WITT COHOMOLOGY, MAYER-VIETORIS, HOMOTOPY INVARIANCE AND THE GERSTEN CONJECTURE

Paul Balmer

Abstract. We establish a Mayer-Vietoris long exact sequence for Witt groups of regular schemes. We also establish homotopy invariance for Witt groups of regular schemes. For this, we introduce *Witt groups with supports* using triangulated categories. Subsequently we use these results to prove the Gersten-Witt Conjecture for semi-local regular rings of geometric type over infinite fields of characteristic different from two.

Dedicated to Professor Manuel Ojanguren on his sixtieth birthday.

0. Introduction

The Witt group $\mathrm{W}(X)$ of a scheme X was defined in Knebusch's 1977 paper [**11**]. In this generality, that is when X is not assumed to be affine, very little is known about the contravariant functor $W(-)$. Motivated by K-theory analogies, we may first restrict our attention to regular schemes. Even then, such elementary questions as the existence of a Mayer-Vietoris exact sequence

$$\cdots \longrightarrow \mathrm{W}(U\cup V) \longrightarrow \mathrm{W}(U)\oplus \mathrm{W}(V) \longrightarrow \mathrm{W}(U\cap V) \longrightarrow \cdots$$

and homotopy invariance

$$\mathrm{W}(X) \cong \mathrm{W}(\mathbb{A}^1_X)$$

appear to be out of range of classical methods. These are the main results of the present paper. Mayer-Vietoris is Theorem 2.5. Homotopy invariance is Theorem 3.4. We shall work over regular noetherian separated schemes on which 2 is everywhere invertible.

Over affine schemes, homotopy invariance is a famous theorem of Karoubi [**9**, Corollary 3.10] and a Mayer-Vietoris exact sequence can be found in Ranicki [**18**, Chapter 6]. Those authors work, more generally, over rings with involution, but not over global schemes.

These two results are obviously of great importance. The first basic reason is purely practical. In any theory for which such theorems hold, they immediately provide computations by means of geometric decomposition, without needing to know much about explicit bundles, symmetric forms, and so on. But Mayer-Vietoris and homotopy invariance have other, more far-reaching applications. One of them was mentioned to the author by Bruno Kahn, namely a formal proof of the Gersten conjecture over infinite fields. We now recall what this conjecture is about.

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Given a scheme X, we denote by $X^{(p)}$ its points of codimension p and for any point $x \in X$ by $\kappa(x)$ the residue field at x. If X is regular separated and noetherian, and if X has finite Krull dimension n, we want to study complexes of the form:

$$0 \to \mathrm{W}(X) \longrightarrow \bigoplus_{x \in X^{(0)}} \mathrm{W}(\kappa(x)) \longrightarrow \bigoplus_{x \in X^{(1)}} \mathrm{W}(\kappa(x)) \longrightarrow \cdots \longrightarrow \bigoplus_{x \in X^{(n)}} \mathrm{W}(\kappa(x)) \to 0.$$

Such a *Gersten-Witt complex* has been constructed by several authors, under more restrictive hypotheses, mainly the existence of a ground field. In the above general setting, such a complex is obtained in [**3**], as the E_1-term of a Gersten-Witt spectral sequence.

The Gersten Conjecture for Witt groups (Pardon [**15**]) claims the existence and the exactness of such a complex for $X = \mathrm{Spec}(R)$ where R is a regular local ring. This conjecture was inspired by the K-theoretic analogue. The reader will see that our proof stems from Quillen's original proof, which was given for R semi-local, regular and of geometric type over a base field k. It is for such rings, with k infinite and of characteristic different from 2, that we prove the above Witt group version of the conjecture in Section 4.

The following is an overview of the present article.

The treatment of Witt groups of global schemes relies upon triangulated categories and their Witt groups, or in short, *Triangular Witt Groups*. A detailed and elementary introduction to this framework can be found in [TWG, Part I, Section 2]. To avoid alzheimerizing the reader, we choose not to repeat it here. In Theorem 1.1, we give a summary of the material needed to understand this paper. In the remainder of Section 1 we apply this abstract theory to schemes, building a *cohomology theory with supports* in the sense of Colliot-Thélène, Hoobler and Kahn [CT-H-K].

Mayer-Vietoris is a corollary of flat excision, which is in turn an easy consequence of [TWG] and some considerations about triangulated categories. This is done in the very short Section 2. Throughout the paper, we use well known results about derived categories. They can all be deduced from [**5**] or [**20**], if necessary, but are much simpler because of our regularity hypotheses. In every case, we include either a reference or a sketch of the proof.

Section 3 is dedicated to global homotopy invariance. We obtain it by means of Mayer-Vietoris glueing, as soon as we have the affine homotopy invariance of *all* the higher and lower Witt groups W^n which appear in the general theory (see [TWG] or Theorem 1.1). Note that Karoubi's theorem gives us the affine result only for the usual Witt group. The higher Witt groups he constructs and for which he also obtains homotopy invariance (see [**9**]) are not the same as ours. In this regard, the reader is referred to [**12**], [**2**] and [**22**]. On the other hand, L-theorists also have homotopy invariance results (see [**7**] and Chapter 5 of [**18**]). This might lead to another proof of our Theorem 3.1 below.

Nevertheless, the reader is not assumed to have extensive knowledge of L-theory. Moreover, since we are only interested in affine coverings of regular schemes, we need only homotopy invariance for regular rings. In fact, we are able to outline a very elementary proof that $\mathrm{W}^i(R) = \mathrm{W}^i(R[t])$ for R regular. This is done in Section 3.

Section 4 illustrates the strength of the above theorems with a short demonstration of the Gersten Conjecture, which improves on Quillen's fundamental ideas. The improvements needed here are due primarily to Gabber and also to Colliot-Thélène, Hoobler and Kahn [CT-H-K], who showed that Gabber's arguments apply to very general situations. Moreover, they formalized their proofs under some simple axioms, among which étale excision (COH 1) and homotopy invariance (COH 3). We do not state the list of all possible axioms (COH 1 . . . 6) that a cohomology theory can satisfy, nor the dependences between them, because this can be found in [CT-H-K, Part 2]. On the other hand, we state those of their axioms that we prove to be true for Witt groups, namely (COH 1) in Remark 2.4 and (COH 3) in Remark 3.5.

