Page 148, Problem 1.

Page 129, Problem 2. If two contours $\Gamma_0$ and $\Gamma_1$ are respectively shrunkable to single points in a domain $D$, then they are continuously deformable to each other.

Proof.

Formal Solution. Two curves $\gamma_0 : t \in [0, 1] \mapsto z_0(t) \in G$ and $\gamma_1 : t \in [0, 1] \mapsto z_1(t) \in G$ in a domain $G$ is called homotopic if the curve $\gamma_0$ can be continuously demormable to the second curve $\gamma_1$, which means that there exists a continuous mapping $z : (s, t) \in [0, 1] \times [0, 1] \mapsto z(s, t) \in G$ such that
\begin{align*}
\begin{cases}
  z(0, t) = z_0(t), \\
  z(1, t) = z_1(t).
\end{cases}
\end{align*}

We write this fact $\gamma_0 \sim \gamma_1$. Now we claim the following facts about this relation on curves:

\[
\begin{align*}
  &\gamma \sim \gamma \quad \text{for every curve } \gamma \text{ in } G; \\
  &\gamma_0 \sim \gamma_1 \quad \Rightarrow \quad \gamma_1 \sim \gamma_0; \\
  &\gamma_0 \sim \gamma_1, \quad \gamma_1 \sim \gamma_2 \quad \Rightarrow \quad \gamma_0 \sim \gamma_2.
\end{align*}
\]

The first fact can be seen easily by setting: $z(s, t) = z(t), 0 \leq s, t \leq 1$, with $\gamma : t \in [0, 1] \mapsto z(t) \in G$. The second fact also easily check by setting $z'(s, t) = z(1 - s, t), 0 \leq s, t \leq 1$, where $z : (s, t) \in [0, 1] \times [0, 1] \mapsto z(s, t) \in G$ gives $\gamma_0 \sim \gamma_1$, i.e., $z(0, t) = z_0(t)$ and $z(1, t) = z_1(t)$. The third fact appears to be complicated. But it is not hard either. Let $z^1 : (s, t) \in [0, 1] \times [0, 1] \mapsto z^1(s, t) \in G$ give the homotopy: $\gamma_0 \sim \gamma_1$ and $z^2 : (s, t) \in [0, 1] \times [0, 1] \mapsto z^2(s, t) \in G$ give the homotopy $\gamma_1 \sim \gamma_2$. Then set

\[
z(s, t) = \begin{cases} 
  z^1(2s, t), & 0 \leq s \leq \frac{1}{2}, 0 \leq t \leq 1; \\
  z^2(2s - 1, t), & \frac{1}{2} \leq s \leq 1, 0 \leq t \leq 1.
\end{cases}
\]

Then the function $z$ of $(s, t)$ is continuous and $z(0, t) = z_0(t), z(1/2, t) = z_1(t)$ and $z(1, t) = z_2(t)$.

Now suppose curves $\gamma_0$ is homotopic to a point curve $\alpha_0 : t \in [0, 1] \mapsto w_0(t) = \alpha_0 \in G$ and $\gamma_1$ is homotopic to another point curve $\alpha_1 : t \in [0, 1] \mapsto w_1(t) = \alpha_1 \in G$. So, our assumption means that $\gamma_0 \sim \alpha_0$ and $\gamma_1 \sim \alpha_1$. Connect $\alpha_0$ and $\alpha_1$ by a continuous curve: $z : t \in [0, 1] \mapsto z(t) \in G$ so that $z(0) = \alpha_0$ and $z(1) = \alpha_1$. Set $\bar{z}(s, t) = z(s), 0 \leq s, t \leq 1$. Then $\bar{z}(0, t) = w_0(t) = \alpha_0$ and $\bar{z}(1, t) = z(1) = \alpha_1 = w_1(t)$. Thus we get $\gamma_0 \sim \alpha_1$ and

\[
\gamma_0 \sim \alpha_0; \quad \alpha_0 \sim \alpha_1; \quad \alpha_1 \sim \gamma_1 \quad \Rightarrow \quad \gamma_0 \sim \gamma_1.
\]

\begin{center}
\begin{tikzpicture}
\fill[dashed] (0,0) rectangle (1,1);
\fill[gray!50!white] (0.5,0) -- (0.5,1) -- (1,1) -- (1,0) -- cycle;
\draw[very thick] (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\end{tikzpicture}
\end{center}

\medskip

\textbf{Page 149, Problem 3. \textit{In the domain }$D = \{z \in \mathbb{C} : 1 < |z| < 5\}$, \textit{which of the following curves are homotopic to the curve }$\Gamma, z(\theta) = 3 + e^{i\theta}, 0 \leq \theta \leq 2\pi$:

\begin{enumerate}
\item[a)] $\Gamma_a : z(\theta) = 3 + e^{i(\frac{\pi}{2} + \theta)}, 0 \leq \theta \leq 2\pi$;
\item[b)] $\Gamma_b : z(t) = 3i, 0 \leq t \leq 1$;
\item[c)] $\Gamma_c : z(t) = 2e^{it}, 0 \leq t \leq 2\pi$;
\item[d)] $\Gamma_d : z(t) = -3 + e^{it}, 0 \leq t \leq 2\pi$;
\item[e)] $\Gamma_e : z(t) = 3 + e^{-it}, 0 \leq t \leq 4\pi$.
\end{enumerate}
Except the curve $\Gamma_c$, all curves are homotopic to a point curve. As $\Gamma_c$ encloses the hole $|z| < 1$ of the domain $D$, it cannot be shrunk to a point, consequently it cannot be homotopic to any curve which is shrinkable to a point in $D$. ◦

Page 160, Problem 3. Evaluate the integrals over the positively oriented circle $C$, $|z| = 2$:

$$
\begin{align*}
\text{a)} & \quad I_a = \oint_{C} \frac{\sin 3z}{z - \frac{\pi}{2}} \, dz; \\
\text{b)} & \quad I_b = \oint_{C} \frac{ze^z}{2z - 3} \, dz; \\
\text{c)} & \quad I_c = \oint_{C} \frac{\cos z}{z^3 + 9z} \, dz; \\
\text{d)} & \quad I_d = \oint_{C} \frac{5z^2 + 2z + 1}{(z - i)^3} \, dz; \\
\text{e)} & \quad I_e = \oint_{C} \frac{e^{-z}}{(z + 1)^2} \, dz; \\
\text{f)} & \quad I_f = \oint_{C} \frac{\sin z}{z^2(z - 4)} \, dz.
\end{align*}
$$

Answer. a) Set $f(z) = \sin 3z$. As $\pi / 2$ falls inside the circle $C$, we have

$$
I_a = 2\pi i \cdot f\left(\frac{\pi}{2}\right) = 2\pi i \sin \left(\frac{3\pi}{2}\right) = -2\pi i.
$$

b) Set $f(z) = \frac{1}{2} ze^z$. Then $3/2$ is inside of $C$. So we have

$$
I_b = \oint_{C} \frac{1}{2} \frac{ze^z}{z - \frac{3}{2}} \, dz = 2\pi i \cdot f\left(\frac{3}{2}\right) = 2\pi i \cdot \frac{3}{2} e^{\frac{3}{2}} = \frac{3\pi i}{2} e^{\frac{3}{2}}.
$$

c) First observe that

$$
\frac{\cos z}{z^3 + 9z} = \frac{\cos z}{z(z^2 + 9)} = \frac{\cos z}{z(z - 3i)(z + 3i)},
$$

and that the singularities $z = \pm 3i$ lie outside of $C$ and the singularity $z = 0$ sits inside of $C$. So let

$$
f(z) = \frac{\cos z}{z^2 + 9}
$$

and conclude

$$
I_c = 2\pi i f(0) = \frac{2\pi i}{9}.
$$

d) The singularity $z = i$ sits inside of $C$. So we get with $f(z) = 5z^2 + 2z + 1$:

$$
I_d = \frac{1}{2!} 2\pi i f''(i) = \pi i \cdot 10 = 10\pi i.
$$

e) The singularity $z = -1$ falls inside of $C$, so that with $f(z) = e^{-z}$ we have

$$
I_e = 2\pi i f'(-1) = -2\pi ie.
$$
f) The point $z = 0$ is the only singularity of the integrand inside of $C$. So with

$$ f(z) = \frac{\sin z}{z - 4}, $$

obtain

$$ f'(z) = \frac{(z - 4) \cos z - \sin z}{(z - 4)^2}; $$

$$ I_e = 2\pi i f'(0) = 2\pi i \frac{-4}{16} = -\frac{\pi i}{2}. $$

Page 160, Problem 4. Compute the contour integrals:

$$ \oint_C \frac{z + i}{z^3 + 2z^2} dz $$

along the following counterclockwise traveled circles once:

