An Introduction to *HT*-fields

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Hardy fields

- Given functions $f, g : \mathbb{R} \to \mathbb{R}$, say $f \sim g$ if there is $a \in \mathbb{R}$ with $f \restriction_{(a,\infty)} = g \restriction_{(a,\infty)}$ (this is an equivalence relation).
- Let $\mathcal{G}$ be the collection of equivalence classes of these functions with respect to $\sim$ (germs at infinity).
- A *Hardy field* is a subfield $\mathcal{H}$ of $\mathcal{G}$ which is closed under differentiation.
- If $[f]_\sim \in \mathcal{H}$, then f is eventually positive, 0, or negative. This gives us a total order on $\mathcal{H}$.
- $\mathbb{R}(x)$ and $\mathbb{R}(x, \exp x, \log x)$ are examples of Hardy Fields.
- More generally, given an o-minimal expansion of $(\mathbb{R}, +, \cdot, <)$, the germs at infinity of definable unary functions form a Hardy field.

Valued fields

A *valuation* on a field K is given by an ordered group Γ and a surjective group homomorphism $v : K^\times \to \Gamma$ where $v(x+y) \geq \min\{v(x), v(y)\}$ when $x + y \neq 0$.

- We can extend v to all of K by setting $v(0) = \infty$.
- We set $\mathcal{O} := \{x \in K : v(x) \geq 0\}$ and $\mathcal{o} := \{x \in K : v(x) > 0\}$.
- The field $\operatorname{res}(K) := \mathcal{O}/\mathcal{o}$ is the *residue field*.
- We get a dominance relation $\preccurlyeq$ on K given by $x \preccurlyeq y$ if $v(x) \geq v(y)$. We define $\prec$, $\asymp$ analogously.
- An extension $L \supseteq K$ of valued fields is called *immediate* if $\Gamma_L = \Gamma_K$ and $\operatorname{res}(L) = \operatorname{res}(K)$.

Examples of valued fields

- $\mathbb{Q}$ with the p-adic valuation (value group is $\mathbb{Z}$, this gives rise to $\mathbb{Q}_p$).
- $\mathbb{R}((t))$ with $v\left(\sum_{i=k}^{\infty} r_i t^i\right) = k$.
- For an ordered field K, set $x \prec y$ if $n|x| < |y|$ for all $n \in \mathbb{N}$.
- For a Hardy field $\mathcal{H}$, set $f \preccurlyeq g$ if there is $r \in \mathbb{R}$ with $|f/g| < r$ eventually.

Valued differential fields

A *derivation* on a field K of characteristic 0 is an additive map $\partial : K \to K$ with $\partial(xy) = x\partial(y) + y\partial(x)$. Call $\ker(\partial)$ the *constant field* of K. We will denote this by C.

- We will often denote $\partial(x)$ by x' and $\frac{\partial(x)}{x}$ by $x^\dagger$.
- If K differential field and a valued field with $\mathbb{Q} \subseteq \mathcal{O}$, then we say K is a *valued differential field*.
- $\mathbb{R}((t))$ is a valued differential field with $\partial\left(\sum_{i=k}^{\infty} r_i t^i\right) = \sum_{i=k}^{\infty} i r_i t^{i-1}$.
- Any Hardy field is a valued differential field with $C \subseteq \mathbb{R}$ given by eventually constant functions.

Transseries

The field $\mathbb{T}$ of logarithmic-exponential transseries are transfinite series with reverse well-ordered support consisting of elements like

$$e^{e^{e^x}} + 12xe^{x^2+\pi x} + x^{1/2} + x^{1/3} + \ldots + \log(\log(x)) + e^{-x} + e^{-x^2} + e^{-x^3} + \ldots$$

- The terms in the sum are the product of a real number and a *transmonomial*.
- There is a natural order on transmonomials. This gives rise to a valuation by comparing leading transmonomials.
- $\mathbb{T}$ is also an ordered field (look at the leading coefficient) and has a natural derivation which commutes with the infinite sums.
- $\mathbb{T}$ is closed under exponentiation, integration, composition, compositional inversion, and the resolution of some algebraic differential equations.

H-fields

An H-field is an ordered differential field K such that:

(i) For all $f \in K$, if $f > C$ then $f' > 0$;

(ii) $\mathcal{O} = C + \mathcal{o}$ where $\mathcal{O} = \{g \in K : |g| \leq c \textit{ for some } c \in C\}$ and $\mathcal{o}$ is the maximal ideal of $\mathcal{O}$.

- We always construe H-fields as ordered *valued* differential fields with valuation ring $\mathcal{O}$.
- Any Hardy field containing $\mathbb{R}$ is an H-field.
- $\mathbb{T}$ is an H-field.