The assumption that the ground field k is infinite should not be considered as crucial. The case of a finite k requires the existence of "transfers" in the cohomology theory under consideration. See for instance axiom (COH 6), *loc. cit.* General transfers for Witt groups will be treated in a forthcoming work in collaboration with Charles Walter [**4**].

Regarding generalization beyond the hypothesis that our semi-local ring is of geometric type, the reader is referred to the recent preprint of Panin [**14**], where the equicharacteristic case is obtained by an argument which seems likely to generalize to Witt groups.

Several people have already discussed the Gersten-Witt Conjecture in the past : Pardon in [**16**]; Barge, Sansuc and Vogel in a series of talks and notes at the beginning of the 80's; more recently Rost mentioned to the author that the conjecture should be a consequence of Schmid's work [**19**] and general considerations along the lines of Rost's *Chow groups with coefficients*; Panin has also mentioned that he knows a proof as soon as homotopy invariance holds. Finally, Pardon has a very recent preprint [**17**] on the subject.

In a nutshell, the present article connects the abstract results of [TWG] with the geometric results of [CT-H-K]. In the author's point of view, putting the Witt groups into the big picture of cohomology theories and having at hand such tools as Mayer-Vietoris and global homotopy invariance, is probably as important as the Gersten-Witt Conjecture itself.

1. Witt cohomology with supports

1.1. Theorem. *There exists a family of covariant "Witt groups" functors* $(\mathrm{W}^n)_{n\in\mathbb{Z}}$ *defined on triangulated categories with duality, taking their values in abelian groups, and satisfying the following properties :*

(1) *If* $\mathcal{E}$ *is an exact category with duality in which 2 is invertible and if* $\mathrm{D^b}(\mathcal{E})$ *is endowed with the induced duality, we recover the usual Witt group of* $\mathcal{E}$ *among the Witt groups of* $\mathrm{D^b}(\mathcal{E})$ *as follows :*
$$\mathrm{W_{us}}(\mathcal{E}) \cong \mathrm{W}^0\big(\mathrm{D^b}(\mathcal{E})\big).$$

(2) *If* $0 \to J \to K \to L \to 0$ *is a short exact sequence of triangulated categories with duality, there exist connecting homomorphisms* $\partial^n : \mathrm{W}^n(L) \to \mathrm{W}^{n+1}(J)$ *for all* $n \in \mathbb{Z}$ *and a long exact sequence*
$$\cdots \longrightarrow \mathrm{W}^{n-1}(L) \xrightarrow{\partial^{n-1}} \mathrm{W}^n(J) \longrightarrow \mathrm{W}^n(K) \longrightarrow \mathrm{W}^n(L) \xrightarrow{\partial^n} \mathrm{W}^{n+1}(J) \longrightarrow \cdots$$
which is natural with respect to morphisms of short exact sequences.

(3) *There is an isomorphism of functors* $\mathrm{W}^n \simeq \mathrm{W}^{n+4}$ *for all* $n \in \mathbb{Z}$*, making the above long exact sequence 12-periodic.*

1.2. Explanations and references. This theorem is the material of [TWG], where triangulated categories with duality and the functors W^n are defined, see [**1**, Section 2]. All the categories under consideration are supposed to be $\mathbb{Z}[\frac{1}{2}]$-categories, which means that all the Hom-groups are uniquely 2-divisible.

Part (1) is established in [**2**, Theorem 4.3] for exact categories in which split epimorphisms are admissible. Note that all the exact categories used in the present article satisfy this rather weak assumption. Moreover, Charles Walter found an elegant way to go from that case to the general case of any exact category, as explained in [**3**, after 1.4].

In part (2), a *short exact sequence of triangulated categories with duality*

$$0 \to J \to K \to L \to 0$$

refers to the situation where J is a saturated triangulated subcategory of K (see [**21**, II.2.1.6]) on which the duality restricts, and $L = K/J$, with the induced duality. Equivalently, L is a localization of K compatible with the duality and J is the "kernel" of this localization, that is the full subcategory of K on those objects which become isomorphic to zero in L. The long exact sequence is the main result of [**1**, Theorem 6.2] under the assumption that K is weakly cancellative ($A \oplus B \simeq B \Rightarrow A \simeq 0$). Again, the triangulated categories under consideration here do satisfy this hypothesis. Nevertheless, the author found a microscopic improvement of one of the arguments of [**1**] to remove this cancellation assumption and this is presented in [**3**, proof of 2.1]. Naturality of the long exact sequence is a triviality based on the very explicit definition of the connecting homomorphisms ∂^n, see [**1**, Section 5].

Part (3) is very easy in this framework; see [**1**, Proposition 2.14]. Actually and more precisely, if we consider Witt groups of symmetric and skew-symmetric forms, we have an isomorphism between W^{n+2} and the "skew-symmetric" W^n. See also [**2**, Explanation 5.1].

1.3. Notation. Let X be a scheme. We will always suppose that X contains $\frac{1}{2}$ which naturally means that 2 is everywhere invertible on X.

We denote by $\mathcal{E}(X)$ the exact category of locally free coherent $\mathcal{O}_X$-modules. We denote by $\mathrm{D^b}(X)$ the derived category of bounded complexes of $\mathcal{E}(X)$. This is endowed with a duality

$$(-)^* : \mathrm{D^b}(X) \to \mathrm{D^b}(X)$$

coming from the trivial one on $\mathcal{E}(X)$. In short, this duality is the total derived functor of $\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{O}_X)$. More details can be found in [TWG].

1.4. Definition. Throughout this article, we consider *pairs* (X, Z) where X is a scheme containing $\frac{1}{2}$, and where Z is a closed subset of X. A *morphism* of pairs $(X', Z') \to (X, Z)$ is a morphism $f : X' \to X$ such that $f^{-1}(Z) \subset Z'$. We will say that the pair (X, Z) is a *regular pair* if X is regular noetherian and separated.