- a) $C_a : |z| = 1.$
- b) $C_b : |z + 2 - i| = 2.$
- c) $C_c : |z - 2i| = 1.$

Answer. First locate the singularities of the integrand:

$$ \frac{z + i}{z^3 + 2z^2} = \frac{z + i}{z^2(z + 2)}.$$

From the above inspection, we conclude that $z = 0$ and $z = -2$ are the singularities and that $z = 0$ falls inside of $C_a$ and outside of both $C_b$ and $C_c$ and $z = -2$ falls inside of $C_b$ and outside of both $C_a$ and $C_c$. So $C_c$ encloses no singularity of the integrand, so that

$$ \oint_{C_c} \frac{z + i}{z^3 + 2z^2} dz = 0. $$

Now with

$$ f(z) = \frac{z + i}{z^2}; \quad g(z) = \frac{z + i}{z + 2}; \quad g'(z) = \frac{(z + 2) - (z + i)}{(z + 2)^2} = \frac{2 - i}{(z + 2)^2} $$

$$ \oint_{C_a} \frac{z + i}{z^3 + 2z^2} dz = 2\pi i g'(0) = \frac{\pi i (2 - i)}{2} = \pi \left( \frac{1}{2} - i \right); $$

$$ \oint_{C_b} \frac{z + i}{z^3 + 2z^2} dz = 2\pi i f(-2) = 2\pi i \frac{i - 2}{4} = -\pi \left( \frac{1}{2} + i \right). $$
Page 160, Problem 5. Let \( C \) be the ellipse
\[
\frac{x^2}{4} + \frac{y^2}{9} = 1
\]
traversed once on the positive direction and set
\[
G(z) := \int_C \frac{\zeta^2 - \zeta + 2}{\zeta - z} \, dz.
\]
Evaluated \( G(1), G'(i) \) and \( G''(-i) \).

Answer. Let
\[
g(z) = z^2 - z + 2, \quad z \in \mathbb{C}.
\]
By the Cauchy Integral Formula, \( 2\pi i g \) and \( G \) agree inside the ellipse \( C \). Therefore,
\[
G(1) = 2\pi i g(1) = 2\pi i(1^2 - 1 + 2) = 4\pi i;
\]
\[
G'(i) = 2\pi i g'(i) = 2\pi i(2i - 1) = -2\pi(2 + i);
\]
\[
G''(-i) = 2\pi i g''(-i) = 4\pi i.
\]

Page 160, Problem 6. Evaluate
\[
\oint_{|z|=3} \frac{e^{iz}}{(z^2 + 1)^2} \, dz.
\]

Answer. We first prepare the factorization:
\[
\frac{e^{iz}}{(z^2 + 1)^2} = \frac{e^{iz}}{(z + i)^2(z - i)^2}
\]
and split the integration along the circle $\Gamma : |z| = 3$ to the sum of the integrations over the smaller two circles $\Gamma_0 : |z - i| = \frac{1}{2}$ and $\Gamma_1 : |z + i| = \frac{1}{2}$. With

$$f_0(z) = \frac{e^{iz}}{(z + i)^2}; \quad f_1(z) = \frac{e^{iz}}{(z - i)^2};$$

$$f_0'(z) = \frac{ie^{iz}(z + i)^2 - 2e^{iz}(z + i)}{(z + i)^4} = \frac{ie^{iz}(z + i) - 2e^{iz}}{(z + i)^3} = \frac{e^{iz}(iz - 3)}{(z + i)^3};$$

$$f_1'(z) = \frac{ie^{iz}(z - i)^2 - 2e^{iz}(z - i)}{(z - i)^4} = \frac{ie^{iz}(z - i) - 2e^{iz}}{(z - i)^3} = \frac{e^{iz}(iz - 1)}{(z - i)^3}$$

we compute

$$\int_{\Gamma} \frac{e^{iz}}{(z^2 + 1)^2} \, dz = \int_{\Gamma_0} \frac{f_0(z)}{(z - i)^2} \, dz + \int_{\Gamma_1} \frac{f_1(z)}{(z + i)^2} \, dz$$

$$= 2\pi i (f_0'(i) + f_1'(-i)) = 2\pi i \left( \frac{e^{-1}(-1 - 3)}{-8i} + \frac{e(i \cdot i - 1)}{8i} \right)$$

$$= \pi \left( \frac{1}{e} - \frac{e}{2} \right).$$

---

**Page 161, Problem 7.** Compute the contour integral:

$$\int_{\Gamma} \frac{\cos z}{z^2(z - 3)} \, dz$$

along the contour $\Gamma$ shown in the figure on the right.

**Proof.** The contour $\Gamma$ is not a simple closed curve. But the loop inside does not enclose any singularity of the integrand, so that it does not contribute to the integral by the Cauchy Fundamental Theorem. So the integral along $\Gamma$ is the same as the integral along the curve $\Gamma'$ obtained by removing the inner loop from $\Gamma$. Thus
the new contour $\Gamma'$ looks like the figure on the right. The singularity $z = 3$ falls outside of $\Gamma'$, so it does not contribute to the integral. With

$$f(z) = \frac{\cos z}{z - 3}; \quad f'(z) = \frac{-(z - 3) \sin z - \cos z}{(z - 3)^2}$$

we have

$$\oint_{\Gamma} \frac{\cos z}{z^2(z - 3)} \, dz = 2\pi i f'(0) = -\frac{2\pi i}{9}.$$  

Page 161, Problem 8. If $f$ is analytic inside and on the circle $C : |z - \alpha| = r$, then

$$f^{(n)}(\alpha) = \frac{n!}{2\pi ir^n} \int_0^{2\pi} f(\alpha + re^{i\theta})e^{-in\theta} \, d\theta, \quad n = 0, 1, 2, \cdots.$$ 

**Proof.** Parameterize the circle $C$ by $C : z(\theta) = re^{i\theta}, 0 \leq \theta \leq 2\pi$, and compute the Cauchy Integral Formula for the $n$-th derivative:

$$f^{(n)}(\alpha) = \frac{n!}{2\pi ir^n} \oint_C \frac{f(z)}{(z - \alpha)^{n+1}} \, dz = \frac{n!}{2\pi i} \oint_C \frac{f(\alpha + re^{i\theta})}{(re^{i\theta})^{n+1}} rie^{i\theta} \, d\theta$$

$$= \frac{n!}{2\pi ir^n} \int_0^{2\pi} \frac{f(\alpha + re^{i\theta})}{e^{in\theta}} \, d\theta.$$ 

This proves the assertion.  

Page 161, Problem 10. If $f$ is analytic on and inside a simple closed curve $\Gamma$, then

$$\oint_{\Gamma} \frac{f'(z)}{z - z_0} \, dz = \oint_{\Gamma} \frac{f(z)}{(z - z_0)^2} \, dz$$

for all $z_0$ not on the curve $\Gamma$.

**Proof.** Assume that $\Gamma$ is positively oriented. If $z_0$ lies outside of the curve $\Gamma$, then the both sides are zero as the integrands are both analytic on and inside the curve $\Gamma$. If $z_0$ lies inside the curve $\Gamma$, then the Cauchy Integral Formula states

$$\oint_{\Gamma} \frac{f'(z)}{z - z_0} \, dz = 2\pi if'(z_0) = \oint_{\Gamma} \frac{f(z)}{(z - z_0)^2} \, dz.$$ 

If the curve $\Gamma$ is negatively oriented, then the above integrals both give $-2\pi if'(z_0)$ for $z_0$ inside $\Gamma$, and zero for $z_0$ outside of $\Gamma$. This completes the proof.
Page 161, Problem 13. With \( g \) a continuous function on a simple closed curve \( \Gamma \), the limit value of the analytic function:

\[
G(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{g(\zeta)}{\zeta - z} d\zeta
\]

on the boundary \( \Gamma \) of the domain enclosed by \( \Gamma \) need not be the original function \( g \).

Example. Let \( \Gamma : |z| = 1 \) be the positively oriented unit circle and \( g(z) = \frac{1}{z} \). Then we get for all \( z \) with \( |z| < 1 \) that:

\[
G(z) = \frac{1}{2\pi i} \oint_{|\zeta| = 1} \frac{\frac{1}{\zeta}}{\zeta - z} d\zeta = \frac{1}{2\pi i} \oint_{|\zeta| = 1} \frac{1}{\zeta(\zeta - z)} d\zeta
\]

\[
= \frac{1}{2\pi i} \oint_{|\zeta| = 1} \frac{1}{z} \left( \frac{1}{\zeta - z} - \frac{1}{\zeta} \right) d\zeta, \quad z \neq 0,
\]

\[
= \frac{1}{2\pi i} \frac{1}{z} \left( \oint_{|\zeta| = 1} \frac{1}{\zeta - z} d\zeta - \frac{1}{2\pi i} \oint_{|\zeta| = 1} \frac{1}{\zeta} d\zeta \right)
\]

\[
= \frac{1}{2\pi i z} (2\pi i - 2\pi i) = 0.
\]

Thus the function \( G \) is constantly zero inside the unit disk \( \Gamma \). But \( g \) on the circle \( \Gamma \) is not a zero function. Thus \( g(z) \neq \lim_{r \to 1} G(rz) \) for \( z \in \Gamma \).