Subfields of H-fields

A differential subfield K of an H-field L isn't necessarily an H-field, but it is an ordered valued differential field with the following properties:

(i) For all $f, g \in K^{\times}$, if $f \preccurlyeq 1$ and $g \prec 1$ then $f' \prec g^{\dagger}$;

(ii) $\mathcal{O}$ is convex with respect to the ordering on K;

(iii) For all $f \in K$, if $f > \mathcal{O}$ then $f' > 0$.

Call any ordered valued differential field satisfying these three conditions a *pre-H-field*.

Every pre-H-field has an *H-field hull*. Any Hardy field is a pre-H-field.

Model theory of H-fields

The theory of H-fields has a model companion! This is the theory T^{nl} of ω-free newtonian Liouville closed H-fields (Aschenbrenner, van den Dries, van der Hoeven).

- T^{nl} has two completions: one with small derivation ($\partial\mathcal{O} \subseteq \mathcal{O}$) and one with large derivation. $\mathbb{T}$ is a model of $T^{\mathrm{nl}}_{\mathrm{small}}$.
- Liouville closed means real-closed and for each $f \in K$ there is $g, h \in K$, $h \neq 0$ with $g' = f$, $h^{\dagger} = f$.
- ω-free is a technical condition (some pc-sequence doesn't have a pseudo-limit in K).
- Newtonian is sort of like differential-henselian. Under the assumption of ω-free, it implies that K has no immediate differential-algebraic H-field extensions.
- There is a quantifier elimination result for T^{nl} in an extended language.

O-minimality

A structure $\mathcal{R}$ in a language $\{<, \ldots\}$ is said to be *o-minimal* if every definable subset of R is a finite union of points and intervals. A theory is o-minimal of all of its models are. The following are o-minimal:

- RCF (Tarski);
- $\mathrm{Th}(\mathbb{R}, +, \cdot, <, x \mapsto e^x)$ (Wilkie);
- The expansion of the real field by restricted analytic functions (van den Dries, Gabrielov);
- The reals with restricted analytic functions *and* an exponential function (van den Dries, Miller).

What's the point of this project?

$\mathbb{T}$ admits a natural expansion which includes an exponential function and/or restricted analytic functions. Strip away the valuation and derivation and you have a model of $\mathsf{RCF}_{\mathsf{an,exp}}$.

- What does this look like *with* the valuation and derivation?

Let $\mathcal{R}$ be an o-minimal expansion of the field $\mathbb{R}$. Then the germs of definable functions at infinity form a hardy field $\mathcal{H}$. $\mathcal{H}$ as a field expands to a model of $\mathrm{Th}(\mathcal{R})$. As a differential field, it is an *H*-field.

- What happens when we consider both at once?

T-convexity

For a complete o-minimal theory T extending RCF, say that $(\mathcal{R}, V) \models T_{convex}$ if:

(i) $\mathcal{R} \models T$;

(ii) V is a proper convex subring of R

(iii) For every continuous 0-definable function $f : R \to R$, $f(V) \subseteq V$.

Then V is a valuation ring of R, and we also have:

- T_{convex} is complete.
- If T is model complete, then so is T_{convex}.
- T_{convex} admits q.e. if T admits q.e. and is is universally axiomatizable.
- T_{convex} is weakly o-minimal (definable subsets of R are finite unions of convex sets).

Conventions

- For the rest of this talk, T is a complete o-minimal theory extending RCF and $K \models T$.
- Though I will equip K with a derivation and a valuation, *definable* will always mean definable in the T-model K.
- All extensions of models of T will be elementary extensions. For an extension $L \succcurlyeq K$ and $A \subseteq L$, we denote by $K\langle A\rangle$ the elementary substructure of L generated by A over K.
- A definable function f will be said to be $\mathcal{C}^1$ on $A \subseteq K^m$ if there is definable open $U \supseteq A$ and a definable $\mathcal{C}^1$ function $F : U \to K$ with $F \restriction_A = f$.

The chain rule

Equip K with a derivation $x \mapsto x'$. Let $f : U \to K$ be a definable function on $U \subseteq K^m$. We say that f *obeys the chain rule (on U)* if U is open in K^m, f is $\mathcal{C}^1$ on U, and for all $u = (u_1, \ldots, u_m) \in U$ we have

$$f(u)' = \frac{\partial f}{\partial x_1}(u) \cdot u_1' + \cdots + \frac{\partial f}{\partial x_m}(u) \cdot u_m'.$$

More generally, a definable $\mathcal{C}^1$ map $f = (f_1, \ldots, f_n) : U \to K^n$ with $U \subseteq K^m$ is said to obey the chain rule if U is open in K^m and $f(u)' = J_f(u) \cdot u'$.

- $J_f(u) = \left(\frac{\partial f_i}{\partial x_j}(u) \right)_{1 \le i \le n, 1 \le j \le m}$
- $u' = (u_1', \ldots, u_m')^t$.

All $\mathcal{C}^1$ C-definable semialgebraic functions obey the chain rule.