1.5. Definition. Let (X, Z) be a non necessarily regular pair as in 1.4. We denote by $\mathrm{D}^{\mathrm{b}}_Z(X)$ the full subcategory of $\mathrm{D^b}(X)$ on those complexes whose homology is concentrated on Z, that is the "kernel" of the restriction functor $\Lambda : \mathrm{D^b}(X) \to \mathrm{D^b}(X - Z)$. The duality on $\mathrm{D^b}(X)$ gives a duality on $\mathrm{D}^{\mathrm{b}}_Z(X)$, as can be easily deduced from the fact that the restriction functor Λ commutes with the duality.

Let $n \in \mathbb{Z}$. We define the n^{th} *Witt group of X with supports in Z* to be the n^{th} Witt group of the triangulated category with duality $\mathrm{D}^{\mathrm{b}}_Z(X)$:

$$\mathrm{W}^n_Z(X) := \mathrm{W}^n\big(\mathrm{D}^{\mathrm{b}}_Z(X)\big).$$

1.6. Theorem. *Let X be a regular noetherian separated scheme containing $\frac{1}{2}$. Let $Z \subset Y$ be two closed subsets of X. Then there is a natural long exact sequence of Witt groups with supports :*

$$\cdots \to \mathrm{W}^{n-1}_{Y-Z}(X-Z) \longrightarrow \mathrm{W}^n_Z(X) \longrightarrow \mathrm{W}^n_Y(X) \longrightarrow \mathrm{W}^n_{Y-Z}(X-Z) \longrightarrow \mathrm{W}^{n+1}_Z(X) \to \cdots$$

Proof. By Theorem 1.1, part (2), everything amounts to prove that the following is an exact sequence of triangulated categories with duality :

$$0 \longrightarrow \mathrm{D}^{\mathrm{b}}_Z(X) \longrightarrow \mathrm{D}^{\mathrm{b}}_Y(X) \longrightarrow \mathrm{D}^{\mathrm{b}}_{Y-Z}(X-Z) \longrightarrow 0.$$

This is in turn an easy consequence of the following commutative diagram (think of the Snake Lemma !) :

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & & & \mathrm{D}^{\mathrm{b}}_{Y-Z}(X-Z) & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & \mathrm{D}^{\mathrm{b}}_Z(X) & \longrightarrow & \mathrm{D}^{\mathrm{b}}(X) & \longrightarrow & \mathrm{D}^{\mathrm{b}}(X-Z) & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & \mathrm{D}^{\mathrm{b}}_Y(X) & \longrightarrow & \mathrm{D}^{\mathrm{b}}(X) & \longrightarrow & \mathrm{D}^{\mathrm{b}}(X-Y) & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0. & &
\end{array}$$

The two lines and the third column are short exact sequences of triangulated categories with duality. In fact, for any separated noetherian regular scheme X and for any closed subset $Z \subset X$, the open subscheme $X - Z$ is still separated noetherian and regular and the following is a well-known exact sequence of triangulated categories with duality :

$$0 \to \mathrm{D}^{\mathrm{b}}_Z(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(X) \longrightarrow \mathrm{D}^{\mathrm{b}}(X-Z) \to 0.$$

This central result is easy to prove if we use the abelian category of coherent modules instead of locally free ones (see also the first paragraph of the proof of 2.3). Fortunately, those two categories have equivalent derived categories on regular noetherian separated schemes. □

1.7. Remark. In the terminology of [CT-H-K, Definition 5.1.1], Theorem 1.6 precisely says that our Witt groups form a *cohomology theory with supports*. In particular, putting $Y = X$ in the above, we get a long exact sequence

$$\cdots \to \mathrm{W}^{n-1}(X-Z) \longrightarrow \mathrm{W}^n_Z(X) \longrightarrow \mathrm{W}^n(X) \longrightarrow \mathrm{W}^n(X-Z) \longrightarrow \mathrm{W}^{n+1}_Z(X) \to \cdots$$

which shows the $\mathrm{W}^{\bullet}_Z(X)$ as *relative Witt groups* for the regular pair (X, Z).

It is natural to ask if those relative Witt groups can be expressed only in terms of Z, ideally as some Witt groups of Z equipped with the reduced scheme structure and some twisted-shifted duality. We won't give an answer to this very interesting question here, but we will see that even if $\mathrm{W}^n_Z(X)$ does not depend only on Z, it depends on X only in the neighborhood of Z. Here, "neighborhood" can be understood in the Zariski, étale or flat sense.

2. Excision and Mayer-Vietoris

2.1. NOTATIONS. Let X be a noetherian scheme and Z be a closed subset of X. We will denote by $\mathcal{A}(X)$ the abelian category of coherent $\mathcal{O}_X$-modules and by $\mathcal{A}_Z(X)$ the Serre subcategory of $\mathcal{A}(X)$ of those modules with support in Z.

2.2. Theorem. *Let $f : X' \to X$ be a flat morphism of noetherian schemes. Let Z be a closed subscheme of X and let $Z' = Z \times_X X'$. Suppose that f induces an isomorphism $Z' \xrightarrow{\sim} Z$. Then we have an equivalence of categories:*

$$f^* : \mathcal{A}_Z(X) \xrightarrow{\sim} \mathcal{A}_{Z'}(X').$$

PROOF. Consider the commutative diagram (with "exact lines") of abelian categories:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{A}_Z(X) & \longrightarrow & \mathcal{A}(X) & \longrightarrow & \mathcal{A}(X-Z) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle f^*} & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{A}_{Z'}(X') & \longrightarrow & \mathcal{A}(X') & \longrightarrow & \mathcal{A}(X'-Z') & \longrightarrow & 0. \end{array}$$

By Joyet [**8**, Corollary 10], the right-hand square is a cartesian diagram of abelian categories, *i.e.* a pull-back. It is then straigthforward to prove that the "kernels" are equivalent. □

2.3. Corollary (Flat Excision). *Let $f : X' \to X$ be a flat morphism of noetherian separated regular schemes containing $\frac{1}{2}$. Let $Z \subset X$ be a closed subscheme and let $Z' = Z \times_X X'$. Suppose that f induces an isomorphism $f : Z' \xrightarrow{\sim} Z$. Then we have an equivalence of categories:*

$$f^* : \mathrm{D}^{\mathrm{b}}_Z(X) \xrightarrow{\sim} \mathrm{D}^{\mathrm{b}}_{Z'}(X').$$