Page 162, Problem 15. If \( f \) is an analytic function on the unit disk \( \mathbb{D} : |z| \leq 1 \) and \( f(0) = 0 \), then the function \( F \) defined by

\[
F(z) = \begin{cases} 
\frac{f(z)}{z}, & z \neq 0; \\
\frac{f'(0)}{z}, & z = 0,
\end{cases}
\]

is analytic on the disk \( \mathbb{D} \).

Proof. Clearly, the function \( F \) is analytic on the disk \( \mathbb{D} \) possibly except the origin \( z = 0 \). But if we define a new function \( G \) by the integral:

\[
G(z) = \frac{1}{2\pi i} \oint_{|\zeta| = 1} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \oint_{|\zeta| = 1} \frac{f(\zeta)}{\zeta(\zeta - z)} d\zeta
\]

\[
= \frac{1}{2\pi i} \oint_{|\zeta| = 1} \frac{f(\zeta)}{z} \left( \frac{1}{\zeta - z} - \frac{f(\zeta)}{\zeta} \right) d\zeta, \quad z \neq 0,
\]

\[
= \frac{1}{2\pi i z} \oint_{|\zeta| = 1} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \oint_{|\zeta| = 1} \frac{f(\zeta)}{\zeta} d\zeta
\]

\[
= \frac{f(z)}{z} - \frac{f(0)}{z} = \frac{f(z)}{z},
\]

then it is analytic throughout the disk \( \mathbb{D} \) and agrees with \( F(z) \) for non-zero \( z \). The Cauchy Integral Formula tells that if \( z = 0 \), then \( G(0) = f'(0) \). This completes the proof.
Page 162, Problem 16. If an analytic function $f$ is never zero on a simply connected domain $D$, then there exists a single-valued branch of $\log(f(z))$ on the domain $D$. 

Proof. First recall that a simply connected domain $D$ means that every closed curve can be shrunk to a point within the domain $D$. This means that the integral of any analytic function on the domain along every closed contour is zero because the line integral of an analytic function does not change along a continuous deformation of the base curve and the integral over any point curve is zero. This means that every analytic function $f$ over $D$ has an anti-derivative $F$, whose value at $z \in D$ can be defined by the contour integral

$$F(z) = \int_{\Gamma_{z_0}} f(\zeta) d\zeta$$

along a contour $\Gamma_{z_0}$ connecting a fixed point $z_0 \in D$ to the point $z \in D$. Now the function:

$$g(z) = \frac{f'(z)}{f(z)}, \quad z \in D,$$

is analytic throughout the domain $D$ due to the assumption that $f$ does not vanish on $D$. Thus $g$ has the anti-derivative $G$ on the domain $D$ given by

$$G(z) = \int_{\Gamma_{z_0}} g(\zeta) d\zeta = \int_{\Gamma_{z_0}} \frac{f(\zeta)}{f'(\zeta)} d\zeta.$$

The derivative of the function $f(z)e^{-G(z)}$ is zero throughout the Domain $D$, so that $f(z)e^{-G(z)}$ is constant, say $c$. Hence we get $f(z) = ce^{G(z)}$. Thus $G + \text{Log}(c)$ is a branch of $\log(f(z))$. 

Page 162, Problem 17. There exists a single-valued branch of $(z^3 - 1)^{\frac{1}{2}}$ in the disk $\mathbb{D}: |z| < 1$.

Proof. The unit disk $\mathbb{D}$ is simply closed and the function: $z^3 - 1$ does not hit zero on the disk $\mathbb{D}$. Therefore, there exists a single-valued branch $G$ of $\log(z^3 - 1)$ on the disk $\mathbb{D}$. Set

$$g(z) = e^{\frac{1}{2}G(z)}, \quad z \in \mathbb{D},$$

to obtain a single-valued analytic function $g$ such that $g(z)^2 = z^3 - 1$ for all $z \in \mathbb{D}$. This completes the proof.
Page 167, Problem 1. If \( f(z) = \frac{1}{(1-z)^2} \) and \( 0 < R < 1 \), then

i) \[
\max_{|z|=R} |f(z)| = \frac{1}{(1-R)^2};
\]

ii) \[
f^{(n)}(0) = (n+1)! ;
\]

iii) \[
(n+1)! \leq \frac{n!}{R^n(1-R)^2}.
\]

Proof. i) Observe that the maximum of \( |f(z)| \) on the circle \( C_R(0) = \{ z \in \mathbb{C} : |z| = R \} \) means the denominator \( |(1-z)^2| \) is minimum, which is the square of the distance between the circle \( C_R(0) \) and the point \( z = 1 \). Obviously, it is \( (1-R)^2 \). Thus the maximum of \( |f(z)| \) on the circle \( C_R(0) \) is \( \frac{1}{(1-R)^2} \).

ii) It is easy to see that an anti-derivative of \( f(z) \) is

\[
F(z) = \frac{1}{1-z},
\]

which has the power series expansion on the disk \( D_R(0) = \{ z \in \mathbb{C} : |z| < R \} \):

\[
F(z) = 1 + z + z^2 + \cdots = \sum_{n=0}^{\infty} z^n, \quad |z| < 1.
\]

Therefore, we get

\[
f(z) = F'(z) = 1 + 2z + 3z^2 + 4z^3 + \cdots = \sum_{n=0}^{\infty} (n+1)z^n = \sum_{n=0}^{\infty} a_n z^n;
\]

with

\[
\frac{f^{(n)}(0)}{n!} = a_n = (n+1).
\]

Thus, we conclude the second assertion of the problem.
iii) The Cauchy Integral Formula applied to $f(z)$ on the disk $D_R(0)$ around the origin gives the following:

$$\frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \oint_{|z|=R} \frac{f(z)}{z^{n+1}} dz;$$

$$\left| \frac{f^{(n)}(0)}{n!} \right| = \left| \frac{1}{2\pi i} \oint_{|z|=R} \frac{f(z)}{z^{n+1}} dz \right| \leq \frac{1}{2\pi} \max_{|z|=R} |f(z)| \frac{2\pi R}{R^{n+1}} \leq \frac{1}{(1-R)^2 R^n};$$

$$\therefore (n+1)! \leq \frac{n!}{R^n (1-R)^2}.$$

This completes the proof.

Page 167, Problem 2. If $f$ is analytic on the unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ and $|f(z)| < \frac{1}{1-|z|}$, then

$$|f^{(n)}(0)| \leq \frac{n!}{R^n (1-R)}.$$

Proof. The Cauchy Integral Formula applied to $f$, the circle $C_R(0) = \{z \in \mathbb{C} : |z| = R\}, 0 < R < 1$, and the point $\alpha = 0$ gives:

$$\frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \oint_{C_R(0)} \frac{f(z)}{z^{n+1}} dz;$$

$$\left| \frac{f^{(n)}(0)}{n!} \right| = \left| \frac{1}{2\pi i} \oint_{C_R(0)} \frac{f(z)}{z^{n+1}} dz \right| \leq \frac{1}{2\pi} \max_{|z|=R} |f(z)| \frac{2\pi R}{R^{n+1}} \leq \frac{1}{R^n (1-R)};$$

$$\therefore |f^{(n)}(0)| \leq \frac{n!}{R^n (1-R)}.$$

This completes the proof.

Page 168, Problem 4. If $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ and $\max_{|z|=1} |p(z)| = M$, then $|a_k| \leq M, 0 \leq k \leq n$.

Proof. The Cauchy Inegral Formula applied to $p$, the unit circle $C = \{z \in \mathbb{C} : |z| = 1\}$ and the point $\alpha = 0$ gives:

$$a_k = \frac{1}{2\pi i} \oint_{C} \frac{p(z)}{z^{k+1}} dz$$

$$|a_k| = \left| \frac{1}{2\pi i} \oint_{C} \frac{p(z)}{z^{k+1}} dz \right| \leq \frac{1}{2\pi} \max_{|z|=1} |p(z)| 2\pi = M.$$

This completes the proof.
**Page 168, Problem 5.** If $f$ is entire and $\Re f(z) \leq M$, $z \in \mathbb{C}$, then $f$ must be constant.

