Inertial types and automorphic representations with prescribed ramification

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Chapter 1

Introduction

Our two main goals in this work are to study a variant of the theory of
"types" of Bushnell and Kutzko, and to demonstrate an application of this
to the theory of automorphic forms, by proving an existence theorem for
automorphic representations. The problem of establishing the existence of
automorphic forms or representations with prescribed local constraints has
a long history. Until very recently, most results in this area were not very
specific about the factors of automorphic representations at the finite places.
However, the main result of [32] proved that, in the context of automorphic
representations corresponding to Hilbert modular forms, for almost every
twist class of "global inertial type", there exist automorphic representations
of that type. The significance of this is in the fact that the global inertial
types defined in [32] specify the local factors of an automorphic representa-
tion up to inertia at *every* place, and given any arbitrary collection of local
inertial types, subject to a few obvious necessary restrictions, one can con-
struct a global inertial type with those local factors. Thus this theorem is a
sort of local-global principle for automorphic representations corresponding
to Hilbert modular forms. The proof of this theorem relies primarily on
an in-depth study of the local inertial types for the group GL_2. Here we
generalize some of those results to GL_n, for n prime, and use these results
to prove the following:

Theorem. *Let n be a prime, let F be a totally real number field, and let G be
a unitary group of rank n defined over F, such that G is compact at all infi-
nite places of F. Define a global inertial type for G as follows: at every infi-
nite place v of F, choose an irreducible representation of $G(F_v) \cong \mathrm{U}(n)$; for
finitely many split places of F, choose an unramified twist class of irreducible
supercuspidal representation of $G(F_v) \cong \mathrm{GL}_n(F_v)$; at all other places, let the*

type be unramified. Then, modulo twisting by a global character, for all but finitely many such global inertial types, there exist automorphic representations of that type.

For a more precise statement, and in particular a more precise definition of global inertial type, see Sections 4.2 and 4.3. The proof of this theorem can be roughly broken down into two parts: a local part, in which we study the inertial types of $\mathrm{GL}_n(F)$ for a nonarchimedean local field F, and a global part, which pieces together the local data to establish the theorem. In this setting, due to the simplicity of working with a group that is compact at all infinite places, the global part turns out to be quite straightforward. The local part, therefore, is the focus of most of this paper.

We begin by reviewing, in Chapter 2, the general theory of types, due to Bushnell and Kutzko. Section 2.1 is a summary of the main ideas and results of [8], which established that theory. There is nothing new here. In Section 2.2, we consider a variant of the Bushnell-Kutzko notion of a type, which appears to have first been considered for GL_2 by Henniart in [10]. These are the objects referred to above (and in [32]) as inertial types. We refer to them below as K-types, since they are defined on a fixed maximal compact subgroup K of G, and in many respects they seem analogous to Vogan's minimal K-types for a real reductive group. This section culminates in a pair of conjectures, one about "minimal" K-types for GL_n, and another which connects minimal K-types to inertial Weil-Deligne representations, thus explaining the terminology of "inertial type". If proven, these two conjectures should be very useful in establishing much more general existence theorems for automorphic representations, as well as for global Galois representations. We finish Chapter 2 by reviewing the proof of these conjectures for GL_2, and giving a partial proof of them for GL_3.

In Chapter 3, we content ourselves with working with the two major known cases of K-types for GL_n, and we begin working toward our main theorem using only those cases. Section 3.1 is nothing more than a review of the definition of the main objects of [7], known as *simple types*. Once again, there is nothing new here. In Section 3.2, we study supercuspidal K-types, which are merely the induction to K of maximal simple types. The main result of this section, Theorem 3.2.3, gives a bound on the characters of supercuspidal K-types for GL_n, when n is prime. We apply this result in Chapter 4 to prove our main theorem, Theorem 4.3.1.

1.1 Notation

We will use $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ to denote the floor (greatest integer) and ceiling functions, respectively:

$$\lfloor x \rfloor = \max \{ n \in \mathbb{Z} \mid n \leq x \},$$
$$\lceil x \rceil = \min \{ n \in \mathbb{Z} \mid n \geq x \}.$$

For a matrix M, the transpose of M will be denoted by ${}^{\mathrm{t}}M$. For a non-compact group G, we will usually use the terms "character" and "quasicharacter" interchangeably to refer to any continuous homomorphism from G to $\mathbb{C}^\times$. Any time that we require a character to be unitary, we will explicitly refer to it as a unitary character.

All representations considered in this paper will be complex representations. Representations of a group G will usually be denoted by (π, V), where V is the underlying complex vector space. However, we will often use just π, and occasionally just V, to refer to the same representation. For a compact group G and continuous representations π and ρ of G, we will write $\langle \pi, \rho \rangle_G$ for the intertwining number of π and ρ, which is equal to $\dim \mathrm{Hom}_G(\pi, \rho)$. If both π and ρ are finite-dimensional, this is also equal to the inner product of their characters:

$$\langle \pi, \rho \rangle_G = \int_G \mathrm{Tr}\, \pi(g) \overline{\mathrm{Tr}\, \rho(g)} dg.$$

For a finite (or profinite) group G, this integral is merely a finite sum.

For a group G, a subgroup H, and an element $g \in G$, the notation H^g will mean $g^{-1}Hg$. In this setting, if (σ, V) is a representation of G, then σ^g will denote the representation of H^g on the same space V defined by $\sigma^g(x) = \sigma(gxg^{-1})$ for $x \in H^g$.

For a group G and a subgroup H, we will write Ind_H^G for the usual induction functor. We will often use $\mathrm{Res}_H^G(\pi)$ to denote the restriction to H of a representation π of G, but will also interchangeably use the more concise notation $\pi|_H$. For a non-compact topological group G and an open subgroup H, $\mathrm{c}\text{-}\mathrm{Ind}_H^G$ will denote the functor of compact induction (i.e., induction with compact supports). If G is the group of F-points of a reductive algebraic group defined over a field F, P a parabolic subgroup of G, and L the Levi component of P, then ind_P^G (note the capitalization) will denote the functor of *normalized* parabolic induction from L to G via P. Likewise, in the same situation, we may use the (rather non-standard) notation res_P^G to denote the functor of Jacquet restriction. See Section 2.1.1 for details.

Results that are new, or at least for which we have given a proof, are always stated as lemmas or theorems. Any results that are merely quoted from another source are stated as propositions. In the cases where these are not well known results, we provide a reference where the proof may be found.

Chapter 2

Types and K-types for p-adic groups

In this chapter, we review the theory of "types", due to Bushnell and Kutzko, and introduce a variant of this theory that appears to be of significant arithmetic interest.

Throughout this chapter, the following notation will be assumed. Let F be a nonarchimedean local field, with ring of integers $\mathfrak{o}_F$, prime ideal $\mathfrak{p}_F$ (or simply $\mathfrak{p}$ when there is no possibility of confusion), and residue field $\boldsymbol{k}_F = \mathfrak{o}_F/\mathfrak{p}_F$ of cardinality q.

2.1   The theory of types

This section is a brief summary of the main ideas and results of [8], for the reader not already familiar with the theory of types. We begin by letting G be a connected, reductive algebraic group defined over F. We introduce the following abuse of notation, which we will use throughout the rest of this section: We will most often use G to refer to the group of F-points of G, rather than the algebraic group itself. Similarly, when we refer to a parabolic subgroup or Levi subgroup of G, we will generally mean the group of F-points of such an algebraic subgroup.

2.1.1   The representation theory of G

The group G is locally profinite, meaning that its topology can be given by a basis of neighborhoods of the identity consisting of compact open subgroups. (Equivalently, G is locally compact and totally disconnected.) A *smooth*

representation (π, V) of G on the complex vector space V is a homomorphism $\pi : G \to \mathrm{GL}(V)$ such that the stabilizer of any vector $v \in V$ is open. Henceforth, when we refer to representations of any locally profinite group, they will always be assumed smooth. We will denote the category of such representations of G by $\mathfrak{R}(G)$. A smooth representation of G is *admissible* if for every compact open subgroup K of G, the subspace V^K of vectors fixed by K is finite-dimensional.

We fix once and for all a Haar measure μ on G. Let $\mathcal{H}(G)$ be the space of locally constant, compactly supported, complex-valued functions on G. This becomes an associative algebra with the operation of convolution of functions (using the chosen Haar measure). We call $\mathcal{H}(G)$ the Hecke algebra of G. Note that, unless G is compact, $\mathcal{H}(G)$ does not in general have an identity. For a smooth representation (π, V) of G and $f \in \mathcal{H}(G)$, we may define a linear operator $\pi(f)$ on V by

$$\pi(f)v = \int_G f(g)\pi(g)v \, d\mu(g).$$

This gives V the structure of an $\mathcal{H}(G)$-module, and this construction defines an equivalence of categories between $\mathfrak{R}(G)$ and the category $\mathcal{H}(G)$-Mod of modules over $\mathcal{H}(G)$.

Given a parabolic subgroup P of G, we let N be its unipotent radical and L its Levi component (the unique maximal Levi subgroup of G contained in P), so that $P = L \ltimes N$. We will denote the functor of *normalized* parabolic induction of representations from L to G (via P) by

$$\mathrm{ind}_P^G : \mathfrak{R}(L) \to \mathfrak{R}(G).$$

Specifically, if σ is a representation of L, then $\mathrm{ind}_P^G(\sigma)$ is the representation obtained by inflating σ to P, then applying "ordinary" smooth induction to $\delta_P^{-\frac{1}{2}} \otimes \sigma$, where δ_P is the modulus function of P. Recall that this functor has a natural adjoint, often referred to as the Jacquet functor (or Jacquet restriction). For a representation (π, V) of G, let $V(N)$ be the linear span of the set

$$\{\pi(n)v - v \mid n \in N, \ v \in V\}.$$

Then π defines a representation π_N of L on the quotient $V_N = V/V(N)$. The Jacquet restriction functor, which we will denote by

$$\mathrm{res}_P^G : \mathfrak{R}(G) \to \mathfrak{R}(L),$$

is defined by $\pi \mapsto \pi_N$. (Note the lowercase letters we have chosen for both ind_P^G and res_P^G, to distinguish them from ordinary induction and restriction.)

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It is a well known theorem that every irreducible representation π of G occurs as a composition factor of $\text{ind}_P^G(\sigma)$ for some irreducible, supercuspidal representation σ of some Levi subgroup L. (Here we are of course allowing the possibility that $L = G$, so as to include the supercuspidal representations of G itself.) Furthermore, the Levi subgroup L and the representation σ are uniquely determined by π up to conjugation by an element of G. The conjugacy class of the pair (L, σ) is referred to as the *supercuspidal support* (or simply the *support*) of π.

This suggests the following program for classifying the irreducible representations of G:

1. Classify the irreducible supercuspidal representations of all the proper Levi subgroups of G.

2. Understand the decomposition of representations parabolically induced from irreducible supercuspidal representations of proper Levi subgroups of G.

3. Classify the irreducible supercuspidal representations of G itself.

Naturally, there is usually a close connection between the first and third of these. Unfortunately, classifying and/or constructing the supercuspidal representations of a group has proven to be quite difficult in most cases. Furthermore, the second item here is notoriously hard, and at the time of this writing is completely solved only for a few classes of groups.