In particular, for all $n \in \mathbb{Z}$, we have a natural isomorphism:

$$\mathrm{W}^n_Z(X) \xrightarrow{\sim} \mathrm{W}^n_{Z'}(X').$$

PROOF. Given an abelian category $\mathcal{A}$ and a Serre subcategory $\mathcal{B}$, there is usually no equivalence between the derived bounded category of $\mathcal{B}$ and the full subcategory $\mathrm{D}^{\mathrm{b}}_{\mathcal{B}}(\mathcal{A})$ of $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$ whose objects are those complexes which have homology in $\mathcal{B}$. Nevertheless, this is true when $\mathcal{A} = \mathcal{A}(X)$ and $\mathcal{B} = \mathcal{A}_Z(X)$, by Keller [**10**, 1.15, condition (c1)] and the use of the Artin-Rees lemma as in Keller's Example (b), *loc. cit.*

It is now very easy to deduce this corollary from the above theorem. Of course, this uses the regularity of X in the end to have $\mathrm{D}^{\mathrm{b}}(X) \cong \mathrm{D}^{\mathrm{b}}\big(\mathcal{A}(X)\big)$. For the second part, There is nothing left to do by the very definition of $\mathrm{W}^n_Z(X)$ given in 1.5. □

2.4. REMARK. In particular, the above holds for f étale. In the words of the authors of [CT-H-K], this means that our cohomology theory with supports $\mathrm{W}^{\bullet}_{-}(-)$ satisfies the axiom of *étale excision* (COH 1) - the reader might prefer the terminology *Nisnevich excision.* Anyway, we obtain a Mayer-Vietoris long exact sequence by the usual argument. Observe that around $n = 0$ in the theorem below, we have usual Witt groups of schemes by Theorem 1.1, part (1). It is interesting that this very little corollary of the following theorem was not known for usual Witt groups.

2.5. Theorem (Mayer-Vietoris). *Let X be a noetherian separated and regular scheme which contains $\frac{1}{2}$. Let $X = U \cup V$ be an ordered open covering of X. Then there is a natural long exact sequence*

$$\cdots \to \mathrm{W}^{n-1}(U\cap V) \to \mathrm{W}^n(X) \to \mathrm{W}^n(U)\oplus \mathrm{W}^n(V) \to \mathrm{W}^n(U\cap V) \to \mathrm{W}^{n+1}(X) \to \cdots$$

Proof. Consider $Z := X - U$ with any closed subscheme structure. Then the inclusion $f : V \hookrightarrow X$ satisfies the assumptions of Corollary 2.3 and we have $Z' = Z$. Now compare the two long exact sequences (1.7) obtained for the regular pairs (X, Z) and (V, Z) and use excision and a simple chase to prove the result. □

3. Global homotopy invariance

3.1. Theorem. *Let R be a regular noetherian ring containing $\frac{1}{2}$. For all $n \in \mathbb{Z}$, the natural map $\mathrm{W}^n(R) \to \mathrm{W}^n(R[t])$ is an isomorphism.*

Proof. By Theorem 1.1 part (3), we only have four cases to do. The case $n \equiv 0$ mod 4 is Karoubi's Theorem [**9**, Corollary 3.10]. We also refer the reader to Ojanguren and Panin's simple proof of this result [**13**, Theorem 3.1]. As recalled in 1.2 above (see also [**2**, Explanation 5.1]), the group W^2 is nothing but a W^0 of skew-symmetric forms. Its homotopy invariance follows from the references given for the case $n = 0$. We outline a proof for $n = 1$ below, and leave the adaptations for $n = 3$ to the reader. As before, this amounts to "change skew-symmetry into symmetry", that is to replace the identification $\mathrm{Id} \xrightarrow{\sim} *^2$ by its opposite.

The proof requires two lemmas, in which we assume some minimal familiarity with [TWG] and the definitions given there. We start with a general lemma about W^1, not related directly to homotopy invariance. Recall that over affine schemes, the derived category $\mathrm{D^b}(\mathrm{Spec}(R))$ considered since 1.3 is just the homotopy category of bounded complexes of finitely generated projective R-modules $\mathrm{K^b}(\mathcal{P}(R))$. Of course, $\mathrm{W}^1(\mathcal{P})$ stands for $\mathrm{W}^1(\mathrm{K^b}(\mathcal{P}))$.

3.2. Lemma. *For any additive $\mathbb{Z}[\frac{1}{2}]$-category $\mathcal{P}$ with duality $(-)^*$, the following holds.*

(1) *Any element in $\mathrm{W}^1(\mathcal{P})$ is represented by a space (M, ϕ) of the form:*

$$\begin{array}{lccccccccccc}
M := & \cdots 0 & \longrightarrow & 0 & \longrightarrow & P & \xrightarrow{d} & Q & \longrightarrow & 0 & \longrightarrow & 0 \cdots \\
\phi\big\downarrow & & & \big\downarrow & & \varphi\big\downarrow & & \big\downarrow{-\varphi^*} & & \big\downarrow & & \\
M^{\#} = & \cdots 0 & \longrightarrow & 0 & \longrightarrow & Q^* & \xrightarrow[-d^*]{} & P^* & \longrightarrow & 0 & \longrightarrow & 0 \cdots
\end{array}$$

where ϕ is a homotopy equivalence and where Q sits in degree 0 in the complex M.

(2) *Any symmetric space as above is Witt-equivalent to the opposite of the following one:*

$$\begin{array}{lccccccccccc}
M' := & \cdots 0 & \longrightarrow & 0 & \longrightarrow & P & \xrightarrow{\varphi} & Q^* & \longrightarrow & 0 & \longrightarrow & 0 \cdots \\
\phi'\big\downarrow & & & \big\downarrow & & d\big\downarrow & & \big\downarrow{-d^*} & & \big\downarrow & & \\
(M')^{\#} = & \cdots 0 & \longrightarrow & 0 & \longrightarrow & Q & \xrightarrow[-\varphi^*]{} & P^* & \longrightarrow & 0 & \longrightarrow & 0 \cdots
\end{array}$$

obtained by permuting the form (φ) and the differential (d).