**Proof.** Set $g(z) = e^{f(z)}$. Then $g$ is also entire. Now we compute the modulus $|g(z)|$:

$$|g(z)| = |e^{f(z)}| = e^{\Re f(z)} \leq e^M.$$ 

Thus $g$ is a bounded entire function. Liouville’s Theorem states that every bounded entire function is a constant. So $g$ must be a constant, equivalently $f$ is constant. This completes the proof.

**Page 168, Problem 7.** If $f$ is entire and there exists $r_0$ such that $|f(z)| \leq z^2$ for every $z$ with $|z| \geq r_0$, then $f$ is a polynomial of at most degree two.

**Proof.** Fix $z \in \mathbb{C}$ and choose $R > r_0 + |z|$. Apply the Cauchy Integral Formula to $f$, the circle: $C_R(z) = \{w \in \mathbb{C} : |w - z| = R\}$ and the point $z$:

$$\frac{f^{(3)}(z)}{3!} = \frac{1}{2\pi i} \oint_{C_R(z)} \frac{f(w)}{(w - z)^4} dw$$

Observe that if $w \in C_R(z)$, then

$$R + |z| \geq |w| \geq R - |z| \geq r_0,$$

so that

$$|f(w)| \leq |w|^2 = |w|^2 \leq (R + |z|)^2, \quad w \in C_R(z).$$

Thus we get the estimate:

$$\left| \frac{f^{(3)}(z)}{3!} \right| = \left| \frac{1}{2\pi i} \oint_{C_R(z)} \frac{f(w)}{(w - z)^4} dw \right| \leq \frac{1}{2\pi} \frac{(R + |z|)^2}{R^4} 2\pi R \leq \frac{(R + |z|)^2}{R^3} \to 0 \quad \text{as} \quad R \to +\infty.$$

Thus $f^{(3)}(z) = 0$ for every $z$, which means that $f$ is a polynomial of degree at most two.

**Page 168, Problem 9.** If $f$ is analytic on the closed disk $\overline{D}_R(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq R\}$ and $|f(z_0)| \geq |f(z)|$, $z \in C_R(z_0) = \{z \in \mathbb{C} : |z - z_0| = R\}$, then there is no point $z_1$ on the circle $C_R(z_0)$ such that $|f(z_0)| > |f(z_1)|$.

**Proof.** By the maximum modulus principle, $f$ is a constant on the closed disk. This means that there is no point $z$ on the closed disk $\overline{D}_R(z_0)$ such that $f(z_1) \neq f(z_0)$. In particular, there is no point $z_1$ on the circle $C_R(z_0)$ with $|f(z_1)| \neq |f(z_0)|$.
Page 195, Problem 1. Prove the following Taylor expansions:

a) \( e^{-z} = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} = 1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \cdots, \quad z_0 = 0; \)

c) \( \sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots, \quad z_0 = 0; \)

e) \( \log(1 - z) = \sum_{n=1}^{\infty} \frac{-z^n}{n}, \quad z_0 = 0. \)

Proof. a) Let \( f(z) = e^z \) and \( g(z) = e^{-z}. \) Then \( g(z) = f(-z) \) and
\[
\begin{align*}
& f'(z) = f(z) \quad \text{and} \quad g'(z) = -f'(-z) = -f(-z) = -g(z), \quad z \in \mathbb{C}.
\end{align*}
\]
Therefore we get
\[
\begin{align*}
& f^{(n)}(0) = f(0) = 1, \quad n = 1, 2, \cdots ; \\
& g^{(n)}(z) = (-1)^n g(z) \quad \Rightarrow \quad g^{(n)}(0) = (-1)^n g(0) = (-1)^n; \\
& g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^n = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!}.
\end{align*}
\]

b) By definition,
\[
\sinh z = \frac{e^z - e^{-z}}{2}.
\]
Hence we get
\[
\sinh z = \frac{1}{2} \left( \sum_{n=0}^{\infty} \frac{z^n}{n!} - \sum_{m=0}^{\infty} \frac{(-z)^m}{m!} \right) = \frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{z^n - (-z)^n}{n!} \right).
\]
In the summation, all even terms hit zero as \( z^{2n} - (-z)^{2n} = z^{2n} - z^{2n} = 0, \) so that
\[
\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}.
\]
The expansion can also be obtained by observing
\[
\begin{align*}
& \sinh' z = \cosh z \quad \text{and} \quad \cosh' z = \sinh z; \\
& \sinh^{(2n+1)} z = \cosh z \quad \text{and} \quad \sinh^{(2n)} z = \sinh z; \\
& \sin^{(2n+1)}(0) = 1 \quad \text{and} \quad \sin^{(2n)}(0) = 0.
\end{align*}
\]
e) First observe that with \( f(z) = \log(1 - z) \)
\[
f'(z) = -\frac{1}{1 - z} = -\sum_{n=0}^{\infty} z^n, \quad |z| < 1.
\]
Thus
\[
f^{(n+1)}(0) = -n! \quad \Rightarrow \quad \frac{f^{(n)}(0)}{n!} = -\frac{1}{n}, |z| < 1.
\]
Therefore we get the expansion:
\[
f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} = \sum_{n=1}^{\infty} \frac{-z^n}{n}, |z| < 1.
\]
This proves the expansions.

**Page 195, Problem 4.** With \( \alpha \in \mathbb{C} \), if \( (1 + z)^\alpha = e^{\alpha \log(1 + z)} \) for \( |z| < 1 \), then
\[
(1 + z)^\alpha = \sum_{n=0}^{\infty} \frac{\alpha(\alpha - 1) \cdots (\alpha - (n - 1))}{n!} z^n.
\]

**Proof.** First we observe that the function:
\[
f(z) = \frac{1}{1 + z}
\]
is analytic on the open unit disk \( D = \{ z \in \mathbb{C} : |z| < 1 \} \). Thus the integral:
\[
F(z) = \int_{\Gamma(z)} \frac{1}{1 + z} \, dz
\]
along any contour \( \Gamma(z) \) connecting the 0 to the point \( z \in D \) does not depend on the curve \( \Gamma(z) \) but only on the end point \( z \). This function \( F(z) \) is precisely the principal logarithm function \( \log(1 + z) \). Now let
\[
f(z) = e^{\alpha F(z)} = e^{\alpha \log(1 + z)} = (1 + z)^\alpha
\]
and compute its higher derivatives:
\[
f'(z) = \alpha F'(z)e^{\alpha F(z)} = \frac{\alpha}{1 + z} f(z) \quad \Rightarrow \quad f'(0) = \alpha;
\]
\[
f''(z) = -\frac{\alpha}{(1 + z)^2} f(z) + \frac{\alpha}{1 + z} f'(z) = -\frac{\alpha}{(1 + z)^2} f(z) + \left( \frac{\alpha}{1 + z} \right)^2 f(z)
\]
\[
= \frac{\alpha(\alpha - 1)}{(1 + z)^2} f(z) \quad \Rightarrow \quad f''(0) = \alpha(\alpha - 1);
\]
\[
f^{(3)}(z) = -2 \frac{\alpha(\alpha - 1)}{(1 + z)^3} f(z) + \frac{\alpha(\alpha - 1)}{(1 + z)^2} f'(z)
\]
\[
= -2 \frac{\alpha(\alpha - 1)}{(1 + z)^3} f(z) + \frac{\alpha(\alpha - 1)}{(1 + z)^2} \frac{\alpha}{1 + z} f(z)
\]
\[
= \frac{\alpha(\alpha - 1)(\alpha - 2)}{(1 + z)^3} f(z) \quad \Rightarrow \quad f^{(3)}(0) = \alpha(\alpha - 1)(\alpha - 2);
\]
If \( f^{(n)}(z) = \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))}{(1+z)^n} f(z) \), then

\[
f^{(n+1)}(z) = -n \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))}{(1+z)^{n+1}} f(z)
+ \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))}{(1+z)^n} f'(z)
= -n \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))}{(1+z)^{n+1}} f(z)
+ \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))}{(1+z)^n} \frac{\alpha}{1+z} f(z)
= \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))(\alpha-n)}{(1+z)^{n+1}} f(z).
\]

By induction, we get

\[
f^{(n)}(z) = \frac{\alpha(\alpha-1)\cdots(\alpha-(n-1))}{(1+z)^n} f(z)
\]
and

\[
f^{(n)}(0) = \alpha(\alpha-1)\cdots(\alpha-(n-1))
\]
for all \( n \in \mathbb{N} \). This gives the required Taylor expansion of \( f \) on the open unit disk. This completes the proof.