2.1.2 The Bernstein "Center"

Unfortunately, even when σ is an irreducible, supercuspidal representation of L, the parabolically induced representation $\text{ind}_P^G(\sigma)$ is not always irreducible. Furthermore, a major source of difficulty in this subject is that such representations are in general not semisimple either. However, it was first proved by Bernstein in [1] that, in a certain sense, the lack of semisimplicity in the category $\mathfrak{R}(G)$ does not go much farther than this. To make this precise, we first define a weaker notion of "support" for an irreducible representation.

A *quasicharacter* (or simply a *character*) of G is a continuous homomorphism $\chi : G \to \mathbb{C}^\times$. We say that χ is *unramified* if it is of the form $g \mapsto |\xi(g)|^s$ for some $s \in \mathbb{C}$ and some F-rational character $\xi : G \to F^\times$. (Here we are of course using $|\cdot|$ to refer to the absolute value on F, normalized as usual so that $|\varpi| = q^{-1}$ for a uniformizer ϖ of F.) As above, we consider pairs of the form (L, σ), in which L is a Levi subgroup of G and σ is

an irreducible, supercuspidal representation of L. We will say that two such pairs (L_1, σ_1) and (L_2, σ_2) are *inertially equivalent in G* if they are equivalent up to conjugation by an element of G *and* a twist by an unramified character of L_i. To be quite precise, this means that there exists $g \in G$ and an unramified quasicharacter χ of L_2 such that

$$L_2 = L_1^g \text{ and } \sigma_2 \cong \chi \otimes \sigma_1^g,$$

where $L_1^g = g^{-1}L_1 g$ and $\sigma_1^g(x) = \sigma_1(gxg^{-1})$ for $x \in L_1^g$. We write $[L, \sigma]_G$ for the inertial equivalence class of the pair (L, σ) in G. The set of all inertial equivalence classes of such pairs will be denoted by $\mathcal{B}(G)$, and we will often use the symbol $\mathfrak{s}$ to refer to an element $[L, \sigma]_G$ of $\mathcal{B}(G)$.

If (π, V) is an irreducible representation of G, the *inertial support* of π means the inertial equivalence class of the supercuspidal support of π. We will write $\mathcal{I}(\pi)$ for the inertial support of π. We can now extend our definition of inertial equivalence to all irreducible representations of G (or a Levi subgroup L), rather than just the supercuspidal ones, by saying that π_1 is inertially equivalent to π_2 if $\mathcal{I}(\pi_1) = \mathcal{I}(\pi_2)$. Note that if π_1 and π_2 are supercuspidal, this definition agrees with the previous one, so there is no conflict of terminology here.

For $\mathfrak{s} \in \mathcal{B}(G)$, let $\mathfrak{R}^{\mathfrak{s}}(G)$ be the full subcategory of $\mathfrak{R}(G)$ consisting of representations of G, all of whose irreducible subquotients have inertial support $\mathfrak{s}$. According to [1], the category $\mathfrak{R}(G)$ decomposes as the "direct product" of these subcategories:

$$\mathfrak{R}(G) = \prod_{\mathfrak{s} \in \mathcal{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G).$$

To be more precise, this means that if (π, V) is any smooth representation of G, then

$$V = \bigoplus_{\mathfrak{s} \in \mathcal{B}(G)} V^{\mathfrak{s}},$$

where

1. for each $\mathfrak{s} \in \mathcal{B}(G)$, $V^{\mathfrak{s}}$ is the unique maximal G-subspace of V that is an object of $\mathfrak{R}^{\mathfrak{s}}(G)$, and

2. for $\mathfrak{s}_1 \neq \mathfrak{s}_2$, $\mathrm{Hom}_G(V^{\mathfrak{s}_1}, V^{\mathfrak{s}_2}) = 0$.

Note that an immediate consequence is that if (ρ, W) is any other smooth representation of G, then

$$\mathrm{Hom}_G(V, W) = \bigoplus_{\mathfrak{s} \in \mathcal{B}(G)} \mathrm{Hom}_G(V^{\mathfrak{s}}, W^{\mathfrak{s}}).$$

As a result of this product decomposition, the subcategories $\mathfrak{R}^{\mathfrak{s}}(G)$ are often referred to as the *Bernstein components* of G. By a slight abuse of language, we will often use the term "Bernstein component" to refer to an inertial equivalence class $\mathfrak{s} \in \mathcal{B}(G)$, as there is a natural one-to-one correspondence between these objects.

Before we go on, we make a few more basic observations about inertial equivalence classes and their corresponding Bernstein components. If $\mathfrak{s} = [G, \pi]_G$, then clearly every irreducible representation in $\mathfrak{R}^{\mathfrak{s}}(G)$ is supercuspidal. Conversely, if any of the irreducible representations in $\mathfrak{R}^{\mathfrak{s}}(G)$ is supercuspidal, then all of them are unramified twists of a single supercuspidal representation π, and $\mathfrak{s} = [G, \pi]_G$. In such a case, we will simply refer to $\mathfrak{s}$ as a supercuspidal inertial equivalence class, and to $\mathfrak{R}^{\mathfrak{s}}(G)$ as a supercuspidal Bernstein component.

In a similar vein, if $\mathfrak{s} = [L, \sigma]_G$ and χ is any quasicharacter of G, then we may view χ as a quasicharacter of L by restriction, and let $\chi\mathfrak{s}$ be the inertial equivalence class $[L, \chi \otimes \sigma]_G$. (Clearly $\chi\mathfrak{s} = \mathfrak{s}$ if and only if χ is unramified.) Note that we may twist any representation of $\mathfrak{R}^{\mathfrak{s}}(G)$ by χ to obtain a representation of $\mathfrak{R}^{\chi\mathfrak{s}}(G)$, and this defines a functor from $\mathfrak{R}^{\mathfrak{s}}(G)$ to $\mathfrak{R}^{\chi\mathfrak{s}}(G)$ that is clearly an equivalence of categories. Naturally, we will call $\chi\mathfrak{s}$ the twist of $\mathfrak{s}$ by χ, and we may refer to the Bernstein component $\mathfrak{R}^{\chi\mathfrak{s}}(G)$ as the twist by χ of the Bernstein component $\mathfrak{R}^{\mathfrak{s}}(G)$.

2.1.3 Types

The theory of types for reductive p-adic groups is, in brief, an attempt to classify or parametrize all irreducible representations of such a group *up to inertial equivalence*, using irreducible representations of compact open subgroups of G. We will see in the next section that, with such an approach, one cannot hope to achieve a classification much more fine-grained than inertial equivalence.

Let J be a compact open subgroup of G, and let (λ, W) be an irreducible representation of J. Denote by $(\check{\lambda}, \check{W})$ the contragredient of (λ, W). Define $\mathcal{H}(G, \lambda)$ to be the space of compactly supported functions $f : G \to \mathrm{End}_{\mathbb{C}}(\check{W})$ such that

$$f(j_1 g j_2) = \check{\lambda}(j_1) f(g) \check{\lambda}(j_2)$$

for all $j_1, j_2 \in J$ and $g \in G$. As before, this space becomes an associative algebra under the operation of convolution, this time with an identity element (given by the function that takes the value $\check{\lambda}(g)$ for $g \in J$ and 0 elsewhere). $\mathcal{H}(G, \lambda)$ is referred to as the λ-spherical Hecke algebra of G. Note that there

is a similar construction that defines a related algebra, which is actually a subring of the ordinary Hecke algebra $\mathcal{H}(G)$. Define $e_\lambda : G \to \mathbb{C}$ by

$$e_\lambda(g) = \begin{cases} \frac{\dim(\lambda)}{\mu(J)} \operatorname{Tr} \lambda(g^{-1}) & g \in J \\ 0 & \text{otherwise.} \end{cases}$$

Then $e_\lambda \in \mathcal{H}(G)$ is an idempotent, and thus $e_\lambda * \mathcal{H}(G) * e_\lambda$ is a subring of $\mathcal{H}(G)$ with identity e_λ. While it is not in general true that $\mathcal{H}(G, \lambda)$ and $e_\lambda * \mathcal{H}(G) * e_\lambda$ are isomorphic (indeed, this will be true if and only if λ is 1-dimensional), it turns out that there is a canonical isomorphism

$$e_\lambda * \mathcal{H}(G) * e_\lambda \cong \mathcal{H}(G, \lambda) \otimes_\mathbb{C} \operatorname{End}_\mathbb{C}(W). \qquad (2.1.1)$$

Thus, in particular, these two rings are Morita equivalent: their module categories are equivalent.

The relationship between these two algebras goes much deeper. If (π, V) is a representation of G, we define V^λ to be the λ-isotypic subspace of V, and V_λ to be the space of λ-invariants of V:

$$V^\lambda = \pi(e_\lambda)V$$
$$V_\lambda = \operatorname{Hom}_J(W, V).$$

It is clear from this that V^λ has a natural $e_\lambda * \mathcal{H}(G) * e_\lambda$-module structure. Likewise, V_λ carries a natural $\mathcal{H}(G, \lambda)$-module structure, defined as follows. For $\phi \in \mathcal{H}(G, \lambda)$ and $f \in \text{c-Ind}_J^G(\lambda)$, we may define the convolution of ϕ and f in the obvious way, and it is easy to see that $\phi * f \in \text{c-Ind}_J^G(\lambda)$. Thus any fixed $\phi \in \mathcal{H}(G, \lambda)$ defines an endomorphism $f \mapsto \phi * f$ of $\text{c-Ind}_J^G(\lambda)$, and this endomorphism is G-equivariant. This induces a natural algebra isomorphism[1] between $\mathcal{H}(G, \lambda)$ and $\operatorname{End}_G(\text{c-Ind}_J^G(\lambda))$. By the compactly supported version of Frobenius reciprocity, we have $V_\lambda \cong \operatorname{Hom}_G(\text{c-Ind}_J^G(\lambda), V)$. This provides the natural $\mathcal{H}(G, \lambda)$-module structure on V_λ. We have thus defined a pair of functors

$$\mathbf{M}^\lambda : \mathfrak{R}(G) \to e_\lambda * \mathcal{H}(G) * e_\lambda \text{-Mod}$$
$$(\pi, V) \mapsto V^\lambda$$

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[1] The observant reader may notice that what we have actually described here is an isomorphism between $\mathcal{H}(G, \check{\lambda})$ and the usual algebra of *left* G-endomorphisms of $\text{c-Ind}_J^G(\lambda)$. However, to obtain a left module structure on V_λ, we need to consider the algebra of *right* G-endomorphisms of $\text{c-Ind}_J^G(\lambda)$. Fortunately, there is also a natural anti-isomorphism of algebras between $\mathcal{H}(G, \check{\lambda})$ and $\mathcal{H}(G, \lambda)$, given by $f \mapsto \check{f}$, where $\check{f}(g) = f(g^{-1})$. Composing this with the isomorphism described above provides the desired isomorphism.

and

$$\mathbf{M}_\lambda : \mathfrak{R}(G) \to \mathcal{H}(G, \lambda)\text{-Mod}$$
$$(\pi, V) \mapsto V_\lambda,$$

and the isomorphism (2.1.1) connects these.

For a representation (π, V) of G, we define $V[\lambda]$ to be the G-subspace of V generated by the J-subspace V^λ. We may then define a full subcategory $\mathfrak{R}_\lambda(G)$ of $\mathfrak{R}(G)$, consisting of representations (π, V) for which $V[\lambda] = V$. We will abuse notation slightly and refer to the restrictions to $\mathfrak{R}_\lambda(G)$ of the two functors above by the same names. The following proposition summarizes much of Sections 2–4 of [8].

