Midterm 1 Review

Rate of change

We are interested in finding the rate of change of a function. In particular, given a function \( y = f(x) \) we are interested in finding how fast \( y \) is changing with respect to \( x \) at some fixed time \( x = a \). The main way we will do this is to combine two observations.

1. Given a line \( y = mx + b \) the slope is \( m \) and is found by
   \[
   m = \frac{\text{rise}}{\text{run}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}.
   \]
   In other words, for a line the rate of change of \( y \) with respect to \( x \) is found by looking at the slope.

2. For a typical function we will see in our class when we look at the function “near” \( x = a \) it will look like a line, namely the tangent line. (The tangent line is the line which touches “without crossing” the curve.)

So to find the rate of change we will find the slope of the tangent line. But, to calculate the slope we need the limit as \( \Delta x \) approaches 0 which are undefined, so we need some way to handle this.

Limits

The way we handle this is to use limits. Intuitively limits tell us what should happen based on what is happening nearby. So for example

\[
\lim_{x \to c} g(x) = L,
\]

which we read “the limit as \( x \) goes to \( c \) of \( g(x) \) is \( L \)”, means that as \( x \) gets close to \( c \) the function \( g(x) \) is getting close to \( L \) (and staying close!). It is possible that the limit does not exist. For example,

\[
\lim_{x \to 0} \left( \frac{1}{x} \right) = \text{Does not exist}.
\]

To see this we note that the function \( \sin(1/x) \) will do “infinitely” many oscillations between 1 and \(-1\) around \( x = 0 \) and so it does not approach a single fixed \( L \).

One way to guess a limit is to plug in values of \( x \) closer and closer to \( c \) and see if it is approaching some certain value; we can also try plotting a picture of \( g(x) \) near \( x = c \) and seeing how the function is behaving. Both of these methods have shortcomings. In particular, they are hard to do without a calculator and can sometimes be deceiving. So we want to have some methods to deal with these limits. One method is to build up a collection of rules that we can use. For example we have the following two rules

\[
\lim_{x \to c} K = K \quad \text{and} \quad \lim_{x \to c} x = c.
\]

(The first follows by noting that \( K \) is always close to \( K \), and the second says “as \( x \) gets close to \( c \) then \( x \) gets close to \( c \).) On the other hand we have that if

\[
\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M
\]

and \( L,M \) are finite then we have the following rules, which essentially say that limits do what we think they should do.

1. \[ \lim_{x \to c} (f(x) + g(x)) = (\lim_{x \to c} f(x)) + (\lim_{x \to c} g(x)). \]
2. \[ \lim_{x \to c} (kf(x)) = k(\lim_{x \to c} f(x)) \]
3. \[ \lim_{x \to c} (f(x)g(x)) = (\lim_{x \to c} f(x))(\lim_{x \to c} g(x)). \]
4. \[ \lim_{x \to c} \left( \frac{f(x)}{g(x)} \right) = \frac{(\lim_{x \to c} f(x))}{(\lim_{x \to c} g(x))} \quad (\text{when } M \neq 0). \]

In the definition of limits we look at what happens as \( x \to c \), but there are two ways that \( x \) can approach \( c \). Namely we can approach it from below (i.e., \( x < c \)) or we can approach it from above (i.e., \( x > c \)). Sometimes it is convenient to limit ourselves to one direction when evaluating the limit, which leads to one-sided limits.

\[
\lim_{x \to c^-} g(x) \leftrightarrow \text{limit as we approach } c \text{ from below} \\
\lim_{x \to c^+} g(x) \leftrightarrow \text{limit as we approach } c \text{ from above}
\]

These are also known respectively as the left limit and the right limit. As an example we have

\[
\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{-x}{x} = \lim_{x \to 0^+} (-1) = -1; \quad \text{and} \quad \lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{x}{x} = \lim_{x \to 0^-} 1 = 1.
\]
When the left and right limits do not agree then the limit does not exist. So in this case we say that the limit does not exist (this example can be used to show that the derivative of \( y = |x| \) does not exist at \( x = 0 \)). In general, when dealing with piece-wise functions (such as \( |x| \)) it is convenient to use one-sided limits.

[In class we also discussed limits involving \( \infty \), there will be no limits involving \( \infty \) on the test.]

**Continuous functions**

Closely related to the idea of limits is the idea of a continuous function. A function is continuous if it has "no breaks". Another way to say it is the function is continuous at \( x = c \) if what we expect to happen at \( x = c \) is what actually does happen, i.e.,

\[
\lim_{x \to c} f(x) = f(c).
\]

In particular, three things need to happen to be continuous: (1) \( f(c) \) must be defined; (2) the limit must exist; (3) they have to agree. So lots of things can go wrong which leads to different types of discontinuities.

- **Removable discontinuity** The limit exists but either the function is not defined or the value of the function does not match the limit. The name removable comes from the idea that we can redefine the function at the point and we would no longer have a discontinuity.

- **Jump discontinuity** The left and right hand limits exist but are not equal.

- **Infinite discontinuity** The left, right, or both limits are \( \pm \infty \).