**Page 195, Problem 5.** Find and determine the convergence radius of the Taylor series of the following functions around \( z_0 \):

a) \( \frac{1}{1+z} \), \( z_0 = 0 \);

b) \( z^3 \sin 3z \), \( z_0 = 0 \);

g) \( \frac{z}{(1-z)^2} \), \( z_0 = 0 \).

*Answer.* a) \( z = -1 \) is the only singularity of the function \( f(z) = \frac{1}{1+z} \). Thus the largest disk with center 0 which does not contain the singularity is the open unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \). Thus the convergence radius of the Taylor series of \( f \) around 0 is 1 and

\[
\frac{1}{1+z} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-z)^n = \sum_{n=0}^{\infty} (-1)^n z^n \quad \text{for} \quad |z| < 1.
\]

c) The function \( f(z) = z^3 \sin 3z \) is entire, so that the Taylor series converges
everywhere. Now we compute
\[ z^3 \sin 3z = z^3 \frac{e^{3iz} - e^{-3iz}}{2i} = \frac{z^3}{2i} \sum_{n=0}^{\infty} \frac{(3iz)^n - (-3iz)^n}{n!} \]
\[ = \frac{z^3}{2i} \sum_{n=0}^{\infty} \frac{(3iz)2^{n+1} + (3iz)^{2n+1}}{(2n+1)!} \]
\[ = 3z^4 \sum_{n=0}^{\infty} \frac{(-9z^n)}{(2n+1)!} \cdot \frac{(-1)^n3 \cdot 9^n z^{4+2n}}{(2n+1)!} \]
\[ = 3z^4 - 9z^6 + \frac{3 \cdot 9^2 z^8}{5!} - \frac{3 \cdot 9^3 z^{10}}{7!} + \ldots. \]

g) The only singularity of \( f(z) = \frac{z}{(1-z)^2} \) is \( z = 1 \). Thus the largest circle with center \( z_0 = 0 \) which does not encloses the singularity is the unit circle \( |z| = 1 \). The convergence radius of the Taylor series of \( f \) around the origin is 1. Now we compute the expansion with the fact \( \frac{1}{1-z} = \frac{d}{dz} \left( \frac{1}{1-z} \right) \) in mind:
\[ \frac{z}{(1-z)^2} = z \frac{d}{dz} (1 + z + z^2 + z^3 + \ldots) = z(1 + z + 3z^2 + \ldots + nz^{n-1} + \ldots) \]
\[ = z + 2z^2 + 3z^3 + \ldots nz^n + \ldots = \sum_{n=1}^{\infty} nz^n \]

This completes the answer. ♥

Page 195, Problem 6. The Taylor expansion of \( \frac{1}{\zeta - z} \) around \( z_0 \neq z \) is given by
\[ \frac{1}{\zeta - z} = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \]
for \( z \in D_{|\zeta - z_0|}(z_0) = \{ z \in \mathbb{C} : |z - z_0| < |\zeta - z_0| \} \).

Proof.
The singularity of the function \( f(z) = \frac{1}{\zeta - z} \) is only \( z = \zeta \). Thus the largest disk centered at \( z_0 \) which excludes the singularity \( \zeta \) is the open disk \( D_{|\zeta - z_0|}(z_0) \) of radius \( |\zeta - z_0| \). So the Taylor expansion of \( f \) around \( z_0 \) converges on the disk \( D_{|\zeta - z_0|}(z_0) \).

Now we compute the expansion
\[ \frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \left( \frac{z-z_0}{\zeta-z_0} \right)} \]
\[ = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left( \frac{z - z_0}{\zeta - z_0} \right)^n = \sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\zeta - z_0)^{n+1}} \]
to complete the proof. ♥
Page 196, Problem 20. If $f$ is analytic on the closed disk $\overline{D}_R(0)$ of radius $R$ with center $0$, then

$$f(z) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k = \frac{1}{2\pi i} \oint_{|\zeta|=R} \frac{z^{n+1} f(\zeta)}{\zeta^{n+1} (\zeta - z)} d\zeta.$$  

Proof. First observe for $\zeta$ on the unit circle $|\zeta| = 1$, we have

$$\frac{1}{\zeta - z} = \frac{1}{\zeta} - \frac{1}{\zeta - \zeta} = \sum_{k=0}^{n} \frac{z^k}{\zeta^{k+1}} + \sum_{k=n+1}^{\infty} \frac{z^k}{\zeta^{k+1}}$$

$$= \sum_{k=0}^{n} \frac{z^k}{\zeta^{k+1}} + \sum_{k=0}^{\infty} \frac{z^{n+1} z^k}{\zeta^{k+1}} = \sum_{k=0}^{n} \frac{z^k}{\zeta^{k+1}} + \frac{z^{n+1}}{\zeta^{n+2}} \frac{1}{1 - \frac{z}{\zeta}}$$

$$= \sum_{k=0}^{n} \frac{z^k}{\zeta^{k+1}} + \frac{z^{n+1}}{\zeta^{n+1} \zeta - z}.$$  

We then substitute the above to the Cauchy Integral Formula:

$$f(z) = \frac{1}{2\pi i} \oint_{|\zeta|=R} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \oint_{|\zeta|=R} f(\zeta) \frac{1}{\zeta - z} d\zeta$$

$$= \frac{1}{2\pi i} \oint_{|\zeta|=R} f(\zeta) \left( \sum_{k=0}^{n} \frac{z^k}{\zeta^{k+1}} + \frac{z^{n+1}}{\zeta^{n+1} \zeta - z} \right) d\zeta$$

$$= \sum_{k=0}^{n} \frac{z^k}{2\pi i} \oint_{|\zeta|=R} \frac{f(\zeta)}{\zeta^{k+1}} d\zeta + \frac{1}{2\pi i} \oint_{|\zeta|=R} \frac{z^{n+1} f(\zeta)}{\zeta^{n+1} (\zeta - z)} d\zeta$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k + \frac{1}{2\pi i} \oint_{|\zeta|=R} \frac{z^{n+1} f(\zeta)}{\zeta^{n+1} (\zeta - z)} d\zeta, \quad |z| < R.$$  

This completes the proof.

Page 217, Problem 1-a, 1-d. Find the Laurent series for the function $\frac{1}{z + z^2}$ in each of the following domains:

a) $0 < |z| < 1$;

d) $1 < |1 + z|$.
Answer. a) and d) We simply compute:

\[
\frac{1}{z + z^2} = \frac{1}{z} \frac{1}{1 + z} = \frac{1}{z} - \frac{1}{1 + z} = \frac{1}{z} - \left( \sum_{n=0}^{\infty} (-z)^n \right)
\]

\[
= \frac{1}{z} - 1 + z - z^2 + z^3 - \cdots + (-1)^{n-1} z^n + \cdots, \quad |z| < 1;
\]

\[
= \frac{1}{(1+z) - 1} - \frac{1}{1 + z} = \frac{1}{1 + z} - \frac{1}{1 \frac{1}{1+z}} - \frac{1}{1 + z}
\]

\[
= \frac{1}{1 + z} \sum_{n=0}^{\infty} \frac{1}{(1+z)^n} = \frac{1}{1 + z} = \sum_{n=2}^{\infty} \frac{1}{(1+z)^n}, \quad |1 + z| > 1.
\]

This completes the answer.

Page 217, Problem 3. Find the Laurent series of the function:

\[
\frac{z}{(z + 1)(z - 2)}
\]

around the origin.

Answer. First the singularities of the functions are \( z = 1 \) and \( z = 2 \). Thus the Laurent series has three regions to consider:

a) \( |z| < 1 \), b) \( 1 < |z| < 2 \), c) \( 2 < |z| \).