Proof. Part (1) is already in [**2**, Proposition 5.2] and holds for any exact category. More details on odd-indexed Witt groups and groups of formations will be available in [**22**].

For part (2), choose a symmetric homotopy inverse $\chi = (\dots,0,\psi,-\psi^*,0,\dots)$ of ϕ, for a morphism $\psi : Q^* \to P$ and choose a morphism $\epsilon : Q \to P$ such that $\psi\varphi + \epsilon\, d = 1_P$ and $\psi^*\varphi^* + d\,\epsilon = 1_Q$. Define a complex N and a morphism $h : N \to M \oplus M'$ as follows:

$$\begin{array}{ccccccccc}
N := & & \cdots 0 & \longrightarrow & P & \xrightarrow{\quad \varphi^* d \quad} & P^* & \longrightarrow & 0\cdots \\
h\big\downarrow & & & & \begin{pmatrix}\psi\varphi\\ \epsilon d\end{pmatrix}\big\downarrow & & \big\downarrow\begin{pmatrix}\psi^*\\ \epsilon^*\end{pmatrix} & & \\
M \oplus M' = & & \cdots 0 & \longrightarrow & P\oplus P & \xrightarrow{\begin{pmatrix}d & 0\\ 0 & \varphi\end{pmatrix}} & Q\oplus Q^* & \longrightarrow & 0\cdots
\end{array}$$

The reader may check that this is a homotopy equivalence with homotopy inverse being $(1\quad 1)$ in degree 1 and $(\varphi^*\quad d^*)$ in degree 0. Via this h, the form $\phi\perp\phi'$ induces a form $\omega := h^{\#}(\phi\perp\phi')\,h$ on N, which is easily proved to be neutral, *i.e.* $[N,\omega] = 0 \in \mathrm{W}^1(\mathcal{P})$. $\square$

The author first hoped for a categorical expression of $\mathrm{K}^{\mathrm{b}}(\mathcal{P}(R[t]))$ in terms of $\mathrm{K}^{\mathrm{b}}(\mathcal{P}(R))$ and for a proof of homotopy invariance on the level of triangulated categories. We don't know if it is possible or not. Nevertheless, we obtain part of this strategy as presented below.

Given any additive category $\mathcal{P}$, we can define another additive category $\mathcal{P}[t]$ as follows. We take for $\mathcal{P}[t]$ the same objects as those of $\mathcal{P}$, except that we write them $P[t]$ when they are in the new category and we define the morphisms by formal polynomials:

$$\mathrm{Hom}_{\mathcal{P}[t]}(P[t],Q[t]) := \left\{ \sum_{i=0}^{n} f_i t^i \;\middle|\; n\in\mathbb{N},\ f_i \in \mathrm{Hom}_{\mathcal{P}}(P,Q) \text{ for all } i = 0,\dots,n \right\}.$$

It is easy to check that this defines a category (with the obvious composition) and that it is additive. If $\mathcal{P}$ comes equipped with a duality $(-)^*$, we can extend it to $\mathcal{P}[t]$ by $(P[t])^* = P^*[t]$ and $(\sum f_i t^i)^* = \sum f_i^* t^i$.

Applying this to the category $\mathcal{P} = \mathcal{P}(R)$ of finitely generated projective R-modules, $\mathcal{P}[t]$ is the full subcategory of $\mathcal{P}(R[t])$ of those projective $R[t]$-modules which are extended from R. As soon as $\mathrm{K}_0(R) \to \mathrm{K}_0(R[t])$ is surjective, their derived categories are equivalent:

$$\mathrm{K}^{\mathrm{b}}\big(\mathcal{P}(R)[t]\big) \xrightarrow{\sim} \mathrm{K}^{\mathrm{b}}\big(\mathcal{P}(R[t])\big).$$

Therefore, our Theorem 3.1 amounts to prove that for any additive category with duality $\mathcal{P}$, we have an isomorphism $\mathrm{W}^1(\mathcal{P}) \xrightarrow{\sim} \mathrm{W}^1(\mathcal{P}[t])$. To prove this we need a last lemma.

3.3. Lemma. *Let $\mathcal{P}$ be an additive $\mathbb{Z}[\frac{1}{2}]$-category with duality. Consider an element of $\mathrm{W}^1(\mathcal{P}[t])$ of the form $[M,\phi]$ with:*

$$\begin{array}{ccccccccccccc}
M := & & \cdots 0 & \longrightarrow & 0 & \longrightarrow & P[t] & \xrightarrow{\ d\ } & Q[t] & \longrightarrow & 0 & \longrightarrow & 0\cdots \\
\phi\big\downarrow & & & & \big\downarrow & & \varphi\big\downarrow & & \big\downarrow -\varphi^* & & \big\downarrow & & \\
M^{\#} = & & \cdots 0 & \longrightarrow & 0 & \longrightarrow & Q^*[t] & \xrightarrow[-d^*]{} & P^*[t] & \longrightarrow & 0 & \longrightarrow & 0\cdots
\end{array}$$

with d of degree $n \ge 0$ and φ of degree $m \ge 1$. Assume that m is either n or $n+1$.

Then (M,ϕ) is isometric to a space (M',ϕ') of the same type (written with $'$ everywhere), such that $\deg(d') \le n$ *and* $\deg(\varphi') \le m-1$.

Proof. Case (1) : assume that $n \geq 1$. Write $d = d_0 + d_1\, t^n$ where $\deg(d_0) \leq n-1$ and d_1 is constant and $\varphi = \varphi_0 + \varphi_1\, t^m$ where $\deg(\varphi_0) \leq m-1$ and φ_1 is constant. Let $a = m - n$ which is 0 or 1 by hypothesis. Observe that $\varphi^* d = d^* \varphi$ implies $\varphi_1^* d_1 = d_1^* \varphi_1$.