Proposition. *Let J be a compact open subgroup of G, and let λ be an irreducible representation of J. The following are equivalent:*

1. *The subcategory $\mathfrak{R}_\lambda(G)$ of $\mathfrak{R}(G)$ is closed relative to subquotients, i.e., for any (π, V) in $\mathfrak{R}_\lambda(G)$, all subquotients of V are in $\mathfrak{R}_\lambda(G)$ as well.*

2. *The functor $\mathbf{M}^\lambda : \mathfrak{R}_\lambda(G) \to e_\lambda * \mathcal{H}(G) * e_\lambda$-Mod is an equivalence of categories.*

3. *The functor $\mathbf{M}_\lambda : \mathfrak{R}_\lambda(G) \to \mathcal{H}(G, \lambda)$-Mod is an equivalence of categories.*

4. *There exists a finite set $\mathfrak{S} \subset \mathcal{B}(G)$ such that*

$$\mathfrak{R}_\lambda(G) = \prod_{\mathfrak{s} \in \mathfrak{S}} \mathfrak{R}^{\mathfrak{s}}(G).$$

5. *There exists a finite set $\mathfrak{S} \subset \mathcal{B}(G)$ such that, for any irreducible representation (π, V) of G, π is in $\mathfrak{R}_\lambda(G)$ if and only if $\mathcal{I}(\pi) \in \mathfrak{S}$.*

If these conditions hold, then in the last two of these, the sets $\mathfrak{S}$ are the same.

We may now define an $\mathfrak{S}$-*type* to be a representation λ satisfying the equivalent conditions of this proposition, but we will be slightly more specific below. Note that for an irreducible representation (π, V) of G, the condition that π is in $\mathfrak{R}_\lambda(G)$ is equivalent to each of the following: (i) $V^\lambda \neq 0$, (ii) $V_\lambda \neq 0$, and (iii) $\langle \pi, \lambda \rangle_J \neq 0$. For the purposes of this paper, we will tend to favor the last of these, so we make the following definition.

Definition 2.1.1. Let J be a compact open subgroup of G and λ an irreducible representation of J, and let $\mathfrak{s} \in \mathcal{B}(G)$. We say that (J, λ) is a *type for* $\mathfrak{s}$, or more succinctly an $\mathfrak{s}$-*type*, if for every irreducible representation (π, V) of G,

$$\langle \pi, \lambda \rangle_J \neq 0 \iff \mathcal{I}(\pi) = \mathfrak{s}.$$

Corollary. *Let $\mathfrak{s} \in \mathcal{B}(G)$, and let (J, λ) be an $\mathfrak{s}$-type. Then*

1. *The subcategory $\mathfrak{R}_\lambda(G)$ of $\mathfrak{R}(G)$ is closed relative to subquotients.*

2. *The functor $\mathbf{M}^\lambda : \mathfrak{R}_\lambda(G) \to e_\lambda * \mathcal{H}(G) * e_\lambda$-Mod is an equivalence of categories.*

3. *The functor $\mathbf{M}_\lambda : \mathfrak{R}_\lambda(G) \to \mathcal{H}(G, \lambda)$-Mod is an equivalence of categories.*

4. *$\mathfrak{R}_\lambda(G) = \mathfrak{R}^{\mathfrak{s}}(G)$.*

While the preceding proposition already demonstrates that types reveal very interesting information about the structure of the category $\mathfrak{R}(G)$, there are many more applications. Historically, much of the work that led up to the theory of types was motivated largely by the attempt to classify, and if possible construct, the irreducible supercuspidal representations of a reductive p-adic group G. Indeed, one of the primary means of explicitly constructing supercuspidal representations is by inducing (using compactly supported induction) from an open subgroup of G that is compact modulo the center. It is not hard to prove (see [8, Sec. 5]) that if π is an irreducible supercuspidal representation of G, such that $\pi \cong \text{c-Ind}_{\widetilde{J}}^G(\widetilde{\lambda})$ for some open compact-mod-center subgroup $\widetilde{J}$ of G and some irreducible representation $\widetilde{\lambda}$ of $\widetilde{J}$, then any irreducible component of the restriction of $\widetilde{\lambda}$ to the maximal compact subgroup of $\widetilde{J}$ is a type for $\mathfrak{s} = [G, \pi]_G$. Conversely, if $\mathfrak{s} = [G, \pi]_G$ and (J, λ) is an $\mathfrak{s}$-type (satisfying one additional technical condition, cf. [8, (5.2)]), then for any extension $\widetilde{\lambda}$ of λ to ZJ (where Z is the center of G), $\text{c-Ind}_{ZJ}^G(\widetilde{\lambda})$ will be the direct sum of a finite number of unramified twists of π. In all cases known so far, it has always turned out to be possible to further extend $\widetilde{\lambda}$ to a slightly larger subgroup $\widetilde{J}$ in order to get a single unramified twist of π.

Remark. Let χ be any quasicharacter of G, and let (J, λ) be an $\mathfrak{s}$-type for some $\mathfrak{s} \in \mathcal{B}(G)$. Then we may view χ as a character of J by restriction, and it is clear that $(J, \chi \otimes \lambda)$ is a type for $\chi \mathfrak{s}$. Thus, when convenient, we may deal with types only up to twisting.

2.1.4 Covers

As described above, the theory of types attempts to parametrize the irreducible representations of G up to inertial equivalence, and to describe the structure of $\mathfrak{R}(G)$ via the Hecke algebras associated to types. However, the program described in 2.1.1 attempts to classify the irreducible representations of G in terms of parabolic induction. Thus, it may seem at this point that these two programs are not very compatible (beyond the fact that the definition of types is based on inertial equivalence, which is in turn based on the concept of parabolic induction). However, there is one more key ingredient in the theory of types that provides a crucial link between these two programs.

Definition 2.1.2. Let M be a Levi subgroup of G, and let P be a parabolic subgroup of G with Levi component M. Let N be the unipotent radical of P. Let $\overline{P}$ be the opposite parabolic subgroup of P, and $\overline{N}$ the unipotent radical of $\overline{P}$. Let J_M be a compact open subgroup of M and λ_M an irreducible representation of J_M. For a compact open subgroup J of G and an irreducible representation λ of J, we say that (J, λ) is a G-*cover* of (J_M, λ_M) if all of the following hold:

1. J satisfies an Iwahori decomposition with respect to P and J_M, in the sense that $J_M = J \cap M$ and
$$J = (J \cap \overline{N})(J \cap M)(J \cap N).$$

2. λ is trivial on $J \cap \overline{N}$ and $J \cap N$, and $\lambda|_{J_M} \cong \lambda_M$.

3. There exists $f \in \mathcal{H}(G, \lambda)^{\times}$ such that $\operatorname{supp} f = JzJ$, where z is in the center of M and satisfies the following:

   (a) $z(J \cap N)z^{-1} \subset J \cap N$
   (b) $z^{-1}(J \cap \overline{N})z \subset J \cap \overline{N}$
   (c) For any compact open subgroups K_1 and K_2 of N, there exists $m \geq 0$ such that $z^m K_1 z^{-m} \subset K_2$.
   (d) For any compact open subgroups K_1 and K_2 of $\overline{N}$, there exists $m \geq 0$ such that $z^{-m} K_1 z^m \subset K_2$.

Remark 1. The first and second conditions here guarantee that λ is irreducible if and only if λ_M is, and furthermore that λ is defined completely in terms of λ_M and vice-versa. Furthermore, given (J_M, λ_M) as in the definition, it is always possible to construct a pair (J, λ) satisfying these first two

conditions. It is the crucial third condition that both makes this definition highly useful (as we will see below) and simultaneously makes covers rather difficult to construct.

Remark 2. In this section, we are assuming the hypothesis [8, (8.8)]. If this hypothesis fails, then it is necessary to modify the definition above to require that all the same conditions hold for *every* parabolic subgroup P with Levi component M, not just one particular one.

The significance of covers in the theory of types is as follows. Let M be a Levi subgroup of G, and let $\mathfrak{s}_M = [L, \sigma]_M \in \mathcal{B}(M)$. Then L is a Levi subgroup of M, and hence also of G, so we may consider $\mathfrak{s}_G = [L, \sigma]_G \in \mathcal{B}(G)$. Now suppose (J_M, λ_M) is an $\mathfrak{s}_M$-type, and let (J, λ) be a G-cover of (J_M, λ_M). Then by [8, (8.3)], (J, λ) is an $\mathfrak{s}_G$-type. Furthermore, if P is any parabolic subgroup of G with Levi component M, then there exists an algebra homomorphism

$$t_P : \mathcal{H}(M, \lambda_M) \to \mathcal{H}(G, \lambda)$$

that satisfies the following:

1. For any representation (π, V) in $\mathfrak{R}_\lambda(G)$,

$$t_P^* \left(\mathbf{M}_\lambda(\pi) \right) \cong \mathbf{M}_{\lambda_M}(\mathrm{res}_P^G(\pi)).$$

2. For any representation (σ, W) in $\mathfrak{R}_{\lambda_M}(M)$,

$$(t_P)_* \left(\mathbf{M}_{\lambda_M}(\sigma) \right) \cong \mathbf{M}_\lambda \left(\mathrm{ind}_P^G(\sigma) \right).$$

Here t_P^* denotes the pullback functor from $\mathcal{H}(G, \lambda)$-modules to $\mathcal{H}(M, \lambda_M)$-modules, and $(t_P)_*$ denotes its adjoint, defined by

$$(t_P)_*(A) = \mathrm{Hom}_{\mathcal{H}(M, \lambda_M)}(\mathcal{H}(G, \lambda), A)$$

for any $\mathcal{H}(M, \lambda_M)$-module A. In other words, what we are saying here is that the algebra homomorphism t_P transfers the representation-theoretic functors of parabolic induction and Jacquet restriction to the equivalent module categories. To put this graphically, the following two diagrams of functors, in which all of the horizontal arrows are equivalences, commute:

$$
\begin{array}{ccc}
\mathfrak{R}_\lambda(G) & \xrightarrow[\mathbf{M}_\lambda]{\sim} & \mathcal{H}(G, \lambda)\text{-Mod} \\
\Big\uparrow{\scriptstyle \mathrm{ind}_P^G} & & \Big\uparrow{\scriptstyle (t_P)_*} \\
\mathfrak{R}_{\lambda_M}(M) & \xrightarrow[\mathbf{M}_{\lambda_M}]{\sim} & \mathcal{H}(G, \lambda_M)\text{-Mod}
\end{array}
$$

$$\begin{array}{ccc}
\mathfrak{R}_\lambda(G) & \xrightarrow[\mathbf{M}_\lambda]{\sim} & \mathcal{H}(G,\lambda)\text{-Mod} \\
\Big\downarrow{\scriptstyle\mathrm{res}^G_P} & & \Big\downarrow{\scriptstyle t^*_P} \\
\mathfrak{R}_{\lambda_M}(M) & \xrightarrow[\mathbf{M}_{\lambda_M}]{\sim} & \mathcal{H}(G,\lambda_M)\text{-Mod}
\end{array}$$

It is also worth noting that covers are transitive. Let M be a Levi subgroup of G and L a Levi subgroup of M, and let J_L be a compact open subgroup of L and λ_L an irreducible representation of J_L. Then if (J_M, λ_M) is an M-cover of (J_L, λ_L), and (J, λ) is a G-cover of (J_M, λ_M), then (J, λ) is also a G-cover of (J_L, λ_L). Furthermore, if P is a parabolic subgroup of G with Levi component L, then $Q = MP$ is a parabolic subgroup of G with Levi component M, and $R = P \cap M$ is a parabolic subgroup of M with Levi component L, and we have

$$t_P = t_Q \circ t_R.$$

2.1.5   Status of the theory

All of the preceding material in this section makes for a lovely abstract theory, but of course it has little use if one cannot produce types for some group G. The first class of groups for which this theory was at least partially realized was the general linear groups. Types for all supercuspidal representations of $\mathrm{GL}_n(F)$ were first constructed in [7], although the basic idea there goes all the way back to [21], which handled the "tamely ramified" case. Types for the supercuspidal representations of $\mathrm{SL}_n(F)$ were also constructed in [5] and [6], which built heavily on [7]. All of these were written before the theory described here was fully worked out, and in particular before the concept of covers was known.