Second decompose the given function:

\[
\frac{z}{(z - 1)(z - 2)} = z \left( \frac{1}{1 - z} - \frac{1}{2 - z} \right)
\]

a) We compute:

\[
\frac{z}{(z - 1)(z - 2)} = z \left( \sum_{n=0}^{\infty} z^n - \frac{1}{2} \frac{1}{1 - \frac{z}{2}} \right) = z \left( \sum_{n=0}^{\infty} z^n - \frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{z}{2} \right)^n \right)
\]

\[
= z \sum_{n=0}^{\infty} \frac{2^n+1 - 1}{2^{n+1}} z^n = \sum_{n=1}^{\infty} \left( \frac{2^n-1}{2^n} \right) z^n, \quad |z| < 1.
\]

b) We continue:

\[
\frac{z}{(z - 1)(z - 2)} = z \left( \frac{1}{1 - z} - \frac{1}{2 - z} \right) = z \left( \frac{1}{z} \left( \frac{1}{1 - \frac{1}{z}} \right) - \frac{1}{2 - \frac{1}{z}} \right)
\]

\[
= z \left( \frac{1}{z} \left( - \sum_{n=0}^{\infty} \frac{1}{z^n} \right) - \frac{1}{2} \sum_{n=0}^{\infty} \left( \frac{z}{2} \right)^n \right)
\]

\[
= - \sum_{n=0}^{\infty} \frac{1}{z^n} - \sum_{n=1}^{\infty} \frac{z^n}{2^n}
\]

\[
= - \cdots - \frac{1}{z^2} - \frac{1}{z} - 1 - \frac{z}{2} - \frac{z^2}{4} - \cdots, \quad 1 < |z| < 2.
\]
c) For the region \(|z| > 2\), we compute

\[
\frac{z}{(z-1)(z-2)} = z \left( \frac{1}{z-2} - \frac{1}{z-1} \right) = z \left( \frac{1}{z} \cdot \frac{1}{1 - \frac{2}{z}} - \frac{1}{z} \cdot \frac{1}{1 - \frac{1}{z}} \right)
= \sum_{n=0}^{\infty} \left( \left( \frac{2}{z} \right)^n - \left( \frac{1}{z} \right)^n \right) = \sum_{n=0}^{\infty} \frac{2^n - 1}{z^n}, \quad |z| > 2.
\]

This completes the answer.

**Page 218, Problem 5.** Find the Laurent series for

\[
\frac{z + 1}{z(z - 4)^3} \quad \text{in} \quad 0 < |z - 4| < 4.
\]

**Answer.** The singularities of the function are \(z = 0\) and \(z = 4\). The punctured open disk \(\mathbb{D}_4^*(4) = \{z \in \mathbb{C} : 0 < |z - 4| < 4\}\) does not contain any of the singularities. We now compute:

\[
\frac{z + 1}{z(z - 4)^3} = \frac{(z - 4) + 5}{((z - 4) + 4)(z - 4)^3} = \frac{1}{(z - 4)^3} \cdot \frac{(z - 4) + 5}{4 - (4 - z)}
= \frac{1}{(z - 4)^3} \cdot \frac{1}{4} \cdot \frac{1}{1 - \frac{4 - z}{4}}((z - 4) + 5)
= \frac{1}{(z - 4)^3} \cdot \frac{1}{4} \cdot \left( \sum_{n=0}^{\infty} \left( \frac{4 - z}{4} \right)^n \right) ((z - 4) + 5)
= \frac{1}{(z - 4)^3} \left( \sum_{n=0}^{\infty} \left( -\frac{z - 4}{4} \right)^{n+1} + \frac{5}{4} \left( -\frac{z - 4}{4} \right)^n \right)
= \frac{1}{(z - 4)^3} \left( \frac{5}{4} + \sum_{n=1}^{\infty} \frac{1}{4} \cdot (-1)^n \left( \frac{z - 4}{4} \right)^n \right)
= \frac{5}{4} \frac{1}{(z - 4)^3} + \sum_{n=-2}^{\infty} (-1)^{n+1} \frac{(z - 4)^n}{4^{n+4}}
\]

This finishes the expansion.

**Page 218, Problem 10.** The Laurent expansion of the function:

\[
f(z) = \exp \left[ \frac{\lambda}{2} \left( z - \frac{1}{z} \right) \right], \quad z \neq 0
\]
is given by
\[ \sum_{k=-\infty}^{\infty} J_k(\lambda)z^k, \]  

where
\[ J_k(\lambda) = (-1)^k J_{-k}(\lambda) = \frac{1}{2\pi} \int_{0}^{2\pi} \cos(k\theta - \lambda \sin \theta) d\theta. \]

Proof. Obviously the singularity of \( f \) is \( z = 0 \) only. So it has the Laurent expansion of the form (*) which converges everywhere except at the origin. Each coefficient \( J_k(\lambda) \) is computed by the Cauchy integral:
\[ J_k(\lambda) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(z)}{z^{k+1}} dz \]
\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(\lambda i \sin \theta)e^{-ik\theta} d\theta \]
\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\cos(\lambda \sin \theta - k\theta) + i \sin(\lambda \sin \theta - k\theta)) d\theta \]
\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\lambda \sin \theta - k\theta) d\theta, \]

where the last step follows from the fact that the function \( \sin(\lambda \sin \theta - k\theta) \) is an odd function of \( \theta \) and hence its integral over the interval \([-\pi, \pi]\) vanishes.

The Laurent series applied to \(-z\) and \(\frac{1}{z}\) gives:
\[ f(-z) = \sum_{k=-\infty}^{\infty} J_k(\lambda)(-z)^k = \sum_{k=-\infty}^{\infty} (-1)^k J_k(\lambda)z^k; \]
\[ f\left(\frac{1}{z}\right) = \sum_{k=-\infty}^{\infty} J_k(\lambda)z^{-k} = \sum_{k=-\infty}^{\infty} J_{-k}(\lambda)z^k. \]

But \( f(-z) = f(1/z) \), which implies
\[ J_{-k}(\lambda) = (-1)^k J_k(\lambda). \]

This completes the proof. ✷

Page 226, Problem 2. Determine the order of the pole of
\[ f(z) = \frac{1}{(2\cos z - 2 + z^2)^2} \]
at \( z = 0 \).

**Answer.** Setting \( g(z) = \frac{1}{f(z)} \), we will determine the order of the zero of \( g \) at \( z = 0 \).

To determine the degree of the zero of \( g \) at \( 0 \), we consider instead the order of the zero of \( h(z) = 2 \cos z - 2 + z^2 \) at \( 0 \), which is a half of the order of the zero of \( g \) at \( 0 \). So we compute the derivatives:

\[
\begin{align*}
h'(z) &= -2 \sin z + 2z \quad \Rightarrow \quad h'(0) = 0; \\
h''(z) &= -2 \cos z + 2 \quad \Rightarrow \quad h''(0) = 0; \\
h^{(3)}(z) &= 2 \sin z \quad \Rightarrow \quad h^{(3)}(0) = 0; \\
h^{(4)}(z) &= -2 \cos z \quad \Rightarrow \quad h^{(4)}(0) = -2 \neq 0.
\end{align*}
\]

Thus the order of zero of \( h \) at \( 0 \) is 4, so that that of \( g \) at \( 0 \) is 8. Hence the order of the pole of \( f \) at \( 0 \) is 8. This answers the question. \( \heartsuit \)

**Page 226, Problem 9.** Does there exist a function \( f(z) \) with an essential singularity at \( z_0 \) which is bounded along some line segment emanating from \( z_0 \)?

**Answer.** Yes, there exists such a function \( f \). For example, the origin 0 is an essential singularity of the function

\[
f(z) = e^{\frac{1}{z}}.
\]

It is bounded by 1 along the imaginary axis. \( \heartsuit \)

**Page 227, Problem 17 (Schwarz’s Lemma).** If \( f \) is an analytic function on the unit disk \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) and satisfies the condition:

\[
f(0) = 0 \quad \text{and} \quad |f(z)| \leq 1 \quad \text{for all } z \in \mathbb{D},
\]

then \( |f(z)| \leq |z| \) for all \( z \in \mathbb{D} \).

**Proof.** Set

\[
G(z) = \frac{f(z)}{z}, \quad z \neq 0.
\]

First the function \( G \) is analytic on the punctured disk \( \mathbb{D}^* = \{ z \in \mathbb{C} : 0 < |z| < 1 \} \), so that the origin 0 is an isolated singularity of \( G \). But we have

\[
\lim_{z \to 0} G(z) = \lim_{z \to 0} \frac{f(z) - f(0)}{z} = f'(0).
\]

Thus the singularity 0 is removable and the function

\[
F(z) = \begin{cases}
G(z), & z \neq 0 \\
f'(0), & z = 0,
\end{cases}
\]

is analytic on the entire unit disk \( \mathbb{D} \).
For any $\zeta \in C = \{z \in \mathbb{C} : |z| = 1\}$, we have

$$|F(r\zeta)| = \frac{|f(r\zeta)|}{|r\zeta|} = \frac{|f(r\zeta)|}{r} \leq \frac{1}{r}, \quad 0 < r < 1.$$ 

Hence for any $0 < r < 1$, we have

$$\sup_{\zeta \in C} |F(r\zeta)| \leq \frac{1}{r},$$

i.e., the maximum value of $|F(z)|$ on the circle $C_r$ of radius $r$, $0 < r < 1$, with center 0 is at most $1/r$. The maximum modulus principle for analytic functions shows that

$$\sup_{|z| \leq r} |F(z)| \leq \frac{1}{r}, \quad 0 < r < 1,$$

which is equivalent to the assertion of Schwarz’s lemma.

