightarrow \mathcal{D}$ as in (A.7), there exist two (unique) functors $P : \mathcal{D} \longrightarrow M-\mathrm{Mod}_{\mathcal{C}}$ and $Q : \mathcal{C} \longrightarrow L-\mathrm{Comod}_{\mathcal{D}}$

$$\begin{array}{ccc} & \mathcal{C} & \\ {\scriptstyle F}\swarrow\nearrow{\scriptstyle G} & & {\scriptstyle F_M}\searrow\nwarrow{\scriptstyle U_M} \\ \mathcal{D} & \overset{}{\underset{P = P_{F,G,\eta,\epsilon}}{\dashrightarrow}} & M-\mathrm{Mod}_{\mathcal{C}} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathcal{C} & \overset{Q = Q_{F,G,\eta,\epsilon}}{\dashrightarrow} & L-\mathrm{Comod}_{\mathcal{D}} \\ {\scriptstyle F}\searrow\nwarrow{\scriptstyle G} & & {\scriptstyle U^L}\swarrow\nearrow{\scriptstyle F^L} \\ & \mathcal{D} & \end{array} \tag{A.8}$$

which are morphisms of adjunctions, i.e. $P \circ F = F_M$ and $U_M \circ P = G$, whereas $Q \circ G = F^L$ and $U^L \circ Q = F$. Explicitly, $P(d) = \big(G(d)\,,\, G(\epsilon_d)\big)$ for every object $d \in \mathcal{D}$ and $P(g) = G(g)$ for every morphism g in $\mathcal{D}$. Dually, $Q(c) = \big(F(c)\,,\, F(\eta_c)\big)$ for every object $c \in \mathcal{C}$ and $Q(f) = F(f)$ for every morphism f in $\mathcal{C}$.

A.9. *Remark.* Finally, we explicit the comonad L_M on $M-\mathrm{Mod}_{\mathcal{C}}$ associated to a monad M on a category $\mathcal{C}$. We have the Eilenberg-Moore adjunction $F_M : \mathcal{C} \leftrightarrows M-\mathrm{Mod}_{\mathcal{C}} : U_M$ of (A.3). As in (A.7), such an adjunction induces a comonad on the right-hand category, here $M-\mathrm{Mod}_{\mathcal{C}}$, that we denote

$$L_M := F_M \circ U_M \ : \ M-\mathrm{Mod}_{\mathcal{C}} \longrightarrow M-\mathrm{Mod}_{\mathcal{C}}\ .$$

The comonad $(L_M, \lambda_M, \epsilon_M)$ has comultiplication $\lambda_M = F_M\, \eta\, U_M$ and counit ϵ_M as after (A.3). Explicitly, for every M-module (x, ϱ) we have $L_M(x, \varrho) = \big(M(x)\,,\, \mu_x\big)$

whereas $L_M(f) = M(f)$ for every M-linear f. The comultiplication $(\lambda_M)_{(x,\varrho)}$ at (x, ϱ), which is a morphism of M-modules from $L_M(x, \varrho) = \big(M(x)\,,\, \mu_x\big)$ to $(L_M)^2(x, \varrho) = \big(M^2(x)\,,\, \mu_{M(x)}\big)$, is given by the morphism $M(\eta_x) : M(x) \to M^2(x)$ in $\mathcal{C}$. Finally, the counit $(\epsilon_M)_{(x,\varrho)}$ at (x, ϱ), which is a morphism of M-modules from $L_M(x, \varrho) = (M(x), \mu_x)$ to (x, ϱ), is given by the morphism $\varrho : M(x) \to x$ in $\mathcal{C}$.

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Paul Balmer, Mathematics Department, UCLA, Los Angeles, CA 90095-1555, USA
E-mail address: `balmer@math.ucla.edu`
URL: `http://www.math.ucla.edu/∼balmer`