Problem 1. Compute the determinant of this intimidating looking matrix $M = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \\ 21 & 22 & 23 & 24 & 25 \end{pmatrix}$. [Hint: How does row operation help here?] 

Problem 2. Consider the quadrilateral in the $\mathbb{R}^2$ plane formed with these vertices: $(1,5)$, $(2,7)$, $(5,0)$, $(3, -3)$. Find its area.

Problem 3. Consider the system of equation $\begin{pmatrix} 2 & 1 & 8 & 7 \\ 2 & 2 & 6 & 2 \\ 1 & 0 & 2 & 1 \\ 1 & 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

Solve for $z$ directly.

Problem 4. Diagonalize the matrix $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, namely, find an eigenvectors of $\mathbb{R}^3$ for the matrix $A$. Can $A$ be orthogonally diagonalizable? If so, produce an orthonormal basis of $\mathbb{R}^3$ for this matrix $A$.

Problem 5. Consider the matrix $B = \begin{pmatrix} 2 & k \\ 2 & 2 \end{pmatrix}$, where $k$ is some real number. Find all real values of $k$ such that $B$ is invertible. Find all real values of $k$ such that $B$ is diagonalizable over the reals.

Problem 6. Consider the matrix $C = \begin{pmatrix} 1 & k & k^2 \\ 0 & 2 & k^3 \\ 0 & 0 & 3 \end{pmatrix}$. Find all real values of $k$ such that $C$ is invertible. Find all real values of $k$ such that $C$ is diagonalizable.

Problem 7. Let $A$ be a $3 \times 3$ matrix with eigenvalues $\lambda = -2, 1, 2$. Using this information, complete the following chart for each $3 \times 3$ matrices below, if defined at all:

<table>
<thead>
<tr>
<th>$A$</th>
<th>Invertible?</th>
<th>Eigenvalues?</th>
<th>Diagonalizable?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^{-1}$</td>
<td>Yes</td>
<td>$-2, 1, 2$</td>
<td>Yes</td>
</tr>
<tr>
<td>$A^2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A + I$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A - 2I$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A^T$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$A^{-1} + A$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Problem 8. (1) Show that for any $n \times n$ square matrix $A$, both $A$ and its transpose $A^T$ have the same eigenvalues with exactly the same algebraic multiplicity. [Hint: Eigenvalues are from roots of characteristic polynomial.]

(2) Show that if two $n \times n$ matrices $A$ and $B$ are similar, then they have the same eigenvalues with exactly the same algebraic multiplicity.

Problem 9. Suppose $A$ is a square matrix such that $A^2 = A$. Show that any nonzero vector $x \in \text{Im}(A)$ is in fact an eigenvector of $A$. What is the corresponding eigenvalue of this nonzero vector $x \in \text{Im}(A)$?

Problem 10. Find examples of matrices that satisfies the following table:

<table>
<thead>
<tr>
<th>Diagonalizable</th>
<th>Invertible</th>
<th>Not invertible</th>
</tr>
</thead>
<tbody>
<tr>
<td>No diagonalizable</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Problem 11. Let $A$ be an $n \times n$ matrix that is diagonalizable. Show that $A^2$ is also diagonalizable. In fact, show for any positive power $k$, the matrix $A^k$ will be diagonalizable.

Problem 12. Suppose $A$ is $3 \times 3$ such that $A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ and $\ker(A) = \text{span} \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$. Find $A$.

Problem 13. Find a $3 \times 3$ symmetric matrix $A$ such that $A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ and that dim($\ker(A)$) = 2.

Problem 14. Find a $4 \times 4$ symmetric matrix $A$ such that $A \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$ and $A \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

Problem 15. (1) Show that for any $n \times k$ matrix $A$ (not necessarily square!), both matrices $AA^T$ and $A^T A$ are diagonalizable.

(2) Show that any $n \times n$ orthogonal projection matrix $P$ is diagonalizable. [Hint: Recall QR-factorization, and how we can re-express every orthogonal projection matrix, and then use (1) of this problem.]

### On determinants.

Problem 13. Let $A$ be an $n \times n$ symmetric matrix. Suppose that $B$ is a symmetric matrix such that $A - B$ is invertible. Find all real values of $k$ such that $B = kA$ is diagonalizable.
Problem 16. Let $A$ and $B$ both be $n \times n$. (1) Find examples of $A$ and $B$ where both are not invertible, but their sum $A + B$ is invertible.

(2) Find examples of $A$ and $B$ where both are not diagonalizable, but their sum $A + B$ is diagonalizable.