- **Severe** Another possibility is that the limit does not exist (for example \( \sin(1/x) \) at \( x = 0 \)).

A function is left (respectively, right) continuous if

\[
\lim_{x \to c^-} f(x) = f(c) \quad \text{(respectively } \lim_{x \to c^+} f(x) = f(c)\text{)}.
\]

Examples of continuous functions include polynomials, \( x^a \) (in its domain), \( \sin x \), \( \cos x \), \( \tan x \) (away from the vertical asymptotes), \( \ln x \) and \( e^x \) (the last two are functions that we will encounter next quarter and will not be tested on this quarter). These form the building blocks of continuous functions and then we might ask for how can we combine continuous functions together. In particular we have that if \( f(x) \) and \( g(x) \) are continuous then so are \( f(x) + g(x) \), \( k f(x) \), \( f(x) g(x) \), \( f(x)/g(x) \) (when \( g(x) \neq 0 \)) and \( f(g(x)) \) (the composite function also denoted as \( (f \circ g)(x) \)).

All of the functions that we will encounter are built up using the basic building blocks and rules for combining. The only other thing that might happen is that we might have a function which is defined piecewise. When this is the case, the interesting question usually occurs at when two pieces meet.

The nice thing about a limit involving a continuous function is that we can plug in the point we are taking the limit to, i.e., \( c \) into the function. If we get a number out then we are done. If we get \( 0/0 \) then that means we need to work on the limit some more using various techniques such as the Squeeze Theorem of algebraic manipulation.

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**Squeeze Theorem**

One way to find a limit of a function that we do not understand is to put it between two functions that we do understand that come together. In particular if we have

\[
\ell(x) \leq f(x) \leq u(x)
\]

for \( x \) near \( c \) and

\[
\lim_{x \to c} \ell(x) = \lim_{x \to c} u(x) = K \quad \text{then } \lim_{x \to c} f(x) = K.
\]

For example, it is possible to show by an area argument that

\[
cos x \leq \frac{\sin x}{x} \leq 1,
\]

and since

\[
\lim_{x \to 0} \cos x = 1 = \lim_{x \to 0} 1,
\]

we can conclude

\[
\lim_{x \to 0} \frac{\sin x}{x} = 1.
\]

Another example is

\[
\lim_{x \to 0} x^2 \sin \left( \frac{1}{x} \right).
\]

Since the sine function is always bounded between 1 and \(-1\) then we have that

\[-x^2 \leq x^2 \sin \left( \frac{1}{x} \right) \leq x^2
\]

and since \(-x^2\) and \(x^2\) both go to 0 as \( x \to 0 \) then the limit that we are interested in is also 0.

[Note that you will not be asked to produce the proof that the limit of \( \sin(x)/x \) is 1 as \( x \to 0 \) on the test. But you might be asked a question for which using the Squeeze Theorem is very helpful.]

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**Algebraic manipulation of limits**

When our limit is going to 0/0 (or possibly \( \infty/\infty \), \( \infty - \infty \), \( 0 \cdot \infty \)) then we have an ambiguous number since \( 0/0 \) is undefined. Most of the time when we encounter this we will try to manipulate what we are taking the limit of, the goal being to “cancel the 0s” (i.e., rewrite it in such a way that we can cancel a common term from top and bottom so that what remains does not go to \( 0/0 \)). There are three main techniques we will be using.
1. **Rewriting/factoring.** This is usually done when we have a polynomial and we can either expand the polynomials out or factor (or sometimes both). As an example

\[
\lim_{x \to 1} \frac{x^2 + x - 2}{(x+2)^2 - 9} = \lim_{x \to 1} \frac{x^2 + x - 2}{x^2 + 4x - 5} = \lim_{x \to 1} \frac{x^2 + 2}{(x-1)(x+5)} = \lim_{x \to 1} \frac{x+2}{x+5} = \frac{3}{6} = \frac{1}{2}.
\]

2. **Multiplying by the conjugate.** The conjugate of an expression \(a + b\) is \(a - b\). So if we multiply both top and bottom by the conjugate of \(a - b\) and then multiply out we get \(a^2 - b^2\) (this can be helpful for instance in getting rid of square roots). The reason we have to multiply both top and bottom is so that we do not change the limit, i.e., multiplying by 1 does not change the value of the limit. As an example

\[
\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} \left( \frac{\sqrt{x} + 2}{\sqrt{x} + 2} \right) = \lim_{x \to 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{4}.
\]

3. **Using identities.** This is most commonly done with limits involving trigonometry in which case there are often many identities which we can use to rewrite (and hopefully cancel!) the terms. The most commonly used identity being \(\sin^2 x + \cos^2 x = 1\). As an example

\[
\lim_{\theta \to 0} \frac{\sin^2 \theta}{1 - \cos \theta} = \lim_{\theta \to 0} \frac{1 - \cos^2 \theta}{1 - \cos \theta} = \lim_{\theta \to 0} \frac{(1 - \cos \theta)(1 + \cos \theta)}{1 - \cos \theta} = \lim_{\theta \to 0} (1 + \cos \theta) = 2.
\]