\textbf{Page 230, Problem 1-a, 1-g, 1-i.} Classify the behavior of the following functions at the point at infinity:

a) $e^z$; \hspace{1cm} g) $\frac{\sin z}{z^2}$; \hspace{1cm} i) $e^{\tan \frac{1}{z}}$.

\textit{Answer.} For all problems, we consider the behavior of the functions at 0 after replacing $z$ by $1/z$.

a) First let $f(z) = e^z$. Then 0 is an isolated singularity of $f$. When $z$ goes to 0 along the imaginary axis, $f$ stay bounded. But $z$ goes to 0 along the real axis from the positive side, then it goes to $+\infty$. So $f(z)$ does not converge to anywhere nor to $\infty$. This means that 0 is an essential singularity of $f$. Thus the point at infinity is an essential singularity of $e^z$.

b) Let

$$f(z) = \frac{\sin \frac{1}{z}}{(\frac{1}{z})^2} = z^2 \sin \frac{1}{z} = z^2 \frac{e^{\frac{i}{z}} - e^{-\frac{i}{z}}}{2i}.$$ 

As $z \to 0$ along the real axis, the limit of $f(z)$ is 0. But $z \to 0$ along the imaginary axis from the above, $z^2 e^{-\frac{i}{z}} \to 0$, but $z^2 e^{\frac{i}{z}} \to \infty$, so that $f(z) \to \infty$. Thus $f(z)$ does not converge to any point on the extended complex plane $\mathbb{C} \cup \{\infty\}$. Thus the point at infinity is an essential singularity of the function.

i) Let

$$f(z) = e^{\tan z}.$$ 

As $z \to 0$, $\tan z = \frac{\sin z}{\cos z}$ converges to 0, so that

$$\lim_{z \to 0} f(z) = 1,$$

and indeed $f(0) = 1$. Thus the point at infinity is a removable singularity and the assigned value of $e^{\tan \frac{1}{z}}$ at the point at infinity is 0.
Page 231, Problem 2. If $f$ is analytic at $\infty$, then it has a series expansion of the form:

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}$$

which converges uniformly outside some disk.

Proof. With $g(z) = f(1/z)$, $g$ is analytic at the origin 0, so it has a Taylor series:

$$g(z) = \sum_{n=0}^{\infty} a_n z^n$$

which converges uniformly on a disk: $D_r = \{z \in \mathbb{C} : |z| < r\}, r > 0$. Hence the original function $f$ is of the form:

$$f(z) = g\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n \frac{1}{z^n}$$

which converges uniformly outside the closed disk $\overline{D}_\frac{1}{r} = \{z \in \mathbb{C} : |z| \leq \frac{1}{r}\}$, i.e., on the area: $\{z \in \mathbb{C} : |z| > \frac{1}{r}\}$. 

Page 231, Problem 3. Find the power series expansion of the following functions around $\infty$:

a) $\frac{z - 1}{z + 1}$; b) $\frac{z^2}{z^2 + 1}$; c) $\frac{1}{z^3 - i}$.

Answer. As we are concerned with the behavior of the functions near $\infty$, we look at $z$ with large $|z|$ for all problems.

a) We simply compute:

$$\frac{z - 1}{z + 1} = 1 - \frac{2}{z + 1} = 1 - 2\frac{1}{z} \sum_{n=0}^{\infty} \left(-\frac{1}{z}\right)^n = 1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^n}{z^n}, \quad |z| > 1.$$ 

b) We compute:

$$\frac{z^2}{z^2 + 1} = \frac{1}{1 + \frac{1}{z^2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n}}, \quad |z| > 1.$$ 

c) This is also easily computed:

$$\frac{1}{z^3 - i} = \frac{1}{z^3} \frac{1}{1 - \frac{i}{z^3}} = \frac{1}{z^3} \sum_{n=0}^{\infty} \frac{i^n}{(z^3)^n} = \sum_{n=0}^{\infty} \frac{i^n}{z^{3(n+1)}}, \quad |z| > 1.$$ 

This answers the questions.
Page 231, Problem 1. The unit sphere:

\[ S^2 : \ x_1^2 + x_2^2 + x_3^2 = 1 \]

is projected to the extended complex plane \( \mathbb{C} \cup \{ \infty \} \) identified with the \( (x_1, x_2) \)-plane:

\[ \mathbb{P} : \ x_3 = 0 \]

by connecting each point \( (x_1, x_2, x_3) \in S^2 \) to the point on \( \mathbb{P} \) by the straitline line starting from the north pole \( (0,0,1) \), which is called the stereographic projection. Then

\[
\begin{align*}
x_1 &= \frac{2x}{|z|^2 + 1}, & x_2 &= \frac{2y}{|z|^2 + 1}, & x_3 &= \frac{|z|^2 - 1}{|z|^2 + 1}; \\
\end{align*}
\]

\[ z = \frac{x_1 + ix_2}{1 - x_3}. \]

Proof. In general the equation of the line connecting \( (x_1, x_2, x_3) \) and \( (y_1, y_2, y_3) \) is given by

\[
(p_1(t), p_2(t), p_3(t)) = t(x_1, x_2, x_3) + (1-t)(y_1, y_2, y_3), \quad t \in \mathbb{R}.
\]

With \( (y_1, y_2, y_3) = (0,0,1) \) and \( (p_1(t), p_2(t), p_3(t)) = (x,y,0) \) we solve the equation:

\[
x = tx_1, \quad y = tx_2, \quad 0 = tx_3 + (1-t),
\]

equivalently

\[
z = t(x_1 + ix_2), \quad tx_3 + (1-t) = 0.
\]

Hence

\[
|z|^2 = t^2(x_1^2 + x_2^2) = t^2(1 - x_3^2) = t^2 - t^2x_3^2 = t^2 - (t - 1)^2 = 2t - 1.
\]

\[
\therefore \quad t = \frac{|z|^2 + 1}{2}; \quad t - 1 = \frac{|z|^2 - 1}{2}
\]

\[
\therefore \quad x_1 = \frac{2x}{|z|^2 + 1}, \quad x_2 = \frac{2y}{|z|^2 + 1}, \quad x_3 = \frac{t - 1}{t} = \frac{|z|^2 - 1}{|z|^2 + 1}.
\]

The inverse map: \( (x_1, x_2, x_3) \in S^2 \mapsto z = x + iy \in \mathbb{C} \) is given by:

\[
\begin{align*}
tx_3 + (1-t) = 0 \quad &\Rightarrow \quad t(x_3 - 1) + 1 = 0 \quad \therefore \quad t = \frac{1}{1-x_3}; \\
x = tx_1 &= \frac{x_1}{1 - x_3}; & y = tx_x &= \frac{x_2}{1 - x_3}; \\
\therefore \quad z &= \frac{x_1 + ix_2}{1 - x_3}.
\end{align*}
\]

This completes the proof. \( \heartsuit \)
Page 251, Problem 1-a, 1-c, 1-h. Determine all the isolated singularities of each of the following functions and compute the residue of each singularity.

\[ a) \quad \frac{e^{3z}}{z-2}; \quad c) \quad \frac{\cos z}{z^2}; \quad h) \quad \frac{z-1}{\sin z}. \]

Answer. a) The singularities are \( z = 2 \) and \( z = \infty \) by inspection. The singularity \( z = 2 \) is a simple pole. So we get

\[
\text{Res} \left( \frac{e^{3z}}{z-2}, 2 \right) = \lim_{z \to 2} (z-2) e^{3z} = e^{3 \cdot 2} = e^6.
\]

The singularity \( z = \infty \) is an essential singularity. So we have to compute the power series expansion around \( \infty \). To this end, let

\[
f(z) = \frac{e^{3z}}{z-2} = \frac{ze^{3z}}{1-2z} = \left( \frac{1}{1-2z} \right) z e^{3z}.
\]

