Recitation/Discussion 0

Sets, Geometric Series, and Integration

accompaniment to MATH 170E: Introduction to Probability and Statistics 1: Probability
at the University of California, Los Angeles

Allen Lin

In this recitation, we will introduce the notion of sets and events. Then we will review geometric series and integration from calculus, as they are useful tools for the rest of the class.

Logistics

Recitations will primarily be reviewing lecture material in more detail and working through exercises, as opposed to teaching supplemental material. This first recitation is an exception since the first lecture is tomorrow. I highly recommend that you attend if you find yourself struggling through the material. Each recitation will conclude with an Exit survey so I can get feedback from you on concepts and check understanding.

Recitation notes will be published shortly before each recitation; they may differ slightly from the content of your recitation due to timing.

Pre-class Survey

If you haven't already, please fill out the pre-class survey.

This is so I get to know you!

Exit Survey

Please fill out the exit survey.

This is so I can get feedback and know what to focus on next time!

Administrivia

My office hours are Tuesdays and Thursdays 3:00 pm–4:00 pm in my office, Mathematical Sciences 2905. Stop by if you have any questions or if you want to say hi.

I think the least confusing directions are: Enter on the fourth floor, take the westmost elevator down to the second floor, and my office should be a few steps to the right of the elevator.

In case you haven't seen already, you have two midterms and a final exam in this class.

Midterm I	26 Oct 2026	
Midterm II	23 Nov 2026	
Final Exam	10 Dec 2026	6:30 pm–9:30 pm

Motivation and Applications

Why study probability? It's cool because you can

1. Predict outcomes.

   You can quantify how likely an event is to happen (such as winning the lottery) and quantify what you expect to happen.

2. It's a powerful problem-solving tool.

   Probability occurs a lot in randomized algorithms. These are algorithms that incorporate a source of randomness (like a coin flip), and as a result you have a tradeoff between runtime and correctness.

Randomized algorithms that have guaranteed correctness and usually quick runtime are called *Las Vegas algorithms*, while those that have guaranteed fast runtime and usually correct results are called *Monte Carlo algorithms*.

3. Model the real world.

   This is a classic example that I love. Suppose that you have a lot of cities, some of them connected by roads. You can use the roads to travel from any city to another. Then an earthquake comes, damaging each road and rendering it unusable with probability $p \in [0, 1]$.

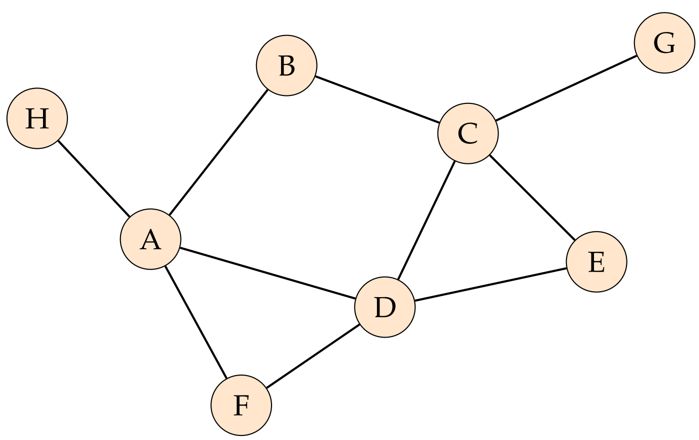

Figure 1: Graph of a network of cities and roads.

   Depending on the strength of the earthquake modeled by p (the bigger p is, the stronger the earthquake), what is the probability that the network is still connected (i.e., you can still travel from any city to any other city)?

   Well, if $p = 0$, then all the roads remain usable, so the probability that the network is connected is 1. And if $p = 1$, then all the roads are destroyed, so the probability that the network is connected is 0. What happens in the middle when p is between 0 and 1? It turns out that there is a sharp dichotomy between "always connected" and "never connected."

It turns out that this narrow window transitioning between "always connected" and "never connected" has length $O(1/\log n)$, where n is the number of cities. This sharp threshold phenomena appears frequently, not just for connectness!

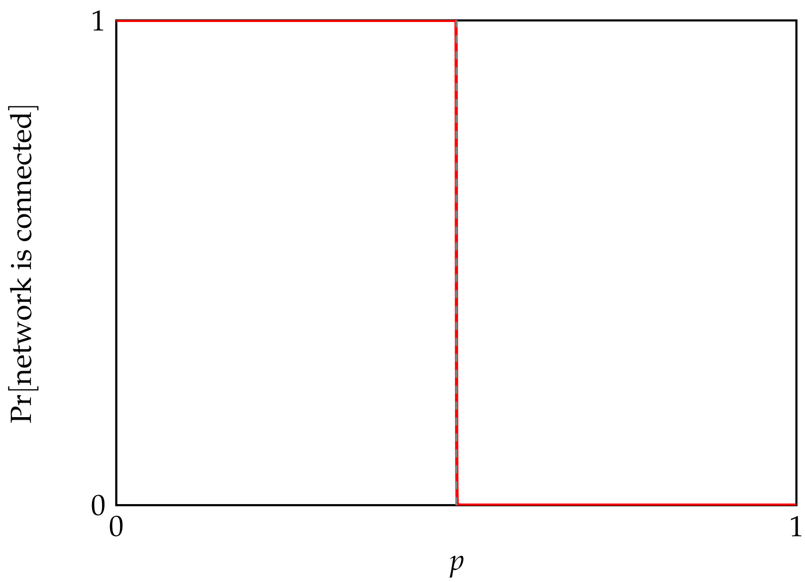

Figure 2: Sharp threshold for disconnectedness.