The first construction of types (as defined here) for a large class of non-supercuspidal representations, and consequently the first usage of covers to construct such types, was achieved in [9], which (building heavily upon [7]) constructed the whole theory described above for $\mathrm{GL}_n(F)$. More recently, the supercuspidal representations of all classical groups over a nonarchimedean local field F have been completely classified via types, in the series of papers [27–30]. At the time of this writing, covers of such types have been constructed only in a few special cases. Thus the existence of types for all non-supercuspidal representations of the classical groups remains an open problem.

2.2 K-types for $\mathrm{GL}_n(F)$

While the theory of types has proven very valuable to the study of the representation theory of reductive p-adic groups, types are not always the most convenient objects to deal with for certain applications. In particular, since the subgroup J appearing in an $\mathfrak{s}$-type (J, λ) is permitted to vary as $\mathfrak{s} \in \mathcal{B}(G)$ varies, there may be no single compact subgroup to which we can restrict arbitrary irreducible representations of G and be guaranteed to find a type. Indeed, we will see shortly that even for $\mathrm{GL}_2(F)$, this is the case. Thus we now introduce a variant of the above theory, which we will study and use throughout the rest of this paper.

In this section (as well as in Section 4.3 below), we will make use of the following general proposition, which is a well known application of Mackey theory.

Proposition 2.2.1. *Let G be any locally profinite group and K an open subgroup of G. Let P be a closed subgroup of G, and let (ρ, W) be a smooth representation of P. Then*

$$\mathrm{Res}_K^G \mathrm{Ind}_P^G(\rho) \cong \bigoplus_{g \in P \backslash G / K} \mathrm{Ind}_{P^g \cap K}^K \mathrm{Res}_{P^g \cap K}^{P^g}(\rho^g),$$

where $P^g = g^{-1}Pg$ and $\rho^g(x) = \rho(gxg^{-1})$ for $x \in P^g$.

We continue to use much of the notation of the previous section. In particular, F is a nonarchimedean local field, together with all of its associated notation. However, in this section, we will restrict our attention to the group $G = \mathrm{GL}_n(F)$. Much of the basic material here should generalize nicely to other groups, but for now that will have to wait.

2.2.1   Definition and basic results

We begin by fixing a choice of maximal compact subgroup of $G = \mathrm{GL}_n(F)$, so we let $K = \mathrm{GL}_n(\mathfrak{o}_F)$. Since it is not always possible to find an $\mathfrak{s}$-type of the form (K, λ) for all $\mathfrak{s} \in \mathcal{B}(G)$, we will weaken the definition of a type as follows. (cf. Definition 2.1.1 above.)

Definition 2.2.2. Let $\mathfrak{s} \in \mathcal{B}(G)$. A K-*type for* $\mathfrak{s}$ is an irreducible representation τ of K such that, for every irreducible representation (π, V) of G,

$$\langle \pi, \tau \rangle_K \neq 0 \implies \mathcal{I}(\pi) = \mathfrak{s}.$$

We make a few immediate observations about K-types. First, note that if (J, λ) is a type for $\mathfrak{s}$ such that $J \subset K$, then by Frobenius reciprocity,

$$\langle \pi, \mathrm{Ind}_J^K(\lambda) \rangle_K \neq 0 \iff \langle \pi, \lambda \rangle_J \neq 0 \iff \mathcal{I}(\pi) = \mathfrak{s}$$

for any irreducible representation π of G. Thus every irreducible component of $\mathrm{Ind}_J^K(\lambda)$ is a K-type for $\mathfrak{s}$, and every irreducible representation π in $\mathfrak{R}^{\mathfrak{s}}(G)$ contains one of these K-types. This also means, of course, that if $\mathrm{Ind}_J^K(\lambda)$ is irreducible, then it is actually a *type* for $\mathfrak{s}$, rather than just a K-type for $\mathfrak{s}$. Since it has been proven for $G = \mathrm{GL}_n(F)$, in [7] and [9], that there exists an $\mathfrak{s}$-type for every $\mathfrak{s} \in \mathcal{B}(G)$, we can immediately conclude that every irreducible representation of G contains a K-type.

Second, note that just as with types, we may deal with K-types only up to twisting. Specifically, let χ be any quasicharacter of G, and let τ be a K-type for some $\mathfrak{s} \in \mathcal{B}(G)$. Then we may view χ as a character of K by restriction, and clearly $\chi \otimes \tau$ is a K-type for $\chi\mathfrak{s}$.
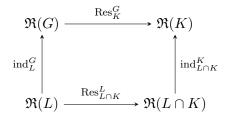
Finally, we observe that if $\mathfrak{s}$ is supercuspidal, i.e., $\mathfrak{s} = [G, \pi]_G$, and if τ is a K-type for $\mathfrak{s}$, then (K, τ) is necessarily a type for $\mathfrak{s}$. Indeed, any irreducible representation π' of G such that $\mathcal{I}(\pi') = \mathfrak{s}$ must be isomorphic to $\chi \otimes \pi$ for some unramified quasicharacter χ of G, whence $\pi'|_K \cong \pi|_K$. This fact generalizes easily to a large class of non-supercuspidal Bernstein components, as we will demonstrate below. First, however, we point out the following theorem of Paskunas (cf. [24]), which will play an important part in what follows.

Proposition 2.2.3 (Paskunas)**.** *Let $\mathfrak{s} \in \mathcal{B}(G)$ be a supercuspidal inertial equivalence class. Then there exists a unique (up to isomorphism) K-type τ for $\mathfrak{s}$. Furthermore, for any irreducible representation π in $\mathfrak{R}^{\mathfrak{s}}(G)$, the multiplicity of τ in $\pi|_K$ is 1.*

At this point, it becomes convenient to specialize our discussion a bit. Let B be the standard Borel subgroup of upper-triangular matrices in G. We will refer to a parabolic subgroup P of G as a *standard* parabolic subgroup if it contains B, and similarly we will call a Levi subgroup *standard* if it is the Levi component of one of the standard parabolics. The standard Levi subgroups are simply the block-diagonal subgroups of G, and the standard parabolic subgroups are the subgroups of block-upper-triangular matrices in G. It is well known that every irreducible representation of $\mathrm{GL}_n(F)$ appears as a composition factor of $\mathrm{ind}_P^G(\sigma)$, where P is a standard parabolic subgroup (and thus σ is an irreducible supercuspidal representation of a standard Levi subgroup). In light of this, we introduce the following abuse

of notation: for a standard Levi subgroup L of G, there is a unique standard parabolic subgroup P whose Levi component is L, so we may write $\mathrm{ind}_L^G(\sigma)$ in place of $\mathrm{ind}_P^G(\sigma)$. When σ is an irreducible supercuspidal representation of a standard Levi subgroup L, we will refer to $\mathrm{ind}_L^G(\sigma)$ as a *standard parabolically induced representation* of G.

We now introduce an analogous concept, which takes place within the maximal compact subgroup K. Let L and P be as in the previous paragraph. Given a representation of $L \cap K$, we may inflate it to $P \cap K$, and induce from $P \cap K$ to K. As this process is just like parabolic induction in G, but restricted to the subgroup K, we will use the notation $\mathrm{ind}_{L \cap K}^K$ for the functor we have just defined. The following lemma is likely well known, and also holds in a much more general context.

Lemma 2.2.4. *Let L be a standard Levi subgroup of G, let σ be a representation of L, and let $\pi = \mathrm{ind}_L^G(\sigma)$. Then $\pi|_K \cong \mathrm{ind}_{L \cap K}^K(\sigma|_{L \cap K})$. In other words, the following diagram of functors commutes:*

$$
\begin{array}{ccc}
\mathfrak{R}(G) & \xrightarrow{\ \mathrm{Res}_K^G\ } & \mathfrak{R}(K) \\[2pt]
\Big\uparrow{\scriptstyle \mathrm{ind}_L^G} & & \Big\uparrow{\scriptstyle \mathrm{ind}_{L \cap K}^K} \\[2pt]
\mathfrak{R}(L) & \xrightarrow{\ \mathrm{Res}_{L \cap K}^L\ } & \mathfrak{R}(L \cap K)
\end{array}
$$

Proof. Let P be the unique standard parabolic subgroup of G whose Levi component is L. Clearly the inflation to $P \cap K$ of the restriction to $L \cap K$ of σ is the same as the restriction to $P \cap K$ of the inflation to P of σ. Thus we may assume we are starting with the inflation of σ to P. By the Iwasawa decomposition $G = PK$, there is only one double coset to consider in $P \backslash G / K$. So when we apply Proposition 2.2.1 to this setting, we have

$$
\begin{aligned}
\pi|_K &= \mathrm{Res}_K^G \mathrm{Ind}_P^G(\delta_P^{-\frac{1}{2}} \otimes \sigma) \\
&\cong \mathrm{Ind}_{P \cap K}^K \mathrm{Res}_{P \cap K}^P(\sigma) \\
&= \mathrm{ind}_{P \cap K}^K(\sigma|_{L \cap K}),
\end{aligned}
$$

since δ_P is trivial on K. $\qquad\square$

Corollary 2.2.5. *Let $\mathfrak{s} \in \mathcal{B}(G)$. Let (L, σ_1) and (L, σ_2) be two representatives of the inertial equivalence class $\mathfrak{s}$ for which L is a standard Levi subgroup. Then*

$$
\mathrm{Res}_K^G \mathrm{ind}_L^G(\sigma_1) \cong \mathrm{Res}_K^G \mathrm{ind}_L^G(\sigma_2).
$$

In brief, any two standard parabolically induced representations in $\mathfrak{R}^{\mathfrak{s}}(G)$ have isomorphic restrictions to K.

In light of this, we make the following definition. This definition also generalizes to a much larger class of groups than just $\mathrm{GL}_n(F)$, and the corollary below should still remain true.

Definition 2.2.6. Let $\mathfrak{s} \in \mathcal{B}(G)$. We will say that $\mathfrak{s}$ is *nondegenerate* if, for every representative (L, σ) of the inertial equivalence class $\mathfrak{s}$, and any parabolic subgroup P of G with Levi component L, $\mathrm{ind}_P^G(\sigma)$ is irreducible. Otherwise, we will say that $\mathfrak{s}$ is *degenerate*.

Equivalently (looking at the situation "from the top" rather than "from the bottom"), $\mathfrak{s}$ is nondegenerate if and only if, for every irreducible representation π in $\mathfrak{R}^{\mathfrak{s}}(G)$, $\pi \cong \mathrm{ind}_P^G(\sigma)$ for some parabolic subgroup P and some irreducible supercuspidal representation σ of the Levi component of P.