We add to our symmetric space (M, ϕ) the following acyclic complex (*i.e.* in $\mathrm{K^b}(\mathcal{P}[t])$ we add a zero object) : $\cdots 0 \to P[t] \oplus P^*[t] \xrightarrow{\alpha} P^*[t] \oplus P[t] \to 0 \cdots$, where α is the hyperbolic isomorphism. We call the orthogonal sum of these spaces (N, ψ), with for instance the zero form on the trivial part. We use a homotopy $Q[t] \oplus P^*[t] \oplus P[t] \to Q^*[t] \oplus P[t] \oplus P^*[t]$ to modify the representant up to homotopy of the form ψ into the one presented below. The homotopy is easy to find because it is almost forced by the fact that α is an isomorphism.

$$\begin{array}{cccccc}
N := & \cdots 0 \longrightarrow & P[t]\oplus P[t]\oplus P^*[t] & \xrightarrow{\begin{pmatrix} d&0&0\\0&0&1\\0&1&0\end{pmatrix}} & Q[t]\oplus P^*[t]\oplus P[t] & \longrightarrow 0\cdots \\
\psi \Big\downarrow & & \begin{pmatrix}\varphi & -\varphi_1 t^a & 0\\ 0&0&0\\ -\varphi_1^* d\, t^a & \varphi_1^* d_1 t^a & 0\end{pmatrix}\Big\downarrow & & \Big\downarrow\begin{pmatrix}-\varphi^* & 0 & d^*\varphi_1 t^a\\ \varphi_1^* t^a & 0 & -d_1^*\varphi_1 t^a\\ 0&0&0\end{pmatrix} & \\
N^{\#} = & \cdots 0 \longrightarrow & Q^*[t]\oplus P[t]\oplus P^*[t] & \xrightarrow[\begin{pmatrix}-d^*&0&0\\0&0&-1\\0&-1&0\end{pmatrix}]{} & P^*[t]\oplus P^*[t]\oplus P[t] & \longrightarrow 0\cdots
\end{array}$$

We now use on N an automorphism, given on $P[t]\oplus P[t]\oplus P^*[t]$ by : $\begin{pmatrix}1&0&0\\ t^n&1&0\\ 0&0&1\end{pmatrix}$ and by Id on $Q[t]\oplus P^*[t]\oplus P[t]$. A direct verification shows that the new space we obtain satisfies the requirement of (1) when $n \geq 1$.

Case (2) : assume that $n = 0$, *i.e.* M is a constant complex ! This is now classical. Write

$$\phi = \phi(0)\cdot(1+\beta\, t)$$

where β is an endomorphism of M such that $(1+\beta\, t)$ is an automorphism. This forces β to be nilpotent and the usual trick of finding a square root to $(1+\beta\, t)$ gives the required isometry :

$$(\sqrt{1+\beta\, t}\,)^*\, \phi(0)\, \sqrt{1+\beta\, t} = \phi(0)\,(1+\beta\, t) = \phi.$$

We use the fact that $\sqrt{1+\beta\, t}$ is a polynomial in $\beta\, t$, with coefficients in $\mathbb{Z}[\frac{1}{2}]$. □

End of the proof of Theorem 3.1. The proof is now an easy induction on the degrees of the form and the differential of (M, ϕ) as in the above lemmas. Indeed, Lemma 3.3 allows us to reduce the degree of φ as long as $\deg(\varphi) \geq \deg(d)$ and Lemma 3.2 (2) allows us to switch φ and d when this inequality does not hold.

By this method we prove the surjectivity of $\mathrm{W}^1(\mathcal{P}) \to \mathrm{W}^1(\mathcal{P}[t])$. This homomorphism is also injective since it has an obvious section, given by $t \mapsto 0$. □

3.4. Theorem. *Let X be a noetherian separated regular scheme on which 2 is everywhere invertible. Then for any $n \in \mathbb{Z}$ the natural map:*

$$\mathrm{W}^n(X) \longrightarrow \mathrm{W}^n(\mathbb{A}^1_X)$$

is an isomorphism. This is in particular true for the usual Witt group: $\mathrm{W}(X) \cong \mathrm{W}(\mathbb{A}^1_X)$.

Proof. This is a simple induction on the number of noetherian affine open subschemes in some covering of X, using of course Theorems 2.5 and 3.1, and paying all due attention to the order of the coverings: when $X = U \cup V$ choose $\mathbb{A}^1_X = \mathbb{A}^1_U \cup \mathbb{A}^1_V$. To apply induction, we need the intersection of two affine open subschemes to be affine. This follows from X separated. The usual group is the $n = 0$ case by Theorem 1.1, part (1). □

3.5. Remark. For any regular pair (X, Z) as in 1.4 and for any $n \in \mathbb{Z}$ the natural map

$$\mathrm{W}^n_Z(X) \longrightarrow \mathrm{W}^n_{\mathbb{A}^1_Z}(\mathbb{A}^1_X)$$

is an isomorphism, where $\mathbb{A}^1_Z$ is the closed subset of $\mathbb{A}^1_X$ above Z. To see this, it suffices to compare the two long exact sequences (1.7) for the regular pairs (X, Z) and $(\mathbb{A}^1_X, \mathbb{A}^1_Z)$ and apply Theorem 3.4 to X and $X - Z$. In the formalism of [CT-H-K], this means that our cohomology theory $\mathrm{W}^\bullet_-(-)$ satisfies the axiom (COH 3).

4. The Gersten Conjecture

The Gersten Conjecture claims the existence and the exactness of a *Gersten-Witt complex*, *i.e.* a complex of the following form:

$$0 \to \mathrm{W}(X) \longrightarrow \bigoplus_{x \in X^{(0)}} \mathrm{W}(\kappa(x)) \longrightarrow \bigoplus_{x \in X^{(1)}} \mathrm{W}(\kappa(x)) \longrightarrow \cdots \longrightarrow \bigoplus_{x \in X^{(n)}} \mathrm{W}(\kappa(x)) \to 0$$

where $X = \mathrm{Spec}(R)$ is the spectrum of a regular local ring of dimension n. Of course, in that case, there is only one point in $X^{(0)}$ and only one point in $X^{(n)}$, but we write the above complex in its general form since it exists not only for regular local rings.