### Tangent lines

Now that we have limits we see that the slope of the tangent line at \(x = a\) can be found by taking the limit of slopes of secant lines between \(a\) and a point that is moving closer to \(a\). We call the slope of the tangent line at \(x = a\) and denote it by \(f'(a)\). So we have

\[f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to a} \frac{f(b) - f(a)}{b - a}.
\]

(The second definition is an alternative definition, but is based on the same idea where we take the limit of slopes of secant lines.) We note that if \(f(x)\) and \(x\) have units then the units of \(f'(a)\) are (the units of \(f\))/the units of \(x\); as an example if \(f\) measures cost in dollars and \(x\) measures number of items then the units of the derivative would be (dollars)/(item) (or “dollars per item”).

Once we have the slope of the tangent line it is easy to find the tangent line (since we also have the point \((a, f(a))\). Namely, since in general a line has the form

\[y - y_0 = m(x - x_0)\]

then we can substitute \((x_0, y_0) = (a, f(a))\) and \(m = f'(a)\) to get

\[y - f(a) = f'(a)(x - a)\] or \[y = f(a) + f'(a)(x - a)\].

Of course we can also recover information if we have the tangent line. For instance, if we know the tangent line, and at what value of \(x = a\) it is tangent at, we can recover \(f(a)\) and \(f'(a)\) (by looking at the \(y\) coordinate of the tangent line at \(x = a\) and the slope, respectively). But it is also important to note that this is all the information that we know about \(f(x)\), i.e., we cannot say anything about \(f\) for \(x\) away from \(a\).

We can also think of the derivative as a function, where evaluating the function at any point gives us the slope of the tangent line. In general we have

\[\frac{d}{dx} (f(x)) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.
\]

(The notation “\(d/dx\)” comes from Leibniz and should be read as “the derivative with respect to \(x\”).)

Actually using the limit definition to calculate the derivative is tedious and can get quickly complicated. So we build up a collection of rules to allow us to calculate the derivatives of functions.

1. \[\frac{d}{dx} (1) = 0.\]
2. \[\frac{d}{dx} (x) = 1.\]
3. \[\frac{d}{dx} (x^a) = ax^{a-1}.\]
4. \[\frac{d}{dx} (f(x) + g(x)) = f'(x) + g'(x).\]
5. \[\frac{d}{dx} (kf(x)) = kf'(x).\]
6. \[\frac{d}{dx} (f(x)g(x)) = f'(x)g(x) + f(x)g'(x).\]
7. \[\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.\]

The first three rules tell us how to take the derivative of \(x^a\), the basic idea is to bring the power of \(x\) down in front and then subtract 1 from the power (note if you ever encounter a \(\sqrt{x}\) or similar term it is best to rewrite as \(x^{1/2}\) before applying the rule). The remaining rules give us ways to take the derivative of combinations of functions. For instance the fourth rule says if we have two functions being added (or subtracted) then to take the derivative we take the
derivative of each piece and add the result. The fifth rule says if we have a constant times a function the derivative is the same constant times the derivative of the function. The sixth rule tells us how to take the derivative of a product, hence it is known as the product rule. Similarly, the seventh rule tells us how to take the derivative of a quotient, hence it is known as the quotient rule.

There is one other way to combine functions and that is by composition. This leads to a rule for derivatives known as the chain rule, but we have not covered that and it will appear on a future exam.

Once we have the derivatives of the basic functions and the rules for how to take derivatives of various ways of combining functions we will be able to take derivatives of all the functions that we encounter. That is why it is important for us to learn these rules!

Using the derivative

The reason that we are going through all of this is that we can use the derivative of a function to give us information about the function. For instance, by knowing the slopes of the tangent lines we can know whether the function is increasing or decreasing (i.e., where the function is going “up” the slopes of the tangent lines, or the derivative, is positive and where the function is going “down” the slopes of the tangent lines, or the derivative, is negative). This helps us to graph a curve and let us identify important features on the curve (we will discuss this in more detail before the next test).

Derivatives arise frequently in physics. Before we discuss an example let us introduce higher order derivatives. Since \( f'(x) \) is a function of \( x \) we can take its derivative, we get a function denoted \( f''(x) \) which is the second derivative of \( x \). While \( f'(x) \) tells us something about how the function is changing (via slopes of tangent lines) then \( f''(x) \) also tells us something about how the function looks by seeing how the slopes of the tangent lines are changing. (This last idea is known as concavity and we will talk about it again in a future lecture; you will not be tested on concavity on the midterm.)

Of course, we can take the derivative of the second derivative and get what is known as the third derivative; and yet another derivative would give us the fourth derivative and so on. In general, the \( n \)th derivative of \( f(x) \) is denoted by \( f^{(n)}(x) \) or in Leibniz notation \( d^n f(x)/dx^n \).

As an example of this, suppose that we have a function \( s(t) \) which measures distance in some units. Then the first derivative, \( s'(t) \), is how position is changing and is called the velocity, while the second derivative, \( s''(t) \) is how the velocity is changing and is called the acceleration.