Note that we may conjugate any parabolic subgroup by some $g \in G$ to obtain a standard parabolic, and $\mathrm{ind}_P^G(\sigma)$ is irreducible if and only if $\mathrm{ind}_{P^g}^G(\sigma^g)$ is. Thus $\mathfrak{s}$ is nondegenerate if and only if, for every representative (L, σ) of $\mathfrak{s}$ in which L is a standard Levi subgroup, $\mathrm{ind}_L^G(\sigma)$ is irreducible. Combining this with the corollary above, we have proved the following.

Corollary 2.2.7. *If $\mathfrak{s} \in \mathcal{B}(G)$ is nondegenerate, then any K-type for $\mathfrak{s}$ is in fact a type for $\mathfrak{s}$.*

Example 1. For $\mathrm{GL}_2(F)$, the only degenerate Bernstein component (modulo twisting by a character) is the one containing the unramified principal series. This is the Bernstein component corresponding to $\mathfrak{s}_{\mathfrak{o}} = [T, 1_T]_G$, where T is the diagonal subgroup of G and 1_T denotes the trivial character. In this case, if χ is any unramified twist of $\delta_B^{\pm\frac{1}{2}}$, then $\mathrm{ind}_T^G(\chi)$ is a standard parabolically induced representation in $\mathfrak{R}^{\mathfrak{s}_0}(G)$ that is reducible. Or equivalently, from the "top-down" view, the one-dimensional unramified representations and the unramified twists of the Steinberg representation are irreducible representations in $\mathfrak{R}^{\mathfrak{s}_0}(G)$, but none of them is isomorphic to a parabolically induced representation. Thus, by the corollary, for every $\mathfrak{s} \in \mathcal{B}(G)$ that is not a twist of $\mathfrak{s}_0$, any K-type is a type. But it turns out that for $\mathfrak{s}_0$, there are two non-isomorphic K-types, and neither one is a type. (See Section 2.3 below for details.) This appears to be the first example of a very general phenomenon.

2.2.2 A partial classification of K-types

We would like to be able to give a complete classification of all the K-types of $\mathrm{GL}_n(F)$, but at this point, that seems to be a hard problem in general. (It has been done by Henniart for $n = 2$ in [10]; see Section 2.3 for a summary.) It would be incredibly helpful here to have some concept for K-types analogous to the concept of a cover in the theory of types (cf. Section 2.1.4 above). So far, we have been unable to find a such a precise construction. However, we can at least be much more specific about where to look for K-types, thanks to Proposition 2.2.3 above.

Let $\mathfrak{s} = [L, \sigma]_G$ be any inertial equivalence class in $\mathcal{B}(G)$. By conjugating L as necessary, we may assume it is a standard Levi subgroup, whence it is isomorphic to $\prod_{i=1}^{r} \mathrm{GL}_{n_i}(F)$ for some partition $n = \sum_{i=1}^{r} n_i$. To simplify notation, we will let $G_i = \mathrm{GL}_{n_i}(F)$ and $K_i = \mathrm{GL}_{n_i}(\mathfrak{o}_F)$ for each i. With this expression for L, we have

$$\sigma \cong \sigma_1 \otimes \cdots \otimes \sigma_r$$

for some irreducible supercuspidal representations σ_i of G_i. For each i, let τ_i be the (unique up to isomorphism) K_i-type occurring in $\sigma_i|_{K_i}$, as guaranteed by Proposition 2.2.3. Note that

$$L \cap K \cong \prod_{i=1}^{r} K_i.$$

Thus we may define a representation τ of $L \cap K$ by

$$\tau = \tau_1 \otimes \cdots \otimes \tau_r.$$

Theorem 2.2.8. *Assume the notation above. Any K-type for $\mathfrak{s}$ must occur in $\mathrm{ind}_{L \cap K}^{K}(\tau)$.*

Proof. Thanks to Lemma 2.2.4 and its corollary, we know that every irreducible representation of K that occurs in $\pi|_K$, for any irreducible (or indeed any) representation π in $\mathfrak{R}^{\mathfrak{s}}(G)$, is an irreducible component of $\mathrm{ind}_{L \cap K}^{K}(\sigma|_{L \cap K})$. Clearly

$$\sigma|_{L \cap K} \cong \sigma_1|_{K_1} \otimes \cdots \otimes \sigma_r|_{K_r}.$$

For each i, we know by Proposition 2.2.3 that τ_i is the only K_i-type occurring in $\sigma_i|_{K_i}$, and it occurs in this representation with multiplicity 1. Thus, for each i, we may write

$$\sigma_i|_{K_i} \cong \tau_i \oplus \bigoplus_{j} \lambda_{i,j}$$

22

where the $\lambda_{i,j}$ are irreducible representations of K_i which are *not* K_i-types. (The j here are taken to be elements of some sufficiently large index sets, which we otherwise do not care about.) Every irreducible component of $\sigma|_{L\cap K}$ must therefore have the form $\bigotimes_{i=1}^{r}\rho_i$, where each ρ_i is isomorphic either to τ_i or to $\lambda_{i,j}$ for some j. The component for which $\rho_i \cong \tau_i$ for all i is of course just τ, and this clearly occurs in $\sigma|_{L\cap K}$ with multiplicity 1. Every other irreducible component of $\sigma|_{L\cap K}$ must therefore be of the form $\rho = \bigotimes_{i=1}^{r}\rho_i$, where at least one of the ρ_i is *not* a K_i-type. In the latter case, we consider the representation $\mathrm{ind}_{L\cap K}^{K}(\rho)$, and we claim that none of its irreducible components may be a K-type.

To prove this, write $\rho = \bigotimes_{i=1}^{r}\rho_i$, and choose an l $(1 \leq l \leq r)$ for which ρ_l is not a K_l-type. Then by definition, there must be an irreducible representation π_l of G_l that is *not* inertially equivalent to σ_l, but such that $\langle \pi_l, \rho_l\rangle_{K_l} \neq 0$. Let

$$\sigma' = \sigma_1 \otimes \cdots \otimes \pi_l \otimes \cdots \otimes \sigma_r.$$

(This is the same as the factorization of σ, but with π_l in place of σ_l in the l place.) Clearly no irreducible subquotient of $\mathrm{ind}_L^G(\sigma')$ can have inertial support $\mathfrak{s}$. However, the restriction of σ' to $L \cap K$ contains ρ. Applying Lemma 2.2.4 again, $\mathrm{Res}_K^G \mathrm{ind}_L^G(\sigma') \cong \mathrm{ind}_{L\cap K}^K(\sigma'|_{L\cap K})$ contains $\mathrm{ind}_{L\cap K}^K(\rho)$. Thus every irreducible component of $\mathrm{ind}_{L\cap K}^K(\rho)$ appears in representations of G with inertial support $\mathfrak{s}$ as well as representations of G with inertial support different from $\mathfrak{s}$, and thus cannot be a K-type. $\square$

It should be true that the K-types for $\mathfrak{s}$ are the irreducible components of $\mathrm{ind}_{L\cap K}^K(\tau)$ appearing at the "lowest level(s)", for some appropriate notion of level. I.e., the K-types for $\mathfrak{s}$ should occur in representations induced from subgroups of K of small index. Of course, the index of a subgroup determines the dimension of an induced representation, so it may be reasonable that the K-types for $\mathfrak{s}$ are precisely the irreducible components of $\mathrm{ind}_{L\cap K}^K(\tau)$ with dimension less than some bound (which will certainly depend on $\mathfrak{s}$). In light of this, we make the following definition.

Definition 2.2.9. Let (π, V) be an irreducible representation of G. A *minimal K-type* for π is a K-type of minimal dimension occurring in $\pi|_K$.

Clearly, since every irreducible representation π of G contains some K-type, π must contain some minimal K-type. Some natural questions to ask, then, are the following:

1. Can there be more than one isomorphism class of minimal K-type occurring in π?

2. Can a minimal K-type occur in π with multiplicity greater than 1?

3. What application is there for a classification of irreducible representations of G in terms of minimal K-types?

After having tediously worked out several examples, it appears that the answer to the first two questions is "no", and that there is a rather interesting number-theoretic answer to the third question. We will state both of these as conjectures, and will discuss the proofs of both conjectures in a few special cases. We state the first conjecture now; the second, which depends on this one, will have to wait until after we have developed a few more prerequisites.

Conjecture 1. *Let (π, V) be an irreducible representation of $\mathrm{GL}_n(F)$, and let $K = \mathrm{GL}_n(\mathfrak{o}_F)$. Then π contains a unique minimal K-type, and it occurs in $\pi|_K$ with multiplicity 1.*

Remark 1. It is not completely clear that the definition of minimality given above (smallest dimension) is the correct one to make this conjecture work in all cases. If not, however, then there should be some natural notion of "level" that makes this conjecture and Conjecture 2 below work correctly. A careful analysis of the various cases when $n = 2$ and $n = 3$ (see Sections 2.3 and 2.4 below) will reveal the motivating idea here.

Note that there is one special case of this conjecture that is readily verified using well known results. Consider the "unramified" Bernstein component of $G = \mathrm{GL}_n(F)$ for $n \geq 2$ (cf. Example 1 above). This is the Bernstein component corresponding to $\mathfrak{s}_\mathfrak{o} = [T, 1_T]_G$, where T is a maximal split torus of G and 1_T denotes the trivial character of T. The irreducible representations in $\mathfrak{R}^{\mathfrak{s}_0}(G)$ include all of the unramified principal series, as well as the unramified quasicharacters of G and the "arithmetically unramified" Steinberg representation.

It is well known that an irreducible representation is in this Bernstein component if and only if it contains the trivial character of an Iwahori subgroup J of G. We may take J to be the subgroup of K consisting of all matrices whose reduction modulo $\mathfrak{p}$ is upper-triangular. Thus, $(J, 1_J)$ is a type for $\mathfrak{s}_0$. Therefore, all of the irreducible components of $\mathrm{Ind}_J^K(1)$ are K-types for $\mathfrak{s}_0$. Obviously, this includes the trivial character 1_K, and it is also well known that an irreducible representation of G contains 1_K if and only if it is unramified. (Indeed, this is often given as the definition of an unramified representation of G.) Furthermore, it is well known that such a representation contains 1_K with multiplicity 1. Clearly no other irreducible component of $\mathrm{Ind}_J^K(1)$ can have dimension 1, so this verifies Conjecture 1

for the unramified representations of G. As mentioned above, we need only deal with K-types up to twisting by characters. So, combining this with Proposition 2.2.3, we have the following.

Lemma 2.2.10. *Let (π, V) be an irreducible representation of G that either is supercuspidal or is a twist of an unramified representation. Then π contains a unique minimal K-type $\tau(\pi)$, and it occurs in $\pi|_K$ with multiplicity 1. If $\pi \cong \chi \otimes \pi'$ for some unramified representation π' and character χ of G, then $\tau(\pi) = \chi|_K$. If π is supercuspidal, then $\tau(\pi) = \mathrm{Ind}_J^K(\lambda)$, where (J, λ) is a type for the Bernstein component of π.*

As previously mentioned, in the supercuspidal case, such types (J, λ) were constructed in [7], and are referred to there as *maximal simple types*. We will review the definition and key properties of maximal simple types in Section 3.1 below.

