Mathematics 33A Review Sheet Brief Answers. (3/12/2016)
(Note these are just answers, and does not show all the work. You want to be able to show all work on an exam.)

BSL

*** On determinants. ***

Problem 1. Answer. 0.
Problem 2. Answer. 35/2.
Problem 3. Answer. z = 0.

*** On diagonalization and spectral theorem.***

Problem 4. Answer. An eigenbasis $\beta = \{(0,1,0), (-1,0,1), (1,0,-1)\}$ diagonalizes $A$, with corresponding eigenvalues $\lambda = 1, 1, -1$. Since $A$ symmetric, we can orthogonally diagonalize $A$. By performing Gram-Schmidt on the eigenbasis we got, we get an orthonormal eigenbasis $\sigma = \{(0,1,0), (-1/\sqrt{2},0,1/\sqrt{2}), (1/\sqrt{2},0,-1/\sqrt{2})\}$.

Problem 5. Answer. For $k > 0$.

Problem 6. Answer. For all values of $k$.

<table>
<thead>
<tr>
<th>Matrix $k$</th>
<th>Invertible?</th>
<th>Eigenvalues?</th>
<th>Diagonalizable?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>Yes</td>
<td>$-2,1,2$</td>
<td>Yes</td>
</tr>
<tr>
<td>$A^{-1}$</td>
<td>Yes</td>
<td>$-1/2,1,1/2$</td>
<td>Yes</td>
</tr>
<tr>
<td>$A^2$</td>
<td>Yes</td>
<td>$4,1,4$</td>
<td>Yes</td>
</tr>
<tr>
<td>$A + I$</td>
<td>Yes</td>
<td>$-1,2,3$</td>
<td>Ye $s$</td>
</tr>
<tr>
<td>$A - 2I$</td>
<td>No</td>
<td>$-4,-1,0$</td>
<td>Yes</td>
</tr>
<tr>
<td>$A^T$</td>
<td>Yes</td>
<td>$-2,1,2$</td>
<td>Yes</td>
</tr>
<tr>
<td>$A^{-1} + A$</td>
<td>Yes</td>
<td>$-5/2,2,5/2$</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Problem 7. Answer.

Problem 8. Answer. (1) Since the eigenvalues and their algebraic multiplicities are deduced from the characteristic polynomial, if we can show that both $A$ and $A^T$ have the same characteristic polynomial, then the must have the same eigenvalues with the same algebraic multiplicities. Indeed, note the characteristic polynomial of $A$ is

$$p_A(\lambda) = \det(A - \lambda I)$$
$$= \det((A - \lambda I)^T)$$
$$= \det(A^T - \lambda I^T) = \det(A^T - \lambda I) = p_{A^T}(\lambda),$$

which is the same as the characteristic polynomial of $A^T$. We are done.

(2) Let $A$ and $B$ be similar, say $A = PBP^{-1}$. To show that they have the same eigenvalues and with the same algebraic multiplicities, we show that they have the same characteristic polynomial. Indeed, note the characteristic polynomial of $A$ is

$$p_A(\lambda) = \det(A - \lambda I)$$
$$= \det(PBP^{-1} - \lambda I)$$
$$= \det(PBP^{-1} - \lambda IP^{-1})$$
$$= \det(PA - \lambda IP)$$
$$= \det(P)\det(A - \lambda I)\det(P^{-1}) = \det(B - \lambda I) = p_B(\lambda),$$

which is the same as the characteristic polynomial of $B$. We are done.

Problem 9. Answer. Let $A$ be such that $A^2 = A$, and $x \in \text{Im}(A)$ where $x \neq 0$. Then note that $x = Ay$ for some vector $y$. Note next that

$$Ax = A(Ay) = A^2y = Ay = x,$$

so that $Ax = 1x$. Hence any nonzero vector $x \in \text{Im}(A)$ is an eigenvector of $A$, with eigenvalue 1.

<table>
<thead>
<tr>
<th>Diagonalizable</th>
<th>Invertible</th>
<th>Not invertible</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\begin{pmatrix} 1 &amp; 0 \ 0 &amp; 1 \end{pmatrix}$</td>
<td>$\begin{pmatrix} 1 &amp; 0 \ 0 &amp; 0 \end{pmatrix}$</td>
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<tr>
<td>$\begin{pmatrix} 1 &amp; 1 \ 0 &amp; 1 \end{pmatrix}$</td>
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</tbody>
</table>

Problem 10. Answer.

Problem 11. Answer. If $A$ is diagonalizable, then we can factor $A = PDP^{-1}$ where $P$ some invertible matrix and $D$ a diagonal matrix. Note next that $A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$. Since $D^2$ is also a diagonal matrix, we see that $A^2$ is similar to the diagonal matrix $D^2$. Hence $A^2$ is diagonalizable.

With the same reasoning, we see that $A^k = PD^kp^{-1}$ and that $D^k$ is still some diagonal matrix. Hence $A^k$ is similar to the diagonal matrix $D^k$, hence diagonalizable.

Problem 12. Answer. $A = \begin{pmatrix} -2 & 2 & 2 \\ -2 & 2 & 2 \\ -2 & 2 & 2 \end{pmatrix}$.

Problem 13. Answer. $A = \begin{pmatrix} 2/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & 2/3 \end{pmatrix}$.

Problem 14. Answer. No such $4 \times 4$ symmetric matrix exists due to spectral theorem: Eigenvectors of different eigenvalues of a symmetric matrix are orthogonal to each other. But it is not the case here.
Problem 15. Answer. (1) Note regardless of what size matrix $A$ is, the matrices $AA^T$ and $A^TA$ are both square and symmetric. Hence by spectral theorem, both are diagonalizable.