The first major difficulty, in contrast to what happens with K-theory, is precisely *to build such a complex.* A well-known classical construction is the second residue homomorphism:

$$\bigoplus_{x \in X^{(0)}} \mathrm{W}(\kappa(x)) \longrightarrow \bigoplus_{x \in X^{(1)}} \mathrm{W}(\kappa(x))$$

and one expects the complex to start with this homomorphism. Note that this first differential is not well defined and depends on a choice of local parameters. This will also be the case for the higher differentials. Nevertheless, the exactness of the complex will not depend on those choices. The most general Gersten-Witt complex is established in [**3**], basically for all regular schemes. For the convenience of the reader, we recall here how it is constructed.

4.1. NOTATION. Let X be a regular noetherian and separated scheme containing $\frac{1}{2}$. We will denote by

$$\mathrm{D}^{(p)}(X) := \bigcup_{\substack{Z\subset X\\ \mathrm{codim}(Z)\geq p}} \mathrm{D}^{\mathrm{b}}_{Z}(X)$$

the full saturated triangulated subcategory with duality of $\mathrm{D}^{\mathrm{b}}(X)$ on those complexes such that the support of their homology is of codimension $\geq p$. In short, the above!

4.2. CONSTRUCTION OF GERSTEN-WITT COMPLEXES. This part is a quick overview of some technical results of [**3**]. The proofs are to be found there.

Let X be as in 4.1 and assume moreover that it is of Krull dimension n. Then we have a finite filtration:

$$0 = \mathrm{D}^{(n+1)}(X) \subset \cdots \subset \mathrm{D}^{(1)}(X) \subset \mathrm{D}^{(0)}(X) = \mathrm{D}^{\mathrm{b}}(X).$$

For simplicity, we write $\mathrm{D}^{(i)}(X) = 0$ for $i \geq n+1$ and $\mathrm{D}^{(j)}(X) = \mathrm{D}^{\mathrm{b}}(X)$ for $j \leq 0$.

This filtration is the triangular analogue of the classical K-theoretic filtration on the level of abelian categories, which is used to produce the classical Gersten complex. For each integer p, we have a short exact sequence of triangulated categories with duality:

$$0 \to \mathrm{D}^{(p+1)} \longrightarrow \mathrm{D}^{(p)} \longrightarrow \mathrm{D}^{(p)}/\mathrm{D}^{(p+1)} \to 0$$

in which we suppressed the mention of X for readibility. By Theorem 1.1, we have a long exact sequence of Witt groups:

$$\cdots \longrightarrow \mathrm{W}^{i}(\mathrm{D}^{(p+1)}) \longrightarrow \mathrm{W}^{i}(\mathrm{D}^{(p)}) \longrightarrow \mathrm{W}^{i}(\mathrm{D}^{(p)}/\mathrm{D}^{(p+1)}) \xrightarrow{\partial} \mathrm{W}^{i+1}(\mathrm{D}^{(p+1)}) \longrightarrow \cdots$$

We use these connecting homomorphisms ∂ (which are natural and explicitly presented in [TWG I, Section 5]) to produce the following complex

$$(\mathrm{GW}_1) \qquad \cdots \to \mathrm{W}^{p}(\mathrm{D}^{(p)}/\mathrm{D}^{(p+1)}) \xrightarrow{d^p} \mathrm{W}^{p+1}(\mathrm{D}^{(p+1)}/\mathrm{D}^{(p+2)}) \to \cdots$$

where the map d^p is defined as the homomorphism $\partial : \mathrm{W}^{p}(\mathrm{D}^{(p)}/\mathrm{D}^{(p+1)}) \to \mathrm{W}^{p+1}(\mathrm{D}^{(p+1)})$ followed by the canonical homomorphism $\mathrm{W}^{p+1}(\mathrm{D}^{(p+1)}) \to \mathrm{W}^{p+1}(\mathrm{D}^{(p+1)}/\mathrm{D}^{(p+2)})$. The fact that (GW_1) is really a complex is a triviality. Now, there is a natural isomorphism:

$$\mathrm{W}^{p}(\mathrm{D}^{(p)}/\mathrm{D}^{(p+1)}) \cong \bigoplus_{x\,\in\, X^{(p)}} \mathrm{W}(\mathcal{O}_x\text{-fl})$$

where $\mathrm{W}(\mathcal{O}_x\text{-fl})$ is the Witt group of finite length $\mathcal{O}_x$-modules with a natural duality. This is again the transposition of the similar equivalence on the level of abelian categories which appears in the construction of the K-theoretic Gersten complex. Using these identifications, we obtain a natural complex:

$$(\mathrm{GW}_2) \qquad \cdots \to \bigoplus_{x\,\in\, X^{(p)}} \mathrm{W}(\mathcal{O}_x\text{-fl}) \longrightarrow \bigoplus_{x\,\in\, X^{(p+1)}} \mathrm{W}(\mathcal{O}_x\text{-fl}) \to \cdots$$

The choices announced at the beginning of the Section appear at this stage. That is, each choice of local parameters in the regular local rings $\mathcal{O}_x$ gives an isomorphism $\mathrm{W}(\kappa(x)) \simeq \mathrm{W}(\mathcal{O}_x\text{-fl})$. In K-theory, there is such an isomorphism, but it does not depend on the choices of parameters (here, the choices actually affect the compatibility of the dualities). Pushing the complex (GW_2) along those last isomorphisms, and introducing $\mathrm{W}(X) = \mathrm{W}^0(D^{(0)})$ at the left end, we obtain the announced Gersten-Witt complex:

$$(\mathrm{GW}) \qquad 0 \to \mathrm{W}(X) \to \bigoplus_{x \in X^{(0)}} \mathrm{W}(\kappa(x)) \to \bigoplus_{x \in X^{(1)}} \mathrm{W}(\kappa(x)) \to \cdots \to \bigoplus_{x \in X^{(n)}} \mathrm{W}(\kappa(x)) \to 0$$

Let us repeat that the maps are well-defined but depend upon choices of local parameters in the local rings. Nevertheless, the exactness of (GW) does not depend on those choices, since all these complexes are isomorphic: they are all isomorphic to the natural (GW_1), by construction. It was also established in [**3**] that the classical second residue homomorphism appears where we expect it to appear, that is between $X^{(0)}$ and $X^{(1)}$.