2.2.3 Inertial Weil-Deligne representations

Let W_F be the Weil group of F, and let I_F be the inertia subgroup. Recall that with its natural topology, W_F is a locally profinite (i.e., locally compact, totally disconnected) group, and thus a smooth representation of W_F will have the same meaning as above. Since we will only be interested here in finite-dimensional representations of W_F, we point out that a finite-dimensional representation of such a group is smooth if and only if it is continuous.

Fix a uniformizer ϖ for F. Let Φ be a geometric Frobenius element of W_F, chosen such that the Artin reciprocity isomorphism $W_F^{\mathrm{ab}} \to F^\times$ of local class field theory maps Φ to ϖ. Then for any $\tau \in W_F$, there exists a unique $n \in \mathbb{Z}$ and $\tau_0 \in I_F$ such that $\tau = \Phi^n \tau_0$. We define $\nu_F(\tau) = n$ and $|\tau| = q^{-n}$. (Note that these definitions are independent of the choice of Φ.) Recall the following important definition.

Definition. A *Weil-Deligne representation* of F is a triple (ρ, V, N), in which (ρ, V) is a finite-dimensional, semisimple, smooth, complex representation of W_F and $N \in \mathrm{End}(V)$ satisfies

$$\rho(\tau)N\rho(\tau)^{-1} = |\tau|\, N \quad \text{for all } \tau \in W_F$$

The N in this definition is referred to as the *monodromy operator* of the representation. Note that the defining relation above implies that N must be nilpotent. Indeed, if λ is an eigenvalue of N with eigenvector v, we see that for all $\tau \in W_F$, $\frac{\lambda}{|\tau|}$ is an eigenvalue with eigenvector $\rho(\tau)v$. If λ

were nonzero, this would yield infinitely many distinct eigenvalues of N, a contradiction. Thus every eigenvalue of N is 0, whence N is nilpotent. The definition also implies immediately that $\operatorname{Ker} N$ is a subspace of V that is stable under the representation ρ, and is nonzero since N is nilpotent. Thus if (ρ, V) is irreducible, we must have $N = 0$.

Morphisms of Weil-Deligne representations are defined in the obvious way: given Weil-Deligne representations (ρ_1, V_1, N_1) and (ρ_2, V_2, N_2), a morphism between them is a map $f : V_1 \to V_2$ that is W_F-equivariant and also satisfies $f N_1 = N_2 f$. Direct sums, tensor products, and contragredients of Weil-Deligne representations are defined by the following formulas:

$$(\rho_1, V_1, N_1) \oplus (\rho_2, V_2, N_2) = (\rho_1 \oplus \rho_2, V_1 \oplus V_2, N_1 \oplus N_2)$$

$$(\rho_1, V_1, N_1) \otimes (\rho_2, V_2, N_2) = (\rho_1 \otimes \rho_2, V_1 \otimes V_2, (N_1 \otimes 1_{V_2}) + (1_{V_1} \otimes N_2))$$

$$(\rho, V, N)^\vee = (\check{\rho}, \check{V}, -{}^t N)$$

Following standard practice, we will say that a Weil-Deligne representation (ρ, V, N) is *semisimple* if $N = 0$, *irreducible* if the representation (ρ, V) is irreducible, and *indecomposable* if it cannot be written as a direct sum of two nonzero Weil-Deligne representations. As noted above, an irreducible Weil-Deligne representation must be semisimple. Likewise, an irreducible Weil-Deligne representation must clearly be indecomposable. Conversely, an indecomposable Weil-Deligne representation is irreducible if and only if it is semisimple.

For $n \geq 1$, let $V = \mathbb{C}^n$ and define a smooth representation ρ of W_F on V by

$$\rho(\tau) = \begin{pmatrix} |\tau|^{\frac{n-1}{2}} & & & & \\ & |\tau|^{\frac{n-3}{2}} & & & \\ & & \ddots & & \\ & & & |\tau|^{\frac{3-n}{2}} & \\ & & & & |\tau|^{\frac{1-n}{2}} \end{pmatrix}$$

for all $\tau \in W_F$, and define $N \in \operatorname{End}(V)$ by

$$N = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}.$$

Then (ρ, V, N) is a Weil-Deligne representation, called the n-dimensional special representation. We will denote this representation by $\mathrm{Sp}(n)$. The following proposition is well known.

Proposition 2.2.11. *Every indecomposable n-dimensional Weil-Deligne representation is isomorphic to one of the form $(\rho, V, 0) \otimes \mathrm{Sp}(m)$, for some divisor m of n and some irreducible $\frac{n}{m}$-dimensional representation (ρ, V) of W_F.*

We will denote by $\mathcal{G}_n(F)$ the set of isomorphism classes of n-dimensional Weil-Deligne representations, by $\mathcal{G}_n^{\mathrm{ss}}(F)$ the subset of semisimple ones, and by $\mathcal{G}_n^0(F)$ the subset of irreducible ones. Similarly, we will denote by $\mathcal{A}_n(F)$ the set of isomorphism classes of irreducible smooth representations of $\mathrm{GL}_n(F)$, and by $\mathcal{A}_n^0(F)$ the subset of supercuspidal ones. The local Langlands correspondence gives, for each $n > 0$, a natural bijection

$$\pi_n : \mathcal{G}_n(F) \to \mathcal{A}_n(F), \tag{2.2.1}$$

which (i) reduces to the Artin reciprocity map for $n = 1$, (ii) is compatible with twisting by characters and taking contragredients, and (iii) preserves L-functions and ϵ-factors. (Note that these properties do not completely characterize the local Langlands correspondence.) For a Weil-Deligne representation σ, we will often write $\pi(\sigma)$ instead of $\pi_n(\sigma)$ when the value of n is clear from the context. Under this correspondence, the irreducible Weil-Deligne representations map to the supercuspidal representations of $\mathrm{GL}_n(F)$:

$$\pi_n^0 : \mathcal{G}_n^0(F) \to \mathcal{A}_n^0(F). \tag{2.2.2}$$

Furthermore, the results of [12] and [33] showed that, in a certain sense, combining irreducible Weil-Deligne representations by taking direct sums and tensoring with $\mathrm{Sp}(m)$ corresponds to combining supercuspidal representations by the "Langlands sum" operation (essentially, applying parabolic induction and taking a composition factor). In particular, this showed that the full correspondence (2.2.1) could be deduced from the restricted map (2.2.2). This also implies the following, which provides the primary justification for the terminology of "inertial equivalence".

Proposition. *Suppose that $\sigma_1 = (\rho_1, V_1, N_1)$ and $\sigma_2 = (\rho_2, V_2, N_2)$ are n-dimensional Weil-Deligne representations, and let $\pi_1 = \pi(\sigma_1)$ and $\pi_2 = \pi(\sigma_2)$ be the corresponding irreducible representations of $\mathrm{GL}_n(F)$. Then $\mathcal{I}(\pi_1) = \mathcal{I}(\pi_2)$ if and only if $\rho_1|_{I_F} \cong \rho_2|_{I_F}$.*

In other words, two irreducible representations of $\mathrm{GL}_n(F)$ are inertially equivalent (i.e., have the same inertial support, or equivalently are in the same Bernstein component) if and only if the corresponding Weil-Deligne representations have isomorphic restrictions to the inertia subgroup I_F. Note that, in this setting, restriction to the inertia subgroup simply forgets the monodromy operator N entirely. In light of this, it is natural to consider a way of restricting to inertia that retains the monodromy operator. The following definition is due to Weinstein, in [32].

Definition 2.2.12. An *inertial Weil-Deligne representation* is a triple (ρ, V, N) in which (ρ, V) is a finite-dimensional, semisimple, smooth (i.e., continuous), complex representation of I_F, and $N \in \mathrm{End}(V)$, such that there exists a Weil-Deligne representation $(\widetilde{\rho}, V, N)$ satisfying $\widetilde{\rho}|_{I_F} = \rho$.

Let $\mathcal{G}_n^I(F)$ denote the set of isomorphism classes of n-dimensional inertial Weil-Deligne representations. Given $\sigma = (\rho, V, N) \in \mathcal{G}_n(F)$, we let $\sigma|_{I_F}$ denote the inertial Weil-Deligne representation $(\rho|_{I_F}, V, N)$. We are now ready to state our second conjecture.

Conjecture 2. *Assume Conjecture 1, and for $\pi \in \mathcal{A}_n(F)$, let $\tau(\pi)$ denote the unique minimal K-type of π. Suppose that $\sigma_1 = (\rho_1, V_1, N_1)$ and $\sigma_2 = (\rho_2, V_2, N_2)$ are n-dimensional Weil-Deligne representations, and let $\pi_1 = \pi(\sigma_1)$ and $\pi_2 = \pi(\sigma_2)$ be the irreducible representations of $\mathrm{GL}_n(F)$ corresponding to them under the local Langlands correspondence. Then $\tau(\pi_1) \cong \tau(\pi_2)$ if and only if $\sigma_1|_{I_F} \cong \sigma_2|_{I_F}$.*

In other words, this conjecture asserts that, assuming minimal K-types do indeed occur uniquely in irreducible representations of $\mathrm{GL}_n(F)$, the minimal K-type of such a representation is uniquely determined by the restriction to inertia of the corresponding Weil-Deligne representation, and vice-versa.

It is well known (from the results of [12] and [33] again, or from [13]) that a given Bernstein component $\mathfrak{R}^{\mathfrak{s}}(G)$ is nondegenerate if and only if, for all irreducible representations π in $\mathfrak{R}^{\mathfrak{s}}(G)$, the Weil-Deligne representation corresponding to π is semisimple. Thus from the proposition above, it is clear that for any nondegenerate Bernstein component $\mathfrak{R}^{\mathfrak{s}}(G)$, if Conjecture 1 is true for any (equivalently every) irreducible representation π in $\mathfrak{R}^{\mathfrak{s}}(G)$, then Conjecture 2 is true for all such π as well.

2.2.4 Status of the theory

Before continuing, a note may be in order on the history of the concept of K-types introduced here. The idea of studying a representation of a reductive

group G by looking at its restriction to a maximal compact subgroup K goes almost as far back as the history of the whole subject. For Lie groups, this and similar approaches have been utilized for decades. Perhaps the most noteworthy of these (and clearly influential to our way of thinking here) is Vogan's theory of minimal K-types for a real reductive Lie group (cf. [31]). For p-adic groups, certain examples of what we have called K-types were constructed for $\mathrm{GL}_2(F)$ in [14] and much more generally for $\mathrm{GL}_n(F)$ in [20]. Since then, the efforts to classify representations of such a group G by its restriction to compact subgroups have generally not dealt exclusively with a maximal compact subgroup K. Obviously, there has been tremendous activity in this area, culminating in the theory of types (as described in Section 2.1).

The exact definition of a K-type for $\mathfrak{s} \in \mathcal{B}(G)$ that we have given here seems to have first appeared in the appendix to [10]. There, Henniart refers to such a representation (in French) as *typique* for $\mathfrak{s}$, and he gives a complete classification of all such representations for $\mathrm{GL}_2(F)$. It is quite easy to see that his classification verifies Conjecture 1 for $n = 2$. Subsequently, Henniart's proof of the uniqueness of K-types for supercuspidal $\mathfrak{s} \in \mathcal{B}(G)$ was generalized (as mentioned above) to $\mathrm{GL}_n(F)$ for arbitrary n in [24]. Therein, an inertial local Langlands correspondence for supercuspidal representations was observed to hold. (Indeed, this case of Conjecture 2 is a trivial corollary of Conjecture 1 and the local Langlands correspondence.) Shortly thereafter, a more or less complete version of Conjecture 2 was stated in [32], although in slightly different terms. Again, this case of the conjecture follows relatively easily from Henniart's classification of the K-types of $\mathrm{GL}_2(F)$. The author is unaware of any more recent developments on this topic.