Problem 17. We know spectral theorem tells us that any real square symmetric matrix is orthogonally diagonalizable. Prove the converse statement, that is: If a real square matrix $A$ has an orthonormal eigenbasis, then it is symmetric. [Hint: If it has an orthonormal eigenbasis, how can we factorize this matrix $A$? What kind of matrices are the “invertible factors” in $A$?]

Problem 18. Consider this intimidating looking matrix $M = \begin{pmatrix} 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 \end{pmatrix}$. Write down its characteristic polynomial. [Hint: Can you do this without actually calculate a determinant?]

Problem 19. [On matrix power and matrix power limits.]

(1) Find a closed form formula for the vector $A^k \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, where $k$ is some positive integer. (This means express $A^k$ as a $3 \times 3$ matrix where each entry is some function of $k$.)

(2) Find a closed form formula for the matrix power limit $\lim_{k \to \infty} A^k$.

Problem 20. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, note $A$ is a transition matrix (why?). Is this transition matrix $A$ a positive transition matrix? Is this transition matrix $A$ a regular transition matrix?

Problem 21. Let $A = \begin{pmatrix} 0 & 0.5 & 1 \\ 1 & 0 & 0 \\ 0 & 0.5 & 0 \end{pmatrix}$, compute the matrix power limit $\lim_{k \to \infty} A^k$. [Hint: Is this matrix a regular transition matrix?]

(2) Compute $\lim_{k \to \infty} \begin{pmatrix} 0.2 \\ 0.3 \\ 0.5 \end{pmatrix}$.

(3) Compute $\lim_{k \to \infty} \begin{pmatrix} 4 \\ 9 \\ 3 \end{pmatrix}$.

Problem 22. Find a regular transition matrix that is not invertible. Find an invertible matrix that is not a regular transition matrix.

Problem 23. Recall that the theorem of regular transition matrix tells us that if a matrix $A$ is a regular transition matrix, then its matrix power limit $\lim_{k \to \infty} A^k$ always exist. This is a sufficient condition, however, but not a necessary condition for the matrix power limit to exist. Meaning: There is a transition matrix that is NOT regular, but its matrix power limit exist. Find an example of such transition matrix. [Hint: Think of the “easiest” example of a transition matrix.]

Problem 24. Show that every $2 \times 2$ regular transition matrix is diagonalizable. [Hint: What do we know about the eigenvalues of a regular transition matrix by the theorem?]

[Note that a regular transition matrix is not in general diagonalizable, here is an example: $A = \begin{pmatrix} 2/5 & 2/5 & 1/5 \\ 1/5 & 2/5 & 1/5 \\ 2/5 & 1/5 & 3/5 \end{pmatrix}$. Confession: I have to look one up.]

*** On complex numbers; matrices with complex eigenvalues. ***

Problem 25. Let $z = 2 + 4i$. Compute $z^3$. [Hint: Polar form and DeMoivre’s formula.]

Problem 26. (1) Find all complex numbers $z$ such that $z^3 = 1$.

(2) Find all complex numbers $w$ such that $w^3 = 3$.

Problem 27. Find all complex numbers $z$ such that $z^2 = 2 + 4i$.

Problem 28. Diagonalize the matrix $A = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}$ over the complex numbers. That is, find an eigenbasis for $\mathbb{C}^2$ for $A$. Factorize $A$ as $PDP^{-1}$ where $D$ is a diagonal matrix.

Problem 29. Factorize the matrix $A = \begin{pmatrix} 2 & -2 \\ 1 & 0 \end{pmatrix}$ as $PRP^{-1}$, where $R = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, a scaling-rotation matrix with both $a$ and $b$ real, and $P$ a real invertible matrix.

*** On quadratic forms. ***

Problem 30. Find a symmetric matrix $S$ such that $q(x) = 3x^2 - 3xy + xz + 4yz + 9y^2 - 3z^2$ can be expressed as $q(x) = x^T S x$, where $x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. 
Problem 31. Suppose we have a quadratic form \( q(\vec{x}) = \vec{x}^T S \vec{x} \) with \( S \) some symmetric matrix is such that \( q(\vec{a}) = 0 \) for some nonzero vector \( \vec{a} \). Can \( S \) be invertible? If so, show an example that it can be. If not, prove why not.

Problem 32. Let \( q(x, y) = x^2 + kxy + y^2 \) be a quadratic form for some real number \( k \). Find all real numbers \( k \) such that \( q(x, y) \) is positive definite.

Problem 33. Does the function \( q(x, y) = 3x^2 + 2xy + 3y^2 \) have a unique minimum? Does it have a unique maximum? If so, find these points \((x, y)\) that \( q \) has a maximum or minimum.

