

Selected solutions and final answers. Below are full solutions and some final answers for selected problems. Things marked “final answer” give just that—the final answer. But, keep in mind that you are expected to show your work and justify your answers. Also keep in mind: it is always better for your learning to go through the entire problem-solving process than to just read someone else’s solution.

Directions. Complete all the problems. Your solutions to the problems should be submitted to Gradescope before the indicated due date above. Please follow rules regarding Gradescope submission as described in the syllabus. If you use any outside source in your solutions, you must cite the source.

Problem 1. Let $n \geq 3$.

- (a) n people are to be randomly seated in a row of n chairs, with each arrangement equally likely. Among the n people are two friends, Gary and Ash. Find the probability that Gary and Ash will be seated together.

Solution. The answer is $2/n$. We show two ways of solving this.

Method 1: law of total probability. Let N be the event that Gary and Ash are seated next to each other. Let M be the event that Gary is seated in a middle seat, so M' is the event that Gary is seated in an edge seat. If M happens, then there are 2 seats next to Gary, hence $P(N|M) = 2/(n-1)$. On the other hand, if M' happens there is only one seat next to Gary. Thus, $P(N|M') = 1/(n-1)$. By the Law of Total Probability,

$$P(N) = P(N|M)P(M) + P(N|M')P(M') = \frac{2}{(n-1)} \frac{(n-2)}{n} + \frac{1}{(n-1)} \frac{2}{n} = \frac{2}{n}.$$

Method 2: counting. Since all arrangements are equally likely, we can just compute the probability by counting the number of arrangements in which Gary and Ash sit next to each other and divide by the total number of arrangements (which is $n!$). There are $(n-1)$ pairs of chairs next to each other. Once we have picked a pair, there are 2 ways to arrange Gary and Ash in those chairs. Finally, there are $(n-2)!$ ways

to arrangement the other people in the remaining chairs. Thus, the probability Gary and Ash sit next to each other is

$$\frac{2(n-1)(n-2)!}{n!} = \frac{2}{n}.$$

- (b) Now suppose the n people will be seated in a *circle* of n chairs, with each arrangement equally likely. In this scenario, what is the probability that Gary and Ash will be seated next to each other?

Final answer. $2/(n-1)$

Problem 2. Consider an experiment where we roll two fair six-sided dice. One die is red, and the other is blue. Define the following events:

A = the sum of the rolls is 10

B = the red die lands 5

C = the blue die lands 5.

Compute the following conditional probabilities:

$$P(A|B), \quad P(B|A), \quad P(A|B \cup C), \quad P(B \cup C|A).$$

Final answer. $P(A|B) = 1/6$, $P(B|A) = 1/3$, $P(A|B \cup C) = 1/11$,
 $P(B \cup C|A) = 1/3$

Problem 3. We randomly cut a window into a wall. The sample space is

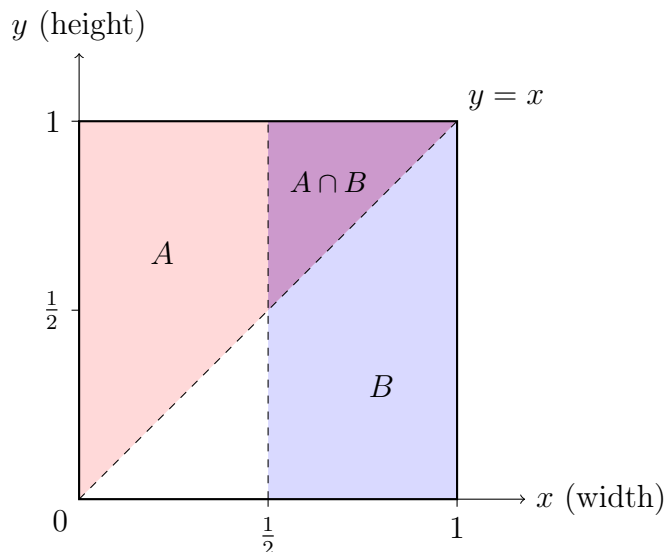
$$S = \{(x, y) \in \mathbb{R}^2 : 0 \leq x, y \leq 1\},$$

where x represents the width of the window and y represents the height, both measured in feet. The probability function is the (continuous) uniform law given by $P(A) = \text{area of } A$.

- (a) What is the conditional probability that the window is taller than it is wide given that its width is at least $1/2$ ft?

Solution. Let A be the event that the window is taller than it is wide. Let B be the event that the width of the window is at least $1/2$. We want $P(A|B)$.

The event B is a rectangle in S with area $1/2$, so $P(B) = 1/2$. The event $A \cap B$ is a right triangle with height and base both equal to $1/2$.



Thus, using $\text{area} = 1/2(\text{base})(\text{height})$, we have $P(A \cap B) = 1/2^3 = 1/8$.
Thus,

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/8}{1/2} = 1/4.$$

- (b) Let A be the event that the window is taller than it is wide. Let B be the event that the perimeter of the window is less than 2 feet. Are A and B independent events? Justify your answer with computations of relevant probabilities.

Final answer. Yes, they are independent. One could show this by establishing that $P(A \cap B) = P(A) \cdot P(B)$.

Problem 4. You have a box with 12 red chips and 14 blue chips. You draw 10 chips from the box one at a time, noting the colors of the chips. Compute the probability that the last 9 draws are all red under the assumption that ...

- (a) ... the chips are drawn with replacement.

Final answer. $\left(\frac{12}{26}\right)^9$

- (b) ... the chips are drawn without replacement.

Final answer. $\frac{12 \cdot 11 \cdots 3}{26 \cdot 25 \cdots 17} + \frac{14 \cdot 12 \cdot 11 \cdots 4}{26 \cdot 25 \cdots 17}$

Problem 5. The Dodgers will face a team in a series of 4 games. Which team they face in the series will be chosen uniformly at random from the Padres, the Giants, and the Rockies. Assume that, given which team is selected, the outcomes of games are mutually independent. Moreover, assume that in any given game, the Dodgers have a 0.6 probability of beating the Padres, 0.7 probability of beating the Giants, and 0.8 probability of beating the Rockies.

Given that the Dodgers won 2 of the 4 games, what is the probability that the selected team was the Padres?

Final answer. $\frac{(.6)^2(.4)^2}{(.6)^2(.4)^2 + (.7)^2(.3)^2 + (.8)^2(.2)^2}$

Recommended practice. The following exercises from the textbook are not assigned and do not have to be turned in. They are recommended to get more practice.

Section 1.3: 1, 8, 11, 16

Section 1.4: 2, 8, 13, 17

Section 1.5: 6, 7, 11

Most of the textbook exercises are very good, so you can always get more practice by working on exercises not mentioned in the above list.