As for the status of the two conjectures above, Conjecture 1 is true for $\mathrm{GL}_2(F)$ (see the next section for a summary) and for all supercuspidal Bernstein components for $\mathrm{GL}_n(F)$, for arbitrary n. Other than that, it appears that very few additional cases have been proved. Conjecture 2 is also true for $\mathrm{GL}_2(F)$ (see the next section for details). We have shown here that Conjecture 2 follows from Conjecture 1 for all nondegenerate Bernstein components. In Section 2.4 we will prove both conjectures for the degenerate Bernstein components of $\mathrm{GL}_3(F)$.

A few words may also be in order on our choice of the terms "K-type" and "minimal K-type". In the opinion of the author, Henniart's choice of terminology ("*typique pour* $\mathfrak{s}$") translates rather poorly into English: the representations being studied are rather special in their relation to $\mathfrak{s}$, and thus quite the opposite of typical. Unfortunately, there is already another kind of representation of a compact open subgroup of G that has been

referred to as a "minimal K-type". These were defined by Moy in [23] and further developed in [19], in the hope of finding a theory of minimal K-types for p-adic groups analogous to Vogan's theory for real groups. While Moy's minimal K-types provided a crucial step in the development of the theory of types, they have more often been referred to as "fundamental strata" (cf. [11]).

2.3 K-types for $\mathrm{GL}_2(F)$

Let $G = \mathrm{GL}_2(F)$ and $K = \mathrm{GL}_2(\mathfrak{o}_F)$. In this setting, a complete classification of all K-types for G was given in the appendix of [10]. From that classification, Conjecture 1 for G is a fairly easy observation. Indeed, for $\mathrm{GL}_2(F)$ there is only one twist class of degenerate Bernstein component. Thus for all other $\mathfrak{s} \in \mathcal{B}(G)$, every K-type for $\mathfrak{s}$ is in fact a type (Lemma 2.2.7), and it turns out that there are at most two of them (and only one when $q \neq 2$). Furthermore, they always occur with multiplicity 1 in each irreducible representation of $\mathfrak{R}^{\mathfrak{s}}(G)$, and it is quite clear that when there are two of them, their dimensions are not equal.

Thus, the only case that needs special attention is the unramified Bernstein component, corresponding to $\mathfrak{s} = [T, 1_T]_G$ (as discussed in Example 1 on p. 21). In this case, there are exactly two K-types, and neither is a type. One is given by the trivial character 1_K, and the other is the inflation of the Steinberg representation from $\mathrm{GL}_2(\mathbf{k}_F)$, which has dimension q. We will refer to the latter by st_2. The irreducible representations in $\mathfrak{R}^{\mathfrak{s}}(G)$ fall into three distinct classes. The generic case is the unramified principal series $\mathrm{ind}_B^G(\chi_1 \otimes \chi_2)$, where B is the Borel subgroup of upper-triangular matrices in G, and χ_1 and χ_2 are unramified characters of $F^\times$ such that $\chi_1 \chi_2^{-1} \neq |\cdot|^{\pm 1}$. Clearly by Lemma 2.2.4, these representations contain *both* 1_K and st_2. The other two classes are the unramified characters of G, which obviously contain only the K-type 1_K, and the unramified twists of the Steinberg representation of G, which we will denote St_2. The Steinberg representation is the unique infinite-dimensional composition factor of $\mathrm{ind}_B^G(\delta_B^{\frac{1}{2}})$. Since the other composition factor is 1_G, it is again clear from Lemma 2.2.4 that the only K-type contained in St_2 is st_2.

Table 2.1 lists all of the isomorphism classes of 2-dimensional Weil-Deligne representations σ of F up to twisting by characters, along with the irreducible representation $\pi(\sigma)$ of G corresponding to σ under the local Langlands correspondence, and in the degenerate cases, the K-types that occur in $\pi(\sigma)|_K$. To aid the reader in verifying that the representations cor-

respond as we have claimed, we have also listed the L-functions and ϵ-factors of both σ and $\pi(\sigma)$. For the sake of brevity, we have also made the following minor abuses of notation: (i) We freely equate abelian characters of W_F with characters of $F^\times$, as they correspond naturally under the Artin reciprocity map of local class field theory. (ii) We denote the trivial character of W_F or $F^\times$ simply by 1.

To verify Conjecture 2 for G, we use Table 2.1. Again, the conjecture is clear for the nondegenerate Bernstein components, so we need only consider rows 1.1, 1.2, and 1.3 in the table, which correspond to the unramified Bernstein component of G. The Weil-Deligne representations in rows 1.1 and 1.2 both have $N = 0$, whereas the special representation in row 1.3 has $N = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Thus the representations in rows 1.1 and 1.2 restrict to the same inertial Weil-Deligne representations, and they also clearly have the same *minimal K-type*. The special representation, on the other hand, restricts to a different inertial Weil-Deligne representation, and it is the only one whose *minimal K-type* is st_2. This completes the proof of Conjecture 2 for $n = 2$.

Table 2.1: 2-dimensional Weil-Deligne representations

	σ	$\pi(\sigma)$	K-types in $\pi(\sigma)$	$L(s,\sigma) = L(s,\pi(\sigma))$	$\varepsilon(s,\sigma,\psi)$ $= \varepsilon(s,\pi(\sigma),\psi)$
1.1	$1 \oplus \chi$, χ unramified, $\chi \neq \lvert\cdot\rvert^{\pm 1}$	$\mathrm{ind}_B^G(1 \otimes \chi)$	$1_K \oplus \mathrm{st}_2$	$(1 - q^{-s})^{-1}$ $\cdot(1 - \chi(\Phi)q^{-s})^{-1}$	$q^{2s-1}\chi(\Phi)^{-1}$
1.2	$\lvert\cdot\rvert^{\frac{1}{2}} \oplus \lvert\cdot\rvert^{-\frac{1}{2}}$	1_G	1_K	$(1 - q^{-s-\frac{1}{2}})^{-1}$ $\cdot(1 - q^{-s+\frac{1}{2}})^{-1}$	q^{2s-1}
1.3	$\mathrm{Sp}(2)$	St_2	st_2	$(1 - q^{-s-\frac{1}{2}})^{-1}$	$-q^{s-\frac{1}{2}}$
2	$1 \oplus \chi$, χ ramified	$\mathrm{ind}_B^G(1 \otimes \chi)$	Unique (except when $q = 2$)	$(1 - q^{-s})^{-1}$	$q^{s-\frac{1}{2}}\varepsilon(s,\chi,\psi)$
3	Irreducible	Supercuspidal	Unique	1	Not specified

2.4 K-types for $\mathrm{GL}_3(F)$

Let $G = \mathrm{GL}_3(F)$ and $K = \mathrm{GL}_3(\mathfrak{o}_F)$. While we will not give here a complete classification of all K-types for G, we will construct all of them for the two classes of degenerate Bernstein components. We will then use this to show that, if Conjecture 1 can be proved for the remaining nondegenerate cases, Conjecture 2 will follow. Let B denote the Borel subgroup of upper-triangular matrices in G, and let T be the diagonal subgroup of B. Let P denote the block-upper-triangular parabolic subgroup that has a 2×2 block and a 1×1 on the diagonal, and let L denote the corresponding block-diagonal Levi subgroup isomorphic to $\mathrm{GL}_2(F) \times F^\times$. Throughout the following discussion, the same minor abuses of notation will be used as mentioned in the previous section.

Given any degenerate $\mathfrak{s} \in \mathcal{B}(G)$, it is possible to twist $\mathfrak{s}$ by a character of $F^\times$ so that it has one of the following two forms:

1. $\mathfrak{s} = [T, 1_T]_G$, or

2. $\mathfrak{s} = [L, 1_{\mathrm{GL}_2(F)} \otimes \chi]_G$ for some ramified character χ of $F^\times$.

2.4.1 First case

Assume first that $\mathfrak{s} = [T, 1_T]_G$. Just as before, a type for this class is $(J, 1_J)$, where J is the Iwahori subgroup of matrices in K that are upper-triangular modulo $\mathfrak{p}$. As this is merely the inflation of the trivial character of the corresponding Borel subgroup $\overline{B}$ of $\overline{G} = \mathrm{GL}_3(\boldsymbol{k}_F)$, the decomposition of $\mathrm{Ind}_J^K(1)$ is given by decomposing $\mathrm{Ind}_{\overline{B}}^{\overline{G}}(1)$ and inflating all of its components to K. This decomposition is well known, for example from [18], but we repeat it here, as the construction will be useful later. Define

$$J_{1,2} = \begin{pmatrix} \mathfrak{o}_F^\times & \mathfrak{o}_F & \mathfrak{o}_F \\ \mathfrak{p}_F & \multicolumn{1}{c}{} & \mathrm{GL}_2(\mathfrak{o}_F) \\ \mathfrak{p}_F & & \end{pmatrix}$$

and

$$J_{2,1} = \begin{pmatrix} \mathrm{GL}_2(\mathfrak{o}_F) & & \mathfrak{o}_F \\ & & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \mathfrak{o}_F^\times \end{pmatrix}.$$

Let $V = \operatorname{Ind}_J^K(1)$, $V_{1,2} = \operatorname{Ind}_{J_{1,2}}^K(1) \subset V$, $V_{2,1} = \operatorname{Ind}_{J_{2,1}}^K(1) \subset V$, and $V_0 = V_{1,2} \cap V_{2,1} = \operatorname{Ind}_K^K(1)$. Then obviously $V_0 \cong 1_K$, and we define

$$\operatorname{st}_{1,2} = V_{1,2}/V_0 \ ,$$
$$\operatorname{st}_{2,1} = V_{2,1}/V_0 \ , \text{ and}$$
$$\operatorname{st}_3 = V_3/(V_{1,2} + V_{2,1}) \ .$$

It is not hard to show that each of these three representations is irreducible and that $\operatorname{st}_{1,2} \cong \operatorname{st}_{2,1}$, and it is thus clear that $V \cong 1 \oplus 2\operatorname{st}_{1,2} \oplus \operatorname{st}_3$. Since any other K-type for $\mathfrak{s}$ must be an irreducible component of $\operatorname{Ind}_B^K(1)$ (Theorem 2.2.8), it is not hard to see by counting dimensions that these are the smallest possible K-types for $\mathfrak{s}$.

We will also need to know exactly which of these K-types is contained in all of the various irreducible representations in $\mathfrak{R}^{\mathfrak{s}}(G)$. We temporarily let $P_{2,1} = P$ and let $P_{1,2}$ be the obvious other block-upper-triangular parabolic subgroup, and we let $L_{2,1}$ and $L_{1,2}$ be the corresponding Levi subgroups. All of the following is summarized in Table 2.2.