You don't need to know any of the above examples, but I thought I'd include them to show you how useful, prevalent, and cool probability is.

If you are curious and want to learn more, let me know!

Set Theory

Definition 1. A *set* is a collection of objects. The objects in a set A are *elements* of A.

For example, $\{0,1\}$ is a set. So is $\left\{5, \pi, \sqrt{2}\right\}$ and $\mathbb{N} = \{1,2,3,\ldots\}$. We use the symbol $\in$ to denote "is an element of" and the symbol $\notin$ to denote "is not an element of." Thus, $1 \in \{0,1\}$ means "1 is an element of $\{0,1\}$." Similarly, $\pi \in \left\{5, \pi, \sqrt{2}\right\}$, and $p \in \mathbb{N}$ for every prime number p. On the other hand, $2 \notin \{0,1\}$, $\sqrt{3} \notin \left\{5, \pi, \sqrt{2}\right\}$, and $\pi \notin \mathbb{N}$.

Some sets like $\mathbb{N}$ can be infinite in size.

Definition 2. We say that a set A is a *subset* of B, denoted by $A \subseteq B$, if every element of A is an element of B.

Example 1. $\{0,1\} \subseteq \{0,1,2\}$, and more generally $\{0,1\} \subseteq \mathbb{N}$.

In this class, we have a special kind of set.

Definition 3. The *universal set* U is the set that contains all the objects under consideration.

Example 2. On a six-sided die, the universal set is the set of all possibilities, so $U = \{1,2,3,4,5,6\}$.

In the context of probability, U usually denotes a sample space (i.e., the space of all possibilities) and a set A is an event if it is a subset of S.

Example 3. Let A be the event of rolling an even number on a six-sided die. Then $A = \{2, 4, 6\}$ and $A \subseteq U$.

The opposite of the universal set is the empty set.

Definition 4. The *empty set* $\varnothing$ is the set that contains nothing.

Example 4. $\varnothing \subseteq A$ for every set A.

We can perform operations on sets.

Definition 5. The *union* $A \cup B$ of A and B is the set that satisfies the following property: $x \in A \cup B$ if and only if $x \in A$ or $x \in B$.

You can actually take unions of more than two sets. What is the generalization?

Definition 6. The *intersection* $A \cap B$ of A and B is the set that satisfies the following property: $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.

You can also take intersections of more than two sets. What is the generalization?

Exercise 1. If $A = \{2, 4, 6\}$ (the event of rolling an even number) and $B = \{3, 4\}$ (the event of rolling a 3 or a 4), then what is $A \cup B$ and $A \cap B$?

Definition 7. The *complement* $U \setminus A$ of A is the set that satisfies the following property: $x \in U \setminus A$ if and only if $x \notin A$.

Exercise 2. What is $U \setminus A$ and $U \setminus B$ in the above example?

Definition 8. Two sets A and B are *disjoint* if $A \cap B = \varnothing$, and we say the events A and B are *mutually exclusive*.

Exercise 3. Find a set C such that A and C are disjoint. Do the same for B.

Geometric Series

Recall that a *geometric series* is a series of the form

$$\sum_{k=0}^{\infty} ar^k,$$

for some initial value $a \in \mathbb{R}$, and this is only defined when $-1 < r < 1$.

Exercise 4. Prove that when $-1 < r < 1$ the geometric series converges to

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.$$

When $|r| \geq 1$, the series diverges.

Exercise 5. Find the value of

$$\sum_{k=0}^{\infty} (0.5)^k$$

by using the formula and by proving it directly.

If you forget the formula it's very handy to remember the trick of the proof.

Exercise 6. Find the value of

$$\sum_{k=0}^{\infty} 2 \cdot 3^k.$$

Exercise 7. Find the value of

$$\sum_{k=0}^{\infty} 2 \cdot (-1/3)^k.$$

Exercise 8. Generalize the geometric series: When $|r| < 1$, compute

$$\sum_{k=n}^{\infty} ar^k.$$

Hint. Apply the same trick in the original proof!

Integration

Here are a few exercises on double integration.

Exercise 9. Let $A = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; 0 \leq x \leq 1 \text{ and } 0 \leq y \leq x\right\}$. Compute

$$\iint_A \left(x + x^3 y^2\right)\, dx\, dy.$$

Exercise 10. Let $A = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\right\}$. Compute

$$\iint_A \frac{1}{2\pi} e^{-\left(x^2+y^2\right)}\, dx\, dy.$$

Hint. Consider polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ and the Jacobian.