*** Some True or False. ***

1. There is a system of linear equations over \( \mathbb{R} \) that has exactly two solutions.
2. If set \( S \) contains only one vector in \( \mathbb{R}^n \), then \( S \) is a linearly independent set.
3. If \( T : \mathbb{R}^k \rightarrow \mathbb{R}^n \) is a linear transformation, and \( \{ v_1, \ldots, v_l \} \) is linearly independent in \( \mathbb{R}^k \), then \( \{ Tv_1, \ldots, Tv_l \} \) is linearly independent in \( \mathbb{R}^n \).
4. If a matrix \( A \) has trivial kernel, i.e., \( \text{Ker}(A) \) contains just the zero vector, then \( A \) is invertible.
5. A basis for \( \mathbb{R}^7 \) can consist of less than 7 vectors.
6. Suppose \( v_1 \) and \( v_2 \) are two vectors in \( \mathbb{R}^3 \). It is possible to get \( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 1/3 \\ 2/3 \\ 2/3 \end{pmatrix} \) as a result of Gram-Schmidt on \( v_1 \) and \( v_2 \) with respect to the dot product as inner product.
7. For any matrix \( A \), we always have \( \text{Im}(A) = \text{Ker}(A^T) \).
8. Given any inconsistent system \( Ax = b \), its least square solution found by solving the normal equation is always unique.
9. There exists an invertible matrix \( A \) whose square \( A^2 \) is the zero matrix.
10. If a square matrix \( A \) has \( \det(A) = 11 \), then we must have \( \det(3A) = 33 \).
11. For any two \( n \times n \) matrices \( A, B \), we have \( \det(A + B) = \det(A) + \det(B) \).
12. If \( A \) is an invertible square matrix with integer entries, then its inverse \( A^{-1} \) must also contain just integer entries.
13. There is a square matrix \( A \) that has only integer entries and determinant 1, but with inverse \( A^{-1} \) having non-integer entries.
14. If we perform row operations to a matrix \( A \), and got some matrix \( B \), then \( A \) and \( B \) will always have the same determinant.
15. If we perform row operations to a matrix \( A \), and got some matrix \( B \), then \( A \) and \( B \) will always have the same image.
16. If we perform row operations to a matrix \( A \), and got some matrix \( B \), then \( A \) and \( B \) will always have the same eigenvalues.
17. If a square matrix \( A \) has a diagonal that are all zeroes, then regardless of what the other entries are, \( A \) has determinant of zero and hence is not invertible.
18. Let \( A \) be a \( 3 \times 5 \) matrix. If the kernel of \( A \) consists of at least 3 linearly independent vectors, then it is possible for \( \text{im}(A) = \mathbb{R}^3 \).
19. If an \( n \times n \) square matrix \( A \) has only one eigenvalue but repeated \( n \) times (that is, algebraic multiplicity is \( n \)), then \( A \) cannot be diagonalizable.
20. A diagonalizable matrix must be a diagonal matrix.
21. If a matrix is invertible, then it must be diagonalizable.
22. The sum of the geometric multiplicities of all the eigenvalues of an \( n \times n \) matrix always equal to \( n \).
23. Zero cannot ever be an eigenvalue of a square matrix.
24. Every vector in an eigenspace of a square matrix \( A \) is an eigenvector of \( A \).
25. There exists a linear transformation \( T : \mathbb{R}^5 \rightarrow \mathbb{R}^5 \) with eigenvalues 1 and \( -2 \), where the eigenspaces \( E_1 \) and \( E_{-2} \) are both dimension 3.
26. There is a square real symmetric matrix that is not diagonalizable.
27. There exists a quadratic form \( q(\vec{x}) = \vec{x}^T S \vec{x} \) that never equals to 0 for all input vector \( \vec{x} \).
28. A quadratic form is a linear transformation.
29. For \( \vec{x} \in \mathbb{R}^2 \), the function \( q(\vec{x}) = \vec{x}^T \begin{pmatrix} 2 & -1 \\ 7 & 8 \end{pmatrix} \vec{x} \) is not a quadratic form.
30. There exists a positive definite quadratic form \( q(\vec{x}) = \vec{x}^T S \vec{x} \) for some symmetric matrix \( S \) such that \( S \) is not invertible.
31. If \( S \) is a symmetric matrix such that \( \det(S) > 0 \), then \( q(\vec{x}) = \vec{x}^T S \vec{x} > 0 \) for all nonzero vector \( \vec{x} \).
32. Suppose \( M \) is a \( 5 \times 5 \) matrix and the matrix \( M - 2I \) is invertible, then 2 is an eigenvalue of \( M \).
33. Suppose \( M \) is a \( 5 \times 5 \) matrix and the matrix \( M - 5I \) has a rank of 3, rank\((M - 5I) = 3\), then 5 is an eigenvalue of \( M \).