(2) Recall that any orthogonal projection matrix $P$ can be written as $P = QQ^T$, where $Q$ is a matrix with orthonormal columns that has image the same as $P$. Hence we see that $P$ is a symmetric matrix, and thus diagonalizable by spectral theorem.

Problem 16. Answer. (1) Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

(2) Take $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$.

Problem 17. Answer. If $A$ has an orthonormal eigenbasis, then we can factorize $A = QDQ^{-1}$ where $D$ is a diagonal matrix and $Q$ an orthogonal matrix. But note that if $Q$ orthogonal projection matrix, $Q^{-1} = Q^T$, so $A = QDQ^T$. Now we compute $A^T = (QDQ^T)^T = QD^TQ^T = QDQ^T = A$. Hence $A$ symmetric.

Problem 18. Answer. $p_M(\lambda) = -\lambda^4(\lambda - 15)$.

*** On matrix power and matrix power limits. ***

Problem 19. Answer. (1) $\begin{pmatrix} 2^{k-1} & 2^{k-1} & 2^{k-1} \\ 0 & 0 & 0 \\ 2^{k-1} & 2^{k-1} & 2^{k-1} \end{pmatrix}$. (2) $6 \begin{pmatrix} 2^{k-1} \\ 0 \\ 2^{k-1} \end{pmatrix}$.

Problem 20. Answer. $A$ is not a positive transition matrix, but it is a regular transition matrix.

Problem 21. Answer. (1) $\begin{pmatrix} 2/5 & 2/5 & 2/5 \\ 2/5 & 2/5 & 2/5 \\ 1/5 & 1/5 & 1/5 \end{pmatrix}$. (2) $\begin{pmatrix} 2/5 \\ 2/5 \\ 1/5 \end{pmatrix}$. (3) $\begin{pmatrix} 2/5 \\ 2/5 \\ 1/5 \end{pmatrix}$.

Problem 22. Answer. Take $M = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$, a regular transition matrix that is not invertible. Take $N = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, an invertible matrix that is not a regular transition matrix.

Problem 23. Answer. Consider the transition matrix $M = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, it is not regular, but its matrix power limit $M^k$ converges to $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

Problem 24. Answer. Let $A$ be a $2 \times 2$ regular transition matrix. Note it has characteristic polynomial $p_A(\lambda)$ that consists of real coefficients. By the theorem of regular transition matrix, $\lambda = 1$ is an eigenvalue of $A$. Now since the characteristic polynomial is real, we must have another real eigenvalue $\lambda = c$ of $A$, so that $p_A(\lambda) = (1 - \lambda)(c - \lambda)$. But again by the theorem of regular transition matrix, all other eigenvalues of $A$ must be strictly less than 1, meaning that $c < 1$, and in particular $c \neq 1$. Hence $A$ has two distinct eigenvalues. As $A$ is $2 \times 2$, we conclude that $A$ is diagonalizable.

*** On complex numbers; matrices with complex eigenvalues. ***

Problem 25. Answer. $z^{33} = \sqrt{20^{33}} \cos(33 \arctan(2)) + i \sin(33 \arctan(2))$.

Problem 26. Answer. (1) $z = 1, \ -\frac{1}{2} + i \frac{\sqrt{3}}{2}, \ -\frac{1}{2} - i \frac{\sqrt{3}}{2}$.

(2) $w = \sqrt{3}, \ -\frac{\sqrt{3}}{2} + i \frac{\sqrt{3}}{2}, \ -\frac{\sqrt{3}}{2} - i \frac{\sqrt{3}}{2}.$

Problem 27. Answer. $\sqrt{1 + \sqrt{5}} + i \frac{2}{\sqrt{1 + \sqrt{5}}}$. And $-\sqrt{1 + \sqrt{5}} - i \frac{2}{\sqrt{1 + \sqrt{5}}}$. We can see that $\lambda = \pm \sqrt{1 + \sqrt{5}}$, with $\lambda = \sqrt{1 + \sqrt{5}}$ having an orthonormal eigenbasis, then we can factorize $A = QDQ^T$, where $D$ is a diagonal matrix and $Q$ an orthogonal matrix. But note that if $D$ a diagonal matrix, then $Q^{-1} = Q^T$, we have $A = QDQ^T$. Now we compute $A^T = (QDQ^T)^T = QD^TQ^T = QDQ^T = A$. Hence $A$ symmetric.

Problem 28. Answer. An eigenbasis for $C^2$ for $A$ is $\beta = \{ (i, 1), (-i, 1) \}$. To produce diagonal factorization $A = PDP^{-1}$, take $P = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 3 + 2i & 0 \\ 0 & 3 - 2i \end{pmatrix}$.

Problem 29. Answer. $P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $R = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

*** On quadratic forms. ***

Problem 30. Answer. $S = \begin{pmatrix} 3 & -3/2 & 1/2 \\ -3/2 & 9 & 2 \\ 1/2 & 2 & -3 \end{pmatrix}$.

Problem 31. Answer. Yes, $S$ can be invertible: Consider $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then for the nonzero vector $w = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ we have $q(w) = 0$.

Problem 32. Answer. $-2 < k < 2$.

Problem 33. Answer. Since $q$ is positive definite (as the corresponding symmetric matrix has eigenvalues 4 and 2), it has unique minimum at $(x, y) = (0, 0)$ with value $q(0, 0) = 0$. It does not have any maximum.

*** Some True or False. ***

Answers. All are false except for one true. Find it.