Problem 1. Let $b$ and $c$ be two real numbers and consider the function $f$ on $\mathbb{R}$ defined by

$$f(x) = x^2 + 2bx + c, \quad x \in \mathbb{R}.$$ 

Let $A$ be the set defined by

$$A = \{ x \in \mathbb{R} : f(x) \geq 0 \}.$$ 

As the set $A$ depends on the parameters $b, c \in \mathbb{R}$, we write $A(b, c)$ for $A$ to indicate the dependence of the set $A$ on $(b, c) \in \mathbb{R}^2$. Answer to the following problems:

1-a) Show that $A(b, c) \neq \emptyset$ for every pair $(b, c) \in \mathbb{R}^2$.

Solution. The graph of the function $f(x) = x^2 + 2bx + c$ is an upward open parabola. The question is asking if the function $f$ takes non-negative value at some point in $\mathbb{R}$. Of course, it takes positive values at a certain value of $x$ regardless of what values of $b$ and $c$ are assigned for $f$. For example if $c \geq 0$, then $f(0) = c \geq 0$, so that $0 \in A(b, c)$, and if $c < 0$ then $b^2 - c > 0$, so that $\sqrt{b^2 - c}$ is a real number and consequently $f(-b + \sqrt{b^2 - c}) = 0$ and $-b + \sqrt{b^2 - c} \in A(b, c)$. Therefore in either case, $A(b, c)$ is non-empty. ☺
1-b) Let $B$ be the set of pairs $(b, c) \in \mathbb{R}^2$ such that $A(b, c) = \mathbb{R}$. Draw the set $B$ on the $bc$-plane:

Solution. First we observe that 

$$A(b, c) = \mathbb{R}$$

means exactly $f(x) \geq 0$ for every $x \in \mathbb{R}$. Thus the set $B$ of $(b, c)$ such that $A(b, c) = \mathbb{R}$ is the exactly the set of pairs $(b, c)$ such that 

$$x^2 + 2bx + c \geq 0$$

for every $x \in \mathbb{R}$. As the function $f$ can be written in the form:

$$f(x) = (x + b)^2 + c - b^2, \quad x \in \mathbb{R},$$

for $f(x)$ to be non-negative everywhere it is necessary and sufficient that $c - b^2 \geq 0$. Thus the graph of the set $B$ looks like the above shaded area with $c$-axis vertical and $b$-axis horizontal. The boundary curve $c = b^2$ is included in the set $B$.

Problem 2. Let $\{a_n\}$ and $\{b_n\}$ be two convergent sequences of real numbers such that 

$$\lim_{n \to \infty} a_n = a = \lim_{n \to \infty} b_n.$$

Define a new sequence $\{c_n\}$ by the following:

$$\left\{ \begin{array}{ll}
c_{2n-1} &= a_n, \quad n = 1, 2, \cdots; \\
c_{2n} &= b_n, \quad n = 1, 2, \cdots.
\end{array} \right.$$ 

Prove that the new sequence $\{c_n\}$ converges to $a$.

Proof. From the convergence: $\lim a_n = a$, it follows that for any $\varepsilon > 0$ there exists $N_a(\varepsilon) \in \mathbb{N}$ such that 

$$|a_n - a| < \varepsilon \quad \text{for every } n \geq N_a(\varepsilon).$$
Similarly for any \( \varepsilon > 0 \) there exists \( N_b(\varepsilon) \in \mathbb{N} \) such that
\[
|b_n - b| < \varepsilon \quad \text{for every } n \geq N_b(\varepsilon).
\]
Suppose \( n \geq 2 \max\{N_a(\varepsilon), N_b(\varepsilon)\} \). Then we have \( n/2 \geq N_b \) for even \( n \) and \( (n + 1)/2 \geq N_a \) for odd \( n \) and also
\[
c_n = \begin{cases} 
a_{(n+1)/2} & \text{if } n \text{ is odd;} \\
b_{n/2} & \text{if } n \text{ is even.}
\end{cases}
\]
Hence we conclude that
\[
|c_n - a| < \varepsilon \quad \text{if } n \geq 2 \max\{N_a(\varepsilon), N_b(\varepsilon)\}.
\]
Thus \( N_c(\varepsilon) = 2 \max\{N_a(\varepsilon), N_b(\varepsilon)\} \) serves as the waiting time for \( \{c_n\} \) to come closer to \( a \) within \( \varepsilon \) distance.

**Problem 3.** Prove the identity
\[
1^3 + 2^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2, \quad n \in \mathbb{N},
\]
by mathematical induction.

**Proof.** For \( n = 1 \), the both sides of the formula equal to 1. So it holds definitively for \( n = 1 \).

Suppose that
\[
1^3 + 2^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2.
\]
We add \( (n+1)^3 \) to the left side and obtain
\[
(1^3 + 2^3 + \cdots + n^3) + (n+1)^3 = (1 + 2 + 3 + \cdots + n)^2 + (n+1)^3
\]
by the induction assumption
\[
= \left\{ \frac{n(n+1)}{2} \right\}^2 + (n+1)^3 \quad \text{by Proposition 1.4.3}
\]
\[
= \left\{ \frac{n^2(n+1)^2}{4} \right\} + (n+1)^2(n+1)
\]
\[
= (n+1)^2 \left\{ \frac{n^2}{4} + (n+1) \right\} = (n+1)^2 \frac{n^2 + 4n + 4}{4}
\]
\[
= (n+1)^2 \frac{(n+2)^2}{4} = \left\{ \frac{(n+1)(n+2)}{2} \right\}^2
\]
\[
= (1 + 2 + 3 + \cdots + n + (n+1))^2 \quad \text{by Proposition 1.4.3}.
\]
The last term is exactly the right hand side of the formula for \( n + 1 \). Therefore the mathematical induction guarantees the validity of the formula for all natural number \( n \).