T-compatible derivations

The derivation of K is said to be T*-compatible* if each 0-definable $\mathcal{C}^1$-function $f : U \to K$ with open $U \subseteq K^m$ obeys the chain rule.

Lemma

Suppose the derivation of K is T-compatible. Then

1. *Each C-definable $\mathcal{C}^1$-function $f : U \to K$ with open $U \subseteq K^m$ obeys the chain rule.*
2. *C is the underlying set of an elementary substructure of the T-model K.*
3. *If $f : U \to K$ is a definable $\mathcal{C}^1$-function on an open set $U \subseteq K^m$, then there is a definable function $f^\partial : U \to K$ such that for all $u \in U$, $f(u)' = f^\partial(u) + J_f(u) \cdot u'$.*

Example

Consider the polynomial

$$P(x) = a_n x^n + \ldots + a_1 x + a_0.$$

We have

$$P(x)' = (a_n' x^n + n a_n x^{n-1} x') + \ldots + (a_1' x + a_1 x') + a_0'$$

$$= (a_n' x^n + \ldots + a_1' x + a_0') + (n a_n x^{n-1} + \ldots + a_1) x'$$

$$= (a_n' x^n + \ldots + a_1' x + a_0') + J_P(x) x'$$

So $P^{\partial}(x) = a_n' x^n + \ldots + a_1' x + a_0'$

HT-fields and pre-*HT*-fields

- We say that K is an *HT-field* if it is equipped with a T-compatible derivation making it an H-field.
- We say that K is a *pre-HT-field* if it is equipped with a T-compatible derivation and a T-convex valuation ring $\mathcal{O}$ making it a pre-H-field.
- If K is an HT-field, then the convex hull of C is a T-convex valuation ring.
- This means that a subfield of an HT-field which is an elementary substructure as a model of T and closed under the derivation is a pre-HT-field.

Examples of *HT*-fields

- $\mathbb{T}$ with restricted analytic functions, or the exponential function, or both.
- The Hardy field generated by an o-minimal structure on $\mathbb{R}$.
- The surreal numbers with the Berarducci-Mantova derivation, equipped with restricted analytic functions, or the exponential function, or both.

Power-bounded theories

- A *power function* in an o-minimal structure $\mathcal{R}$ is a definable endomorphism of the multiplicative group $\mathcal{R}^{>0}$.
- All power functions f look a lot like exponentiation (that is, $f(x) = x^\lambda$ for some $\lambda \in \mathcal{R}$).
- The power functions in RCF, $\mathrm{RCF_{an}}$ are just $x \mapsto x^q$ for $q \in \mathbb{Q}$.
- Call $\mathcal{R}$ *power bounded* if for all definable functions $g : \mathcal{R} \to \mathcal{R}$, there is a power function f with $|g(x)| \leq f(x)$ eventually.
- Power bounded is equivalent to polynomially bounded if $\mathtt{Th}(\mathcal{R})$ has an Archimedean model.
- Every o-minimal theory is power bounded or defines the exponential function (Miller).

Results on power-bounded HT-fields

The goal of this project is to find a model companion for the theory of HT-fields (assuming T is model complete). For now, the scope is restricted to power bounded T. The following theorem shows that this would also be a model companion for pre-HT-fields:

Theorem 1 (K.)

Let K be a pre-HT-field with T power-bounded. Then there is an HT-field $\mathrm{HT}(K)$ extending K such that any embedding of K into an HT-field L extends uniquely to an embedding of $\mathrm{HT}(K)$ into L.

Results on power-bounded *HT*-fields

- Recall that K is said to be *Liouville closed* if for all $f \in K$ there is $g, h \in K$, $h \neq 0$ with $g' = f$, $h^\dagger = f$.
- A *T-Liouville closure* of K is a Liouville closed *HT*-field extension L of K where $C_L = C$ and where for each $a \in L$ there are $t_1, \ldots, t_n \in L$ with $a \in K\langle t_1, \ldots, t_n\rangle$ and for each $i \in \{1, \ldots, n\}$, either $t_i' \in K\langle t_1, \ldots, t_{i-1}\rangle$ or $t_i^\dagger \in K\langle t_1, \ldots, t_{i-1}\rangle$.

Theorem 2 (K.)

Let K be an HT-field with T power-bounded. Then K has either exactly one or exactly two T-Liouville closures up to isomorphism over K. Any embedding of K into a Liouville closed HT-field M extends to an embedding of a T-Liouville closure of K into M.

Future work

(1) Find and axiomatize this model companion (for power bounded T.)
(2) Prove a quantifier elimination result in an extended language.
(3) Is this model companion NIP? Asymptotically o-minimal?
(4) Find a natural model of this model companion when $T = \mathrm{RCF}_{\mathrm{an}}$. (Is $\mathbb{T}_{\mathrm{an}}$ a model? Are the surreals?)
(5) Extend this work to the exponential case.