Notes on Fixed-Point Iteration

Math 151A, Fall 2015

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If for some function $g(x)$, we can find $p$ such that $p = g(p)$, then we say $p$ is a fixed point of $g$.

Given a nonlinear equation $f(x) = 0$, we may first find a function $g$ so that $x = g(x)$ is an equivalent form of $f(x) = 0$. For example, $f(x) = x^2 - x - 2 = 0$ can be rewritten as $x = x^2 - 2$ or $x = \sqrt{x + 2}$ ($g(x) = x^2 - 2$ and $g(x) = \sqrt{x + 2}$, respectively). The fixed-point iteration has the following update rule, given an initial iterate $x_0$:

$$x_n = g(x_{n-1}).$$

(1)

This algorithm only converges when the derivative of $g$, $g'(x)$, has magnitude less than 1 in some neighborhood of the exact solution.

**Theorem.** Let $g \in C[a,b]$, $g(x) \in [a,b]$, then $g$ has at least one fixed point in $[a,b]$. If $|g'(x)| \leq K < 1$ for all $x \in (a,b)$, then for any $x_0 \in [a,b]$, the sequence of iterates defined by $x_n = g(x_{n-1})$ converges to a fixed point in $[a,b]$.

**Proof.** (1) We know $g(a) - a \geq 0, g(b) - b \leq 0$. Then we must have $p \in [a,b]$ such that $g(p) - p = 0$ (Mean Value Theorem), and therefore $p$ is a fixed point.

(2) Let $x^* \in [a,b]$ be an exact solution. Then

$$|x_n - x^*| = |g(x_{n-1}) - g(x^*)|$$

$$= |g'(\xi_n)| \cdot |x_{n-1} - x^*| \quad \text{(Mean Value Theorem)}$$

$$\leq K|x_{n-1} - x^*|$$

for some $\xi_n \in (a,b)$. Applying the above inequality repeatedly, we have $|x_n - x^*| \leq K|x_{n-1} - x^*| \leq \cdots \leq K^n|x_0 - x^*|$. Since $K < 1$, the error $|x_n - x^*|$ goes to zero as $n$ goes to infinity.

**Corollary.** Let $g \in C^1[a,b]$. If $x^* \in (a,b)$ is a fixed point of $g(x)$, and $|g'(x^*)| < 1$, then there exists a sequence of iterates defined by $x_n = g(x_{n-1})$ that converges to $x^*$. 