We want to find the residue of the essential singularity 0 of the function \( f \), which is determined by looking at the Laurent series of \( f \) near 0:

\[
f(z) = \left( \sum_{n=0}^{\infty} (2z)^n \right) z \sum_{n=0}^{\infty} \left( \frac{3^n}{n!z^n} \right)
\]

\[
= \left( 1 + 2z + (2z)^2 + \cdots \right) z \left( 1 + \frac{3}{z} + \frac{9}{2!z^2} + \frac{3^3}{3!z^3} + \cdots + \frac{3^m}{m!z^m} + \cdots \right)
\]

\[
= \left( z + 2z^2 + 4z^3 + \cdots + 2^{n-1}z^n + \cdots \right) \left( 1 + \frac{3}{z} + \frac{3^2}{2!z^2} + \cdots + \frac{3^n}{n!z^n} + \cdots \right)
\]

\[
= \left( \sum_{n=1}^{\infty} b_n z^n \right) \left( \sum_{m=0}^{\infty} c_m z^{-m} \right) = \sum_{n=0}^{\infty} a_n z^n
\]

with \( b_n = 2^{n-1}, \ n \geq 1 \) and \( c_m = \frac{3^m}{m!}, \ m \geq 0 \). The coefficients \( a_k \) are determined by the following:

\[
a_0 = \sum_{n=1}^{\infty} b_n c_{-n} = \sum_{n=1}^{\infty} \frac{2^{n-1}3^n}{n!} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{6^n}{n!} = \frac{e^6 - 1}{2};
\]

\[
a_k = \sum_{n=0}^{\infty} b_{n+k} c_{-n} = \sum_{n=0}^{\infty} \frac{2^{n+k-1}3^n}{n!} = 2^{k-1} \sum_{n=0}^{\infty} \frac{6^n}{n!} = 2^{k-1} e^6, \quad k \geq 1;
\]

\[
a_{-k} = \sum_{n=1}^{\infty} b_n c_{-n} = \sum_{n=1}^{\infty} \frac{2^{n-1}3^{n+k}}{(n+k)!} = \frac{3^k}{2} \sum_{n=1}^{\infty} \frac{6^n}{(n+k)!}, \quad k \geq 1.
\]
In particular, we get

\[
\text{Res}(f; 0) = a_{-1} = \frac{3}{2} \sum_{n=1}^{\infty} \frac{6^n}{(n+1)!} = \frac{3}{2 \cdot 6} \sum_{n=1}^{\infty} \frac{6^{n+1}}{(n+1)!}
\]

\[
= \frac{1}{4} (e^6 - 1 - 6) = \frac{e^6 - 7}{4};
\]

\[
\text{Res} \left( \frac{e^{3z}}{z - 2}; \infty \right) = \frac{e^6 - 7}{4}.
\]

c) The singularities of \( f(z) = \frac{\cos z}{z^2} \) are \( z = 0 \) and \( z = \infty \). As \( \lim_{z \to 0} f(z) = \infty \), the singularity \( z = 0 \) is a pole. The numerator \( \cos z \) does not vanish at \( z = 0 \), the order of the pole 0 is 2 and indeed we have

\[
\lim_{z \to 0} zf(z) = \lim_{z \to 0} \frac{\cos z}{z} = \infty; \quad \lim_{z \to 0} z^2 f(z) = \lim_{z \to 0} \cos z = 1.
\]

So we have the Laurent series expansion of \( f \) near 0:

\[
f(z) = \cos z z^2 = \frac{1 + a_{-1} z + a_0 + a_1 z + \cdots + a_n z^n + \cdots}{z^2} = \frac{z^2}{2} \left( \begin{array}{c} \sum_{n=0}^{\infty} \frac{i^n}{n! z^n} + \sum_{n=0}^{\infty} \frac{(-i)^n}{n! z^n} \end{array} \right)
\]

\[
z^2 f(z) = \cos z = 1 + a_{-1} z + z_0 z^2 + \cdots + a_n z^{n+2} + \cdots
\]

\[
a_{-1} = \frac{d}{dz} \left( z^2 f(z) \right) \bigg|_{z=0} = -\sin z \bigg|_{z=0} = -1;
\]

\[
\text{Res} \left( \frac{\cos z}{z^2}; 0 \right) = -1.
\]

To investigate the singularity \( \infty \) of \( f \), set \( g(z) = f(1/z) \), i.e.,

\[
g(z) = z^2 \cos \frac{1}{z} = \frac{z^2 (e^{\frac{i}{z}} + e^{-\frac{i}{z}})}{2}.
\]

Since \( g \) is bounded as \( z \to 0 \) along the real axis and unbounded as \( z \to 0 \) along the imaginary axis, 0 is an essential singularity of \( g \), i.e., \( \infty \) is an essential singularity of the original function \( f \).

To find the residue of \( f \) at \( \infty \), we expand \( g \) near the origin:

\[
g(z) = \frac{z^2}{2} \left( \sum_{n=0}^{\infty} \frac{i^n}{n! z^n} + \sum_{n=0}^{\infty} \frac{(-i)^n}{n! z^n} \right) = \frac{z^2}{2} \left( \sum_{n=0}^{\infty} \left( \frac{i^n}{n! z^n} + \frac{(-i)^n}{n! z^n} \right) \right)
\]

\[
= \frac{z^2}{2} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! z^{2n}} = \frac{z^2}{2} - \frac{1}{2} + \frac{1}{4! z^2} - \cdots.
\]

Thus a simple inspection yields that \( \text{Res}(g; 0) = 0 \). Hence the residue of \( f \) at \( \infty \) is 0.

h) The denominator \( \sin z \) of the function \( f(z) = \frac{z-1}{\sin z} \) is zero at \( z = n\pi, n \in \mathbb{Z} \) but the numerator \( z - 1 \) is not zero at these points. Hence these are singularities of \( f \) and indeed simple poles as
lim_{z \to n\pi} f(z) = \infty. Also z = \infty is a singularity. But as \lim_{n \to \infty} n\pi = \infty, the singularity \infty is not isolated. Thus it is an essential singularity. The residue of f at a non-isolated singularity is not defined. So we want just to calculate the residue of these poles.

Res \left( \frac{z - 1}{\sin z}; n\pi \right) = \lim_{z \to n\pi} \frac{(z - 1)(z - n\pi)}{\sin z} = (n\pi - 1) \lim_{z \to n\pi} \frac{z - n\pi}{\sin z} \left( \frac{1}{\cos z} \right)_{z = n\pi} = (-1)^n(n\pi - 1).

This completes the answer.

Page 251, 3-a, 3-d. Evaluate the integrals:

a) \oint_{|z|=5} \frac{\sin z}{z^2 - 4} \, dz;

d) \oint_{|z|=3} \frac{e^{iz}}{z^2(z - 2)(z + 5i)} \, dz.

Answer. a) The function \( f(z) = \frac{\sin z}{z^2 - 4} \) has simple poles at \( z = \pm 2 \), which are both inside the contour: \( |z| = 5 \). We first determine the residues of these poles:

\[
\text{Res}(f; 2) = \lim_{z \to 2} (z - 2) \frac{\sin z}{z^2 - 4} = \lim_{z \to 2} \frac{\sin z}{z + 2} = \frac{\sin 2}{4};
\]

\[
\text{Res}(f; -2) = \lim_{z \to -2} (z + 2) \frac{\sin z}{z^2 - 4} = \lim_{z \to -2} \frac{\sin z}{z - 2} = \frac{\sin(-2)}{-4} = \frac{\sin 2}{4}.
\]

By the Residue Theorem, we get

\[
\oint_{|z|=5} \frac{\sin z}{z^2 - 4} \, dz = 2\pi i \left( \text{Res}(f; 2) + \text{Res}(f; -2) \right) = \pi i \sin 2.
\]

d) The denominator’s zero of the function \( f(z) = \frac{e^{iz}}{z^2(z - 2)(z + 5i)} \) are at \( z = 0, 2 \) and \( z = 5i \), of which only \( z = 0 \) and \( z = 2 \) are inside the contour \( |z| = 3 \). The singularity \( z = 0 \) is a double pole of \( f \) and the singularity \( z = 2 \) is a simple pole. So we compute:

\[
\text{Res}(f; 0) = \frac{d}{dz} \left. z^2 f(z) \right|_{z=0} = \frac{d}{dz} \left. \frac{e^{iz}}{(z - 2)(z + 5i)} \right|_{z=0} = \frac{ie^{iz}(z - 2)(z + 5i) + e^{iz}(z - 5i) + e^{iz}(z - 2)}{(z - 2)^2(z - 5i)^2} \bigg|_{z=0} = \frac{i(-2)(-5i) + (-5i) + (-2)}{(-2)^2(-5i)^2} = \frac{12 + 5i}{100};
\]

\[
\text{Res}(f; 2) = \lim_{z \to 2} (z - 2) f(z) = \lim_{z \to 2} \frac{e^{iz}}{z^2(z - 5i)} = \frac{e^{2i}}{4(2 - 5i)} = \frac{e^{2i}(2 + 5i)}{4 \cdot 29}.
\]
Finally we evaluate the integral

\[
\int_{|z|=3} \frac{e^{iz}}{z^2(z - 2)(z - 5i)} \, dz = 2\pi i \left( \text{Res}(f; 0) + \text{Res}(f; 2) \right)
\]

\[
= 2\pi i \left( \frac{12 + 5i}{100} + \frac{e^{2i(2 + 5i)}}{4 \cdot 29} \right) = \pi i \left( \frac{12 + 5i}{50} + \frac{e^{2i(2 + 5i)}}{58} \right).
\]

This completes the answer.