The author has no idea of a simpler description of these homomorphisms, involving the choices. In particular, it is not known if these homomorphisms coincide with Schmid's homomorphisms in the special case where the latter are defined.

4.3. Theorem. *The Gersten conjecture for Witt groups holds for semi-local regular rings of geometric type over an infinite field of characteristic different from 2. In other words, the complex (GW) defined in 4.2 is exact when $X = \mathrm{Spec}(R)$ is the spectrum of such a ring.*

PROOF. The proof occupies the end of the Section. We apply the general strategy of [CT-H-K]. The main point is the following result, where all the subtlety of their "Geometric Presentation Theorem" (3.1.1 *loc. cit.*) is encoded.

4.4. Theorem. *Let k be an infinite field of characteristic different from 2. The Witt cohomology theory with supports $\mathrm{W}^{\bullet}_Z(X)$, considered for regular pairs (X,Z) with X smooth over k, is "stricly effaceable". This means the following: For any smooth scheme X over k and any finite set of points $t_1, \ldots, t_r \in X$, for any $p \geq 0$, for any neighborhood $V \subset X$ of $t_1, \ldots, t_r$ and for any closed subset $Z \subset V$ of codimension $\geq p+1$, there exists an open neighborhood $U \subset V$ of $t_1, \ldots, t_r$ and a closed subset $Z' \subset V$ such that $\mathrm{codim}_V(Z') \geq p$ and such that the map*

$$\mathrm{W}^q_{Z \cap U}(U) \longrightarrow \mathrm{W}^q_{Z' \cap U}(U)$$

is zero for any $q \in \mathbb{Z}$.

PROOF. We want to apply Theorem 5.1.10 of [CT-H-K]. For this we need our cohomology theory to satisfy étale excision (COH 1) and what Colliot-Thélène, Hoobler and Kahn call "the key lemma for cohomology" or (COH 2). We do not re-state this axiom here because it suffices to know that homotopy invariance (COH 3) implies this "key lemma", as proved in Proposition 5.3.2 *loc. cit.* The result follows by Remarks 2.4 and 3.5 above. ☐

4.5. Corollary. *Let X be a smooth scheme over an infinite field k, let $t_1 \ldots, t_r \in X$ be a finite number of points of X and let $Y = \mathrm{Spec}(\mathcal{O}_{X,(t_1 \ldots, t_r)})$ where $\mathcal{O}_{X,(t_1 \ldots, t_r)}$ is the semi-local ring of X at $(t_1 \ldots, t_r)$. Then the following natural homomorphism*

$$\mathrm{W}^i\big(\mathrm{D}^{(p+1)}(Y)\big) \longrightarrow \mathrm{W}^i\big(\mathrm{D}^{(p)}(Y)\big)$$

is equal to zero for any $i \in \mathbb{Z}$.

Proof. This is a formal consequence of the Effacement Theorem 4.2 and the proof goes exactly like in the one of Proposition 2.1.2 of [CT-H-K], *mutatis mutandis*. We use in particular the fact that

$$\mathrm{W}^i\big(\mathrm{D}^{(p)}(X)\big) = \lim_{\substack{Z\subset X \\ \mathrm{codim}(Z)\geq p}} \mathrm{W}^i\big(\mathrm{D}^{\mathrm{b}}_Z(X)\big)$$

which is easily verified, using the definition of $\mathrm{D}^{(p)}(X)$ and the very explicit definitions of the Witt groups; in other words, any symmetric space over $\mathrm{D}^{(p)}(X)$ is already defined over some $\mathrm{D}^{\mathrm{b}}_Z(X)$ and any such space which is metabolic in $\mathrm{D}^{(p)}(X)$ is already metabolic in some $\mathrm{D}^{\mathrm{b}}_Z(X)$. For very similar reasons, the natural map

$$\lim_{\substack{V\subset X \text{ open} \\ t_1,\dots,t_r\in V}} \mathrm{W}^i\big(\mathrm{D}^{(p)}(V)\big) \longrightarrow \mathrm{W}^i\big(\mathrm{D}^{(p)}(Y)\big)$$

is an isomorphism for any $i \in \mathbb{Z}$. Again, this would be true "without Witt groups" on both sides, if we had developped a suitable formalism of direct limit of triangulated categories. We do not want to use this sophistication and leave the details to the reader. □

4.6. End of the proof of Theorem 4.3. The end goes as usual. For our semi-local Y, we obtain long exact sequences associated to the short exact sequences of triangulated categories $0 \to \mathrm{D}^{(p+1)} \to \mathrm{D}^{(p)} \to \mathrm{D}^{(p)}/\mathrm{D}^{(p+1)} \to 0$, as in 4.2. Because of Corollary 4.4, those long exact sequences break up into short exact sequences:

$$0 \to \mathrm{W}^i(\mathrm{D}^{(p)}) \longrightarrow \mathrm{W}^i\big(\mathrm{D}^{(p)}/\mathrm{D}^{(p+1)}\big) \xrightarrow{\partial} \mathrm{W}^{i+1}(\mathrm{D}^{(p+1)}) \to 0$$

for all $i \in \mathbb{Z}$ and all $0 \leq p \leq n = \dim(Y)$. Putting together the above short exact sequences (for $i = p$) and repeating the computations presented in 4.2, we obtain the announced exact sequence:

$$0 \to \mathrm{W}(Y) \to \bigoplus_{x\in Y^{(0)}} \mathrm{W}(\kappa(x)) \to \bigoplus_{x\in Y^{(1)}} \mathrm{W}(\kappa(x)) \to \cdots \to \bigoplus_{x\in Y^{(n)}} \mathrm{W}(\kappa(x)) \to 0.$$

Observe that the proof establishes really the exactness of the natural Gersten-Witt complex (GW_1) presented in 4.2. □

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6. References

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Paul Balmer, SFB 478, Universität Münster, Hittorfstr. 27, 48149 Münster, Germany
E-mail address: `balmer@math.uni-muenster.de`
URL: `www.math.uni-muenster.de/u/balmer`

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