The parabolically induced representation $\operatorname{ind}_B^G(\delta_B^{-\frac{1}{2}}) = \operatorname{ind}_B^G(|\cdot|\otimes 1 \otimes |\cdot|^{-1})$ has four composition factors, which we now briefly describe. Figure 2.1 may help to illuminate this explanation. One composition factor is the trivial character 1_G, which corresponds under the local Langlands correspondence to the Weil-Deligne representation $|\cdot| \oplus 1 \oplus |\cdot|^{-1}$. Obviously the only K-type contained in 1_G is 1_K. The Weil-Deligne representation

$$\left(|\cdot|^{\frac{1}{2}} \otimes \operatorname{Sp}(2)\right) \oplus |\cdot|^{-1} = \left(|\cdot| \oplus 1 \oplus |\cdot|^{-1}, \mathbb{C}^3, \begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix}\right)$$

corresponds to a representation that we will denote $\operatorname{St}_{2,1}$, which is the unique composition factor occurring in both $\operatorname{ind}_{P_{2,1}}^G(|\cdot|^{\frac{1}{2}} \operatorname{St}_2 \otimes |\cdot|^{-1})$ and $\operatorname{ind}_{P_{1,2}}^G(|\cdot| \otimes |\cdot|^{-\frac{1}{2}} 1_{\operatorname{GL}_2(F)})$. Comparing this to the decomposition of $\operatorname{Ind}_J^K(1)$ above, it is clear that the only K-type of the above three that is contained in $\operatorname{St}_{2,1}$ is $\operatorname{st}_{1,2}$, with multiplicity 1. Similarly, the Weil-Deligne representation

$$|\cdot| \oplus \left(|\cdot|^{-\frac{1}{2}} \otimes \operatorname{Sp}(2)\right) = \left(|\cdot| \oplus 1 \oplus |\cdot|^{-1}, \mathbb{C}^3, \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & & 0 \end{pmatrix}\right)$$

is contragredient to the previous one, and corresponds to a representation that we will denote $\operatorname{St}_{1,2}$ (which is thus contragredient to $\operatorname{St}_{2,1}$). $\operatorname{St}_{1,2}$ is

the unique composition factor occurring in both $\mathrm{ind}_{P_{2,1}}^G(|\cdot|^{\frac{1}{2}} 1_{\mathrm{GL}_2(F)} \otimes |\cdot|^{-1})$
and $\mathrm{ind}_{P_{1,2}}^G(|\cdot| \otimes |\cdot|^{-\frac{1}{2}} \mathrm{St}_2)$, and thus also contains only the K-type $\mathrm{st}_{1,2}$ with
multiplicity 1. And finally, the Weil-Deligne representation

$$
\mathrm{Sp}(3) = \left(|\cdot| \oplus 1 \oplus |\cdot|^{-1}, \mathbb{C}^3, \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix} \right)
$$

corresponds to a representation that we will denote St_3, which is the unique
composition factor occurring in both $\mathrm{ind}_{P_{2,1}}^G(|\cdot|^{\frac{1}{2}} \mathrm{St}_2 \otimes |\cdot|^{-1})$ and $\mathrm{ind}_{P_{1,2}}^G(|\cdot| \otimes$
$|\cdot|^{-\frac{1}{2}} \mathrm{St}_2)$. Again, comparing this to the decomposition of $\mathrm{Ind}_J^K(1)$ above,
we see that the only K-type of the above three that is contained in St_3 is
st_3, with multiplicity 1.

Figure 2.1: Decomposition of $\mathrm{ind}_B^G(\delta_B^{-\frac{1}{2}})$. Here G_2 denotes $\mathrm{GL}_2(F)$.

If χ is an unramified quasicharacter of $F^\times$ other than $|\cdot|^{\pm\frac{3}{2}}$, we can
similarly decompose $\mathrm{ind}_B^G(|\cdot|^{\frac{1}{2}} \otimes |\cdot|^{-\frac{1}{2}} \otimes \chi)$ into two composition factors. This
is most easily seen by first parabolically inducing from T to L, where the
decomposition is just as it is for $\mathrm{GL}_2(F)$, then parabolically inducing from L
to G. From this it also becomes clear that one composition factor contains
two of our K-types, and the other factor also contains two. (See Table 2.2 for

details.) And finally, if χ_1 and χ_2 are two unramified quasicharacters of $F^\times$, such that none of χ_1, χ_2, and $\chi_1\chi_2^{-1}$ is equal to $|\cdot|^{\pm 1}$, then $\mathrm{ind}_B^G(1 \otimes \chi_1 \otimes \chi_2)$ is irreducible, and thus clearly contains the K-types $1 \oplus 2\mathrm{st}_{1,2} \oplus \mathrm{st}_3$.

2.4.2 Second case

Now let χ be a ramified character of $F^\times$, and assume $\mathfrak{s} = [L, 1_{\mathrm{GL}_2(F)} \otimes \chi]_G$. Let m be the level of χ (so that χ is trivial on $U^{m+1}(F)$, but not on $U^m(F)$, where $U^k(F)$ is $1 + \mathfrak{p}^k$ for $k > 0$ and $\mathfrak{o}_F^\times$ if $k = 0$). Let J be the group

$$
\begin{pmatrix}
\mathfrak{o}_F^\times & \mathfrak{o}_F & \mathfrak{o}_F \\
\mathfrak{p}_F & \mathfrak{o}_F^\times & \mathfrak{o}_F \\
\mathfrak{p}_F^m & \mathfrak{p}_F^m & \mathfrak{o}_F^\times
\end{pmatrix},
$$

and define a character λ of J by

$$
\begin{pmatrix}
a & * & * \\
* & b & * \\
* & * & c
\end{pmatrix} \mapsto \chi(c).
$$

Then (J, λ) is a type for $\mathfrak{s}$. In fact it is the (unique up to conjugation) semisimple type for $\mathfrak{s}$, but that need not concern us here. For the factorization of $V = \mathrm{Ind}_J^G(\lambda)$, we proceed as before. Let

$$
J_{2,1} = \begin{pmatrix}
\mathrm{GL}_2(\mathfrak{o}_F) & & \mathfrak{o}_F \\
 & & \mathfrak{o}_F \\
\mathfrak{p}_F^n & \mathfrak{p}_F^n & \mathfrak{o}_F^\times
\end{pmatrix},
$$

and note that λ can be extended to $J_{2,1}$. Define $V_{2,1} = \mathrm{Ind}_{J_{2,1}}^G(\lambda)$. A lengthy but more or less standard computation shows that the intertwining number of $V_{2,1}$ with itself is 1, and the intertwining number of V with itself is 2. Hence we define $\mathrm{ps}_{2,\chi} = V_{2,1}$ and $\mathrm{st}_{2,\chi} = V/V_{2,1}$, and it is clear that both of these are irreducible, and that $\mathrm{Ind}_J^G(\lambda) \cong \mathrm{ps}_{2,\chi} \oplus \mathrm{st}_{2,\chi}$. It is also not difficult to see by counting dimensions that these are the smallest possible K-types for $\mathfrak{s}$, just as before.

We can describe these last two factors another way, which will show how they are contained in the various irreducible representations in $\mathfrak{R}^\mathfrak{s}(G)$. Let I denote the standard Iwahori subgroup of $\mathrm{GL}_2(\mathfrak{o}_F)$, and recall from Section 2.3 that $\mathrm{Ind}_I^{\mathrm{GL}_2(\mathfrak{o}_F)}(1) = 1 \oplus \mathrm{st}_2$. We can define the representations $1 \otimes \chi$ and $\mathrm{st}_2 \otimes \chi$ on the block-diagonal subgroup $\mathrm{GL}_2(\mathfrak{o}_F) \times \mathrm{GL}_1(\mathfrak{o}_F)$ of $\mathrm{GL}_3(\mathfrak{o}_F)$. We can then extend each of these trivially to $J_{2,1}$. We will continue

to refer to these extensions as $1 \otimes \chi$ and $\mathrm{st}_2 \otimes \chi$. Then it is clear that $\mathrm{Ind}_J^{J_{2,1}}(\lambda)$ is just the direct sum of these two representations, and thus that $\mathrm{ps}_{2,\chi}$ and $\mathrm{st}_{2,\chi}$ are just the inductions from $J_{2,1}$ to K of $1 \otimes \chi$ and $\mathrm{st}_2 \otimes \chi$ respectively.[2]

From the analogous situation in $\mathrm{GL}_2(F)$ (see Section 2.3), it is clear that $\mathrm{ind}_B^G(|\cdot|^{\frac{1}{2}} \otimes |\cdot|^{-\frac{1}{2}} \otimes \chi)$ has two composition factors. One is $\mathrm{ind}_P^G(1_{\mathrm{GL}_2(F)} \otimes \chi)$, which contains the K-type $\mathrm{ps}_{2,\chi}$, and the other is $\mathrm{ind}_P^G(\mathrm{St}_2 \otimes \chi)$, which contains the K-type $\mathrm{st}_{2,\chi}$. On the other hand, for any unramified quasicharacters χ_1 and χ_2 of $F^\times$ such that $\chi_1 \chi_2^{-1} \neq |\cdot|^{\pm 1}$, we see that $\mathrm{ind}_B^G(\chi_1 \otimes \chi_2 \otimes \chi)$ will be irreducible, and thus will contain both K-types. Again, all of this is summarized in Table 2.2.

2.4.3 Summary

Of the nondegenerate $\mathfrak{s} \in \mathcal{B}(G)$, there is actually only one class that presents some difficulty in rigorously proving Conjecture 1. The supercuspidal case is already done. And given ramified quasicharacters χ_1 and χ_2 of $F^\times$ for which $\chi_1 \chi_2^{-1}$ is also ramified, we can construct a K-type for $\mathfrak{s} = [T, 1 \otimes \chi_1 \otimes \chi_2]_G$ as one of Howe's "principal series" for K (cf. [20]). Its multiplicity in any irreducible π of $\mathfrak{R}^{\mathfrak{s}}(G)$ is 1, and it is again possible to show, by the same sort of dimension-counting arguments as before, that such a K-type is minimal. Up to twisting, then, the one remaining case is when $\mathfrak{s} = [L, 1 \otimes \pi]_G$ for some irreducible supercuspidal representation π of $\mathrm{GL}_2(F)$. In this case, we can construct the semisimple type (J, λ) for $\mathfrak{s}$ (cf. [9]), and it will almost certainly turn out that $\mathrm{Ind}_J^K(\lambda)$ is irreducible and is the unique K-type for $\mathfrak{s}$. However, at this point, proving this seems rather difficult.

Though we must for now leave unproved this one case of Conjecture 1 for $G = \mathrm{GL}_3(F)$, we may at least prove all other cases of Conjecture 2 for G. Table 2.2 lists all of the isomorphism classes of 3-dimensional Weil-Deligne representations σ of F up to twisting by characters, along with the irreducible representation $\pi(\sigma)$ of G corresponding to σ under the local Langlands correspondence, and in the degenerate cases, the K-types that occur in $\pi(\sigma)|_K$. As in the previous section, we have again included the L-functions and ϵ-factors of both σ and $\pi(\sigma)$.

To verify Conjecture 2 for $\mathrm{GL}_3(F)$, we need only consider rows 1.1 – 1.7 in Table 2.2, which correspond to the first case considered above, and rows 2.1 – 2.3, which correspond to the second case above. The Weil-Deligne

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[2]This explains the notation for $\mathrm{st}_{2,\chi}$. The notation $\mathrm{ps}_{2,\chi}$ was chosen because this representation is precisely what Howe defines as a *principal series representation of K* in [20].

representations in rows 1.1, 1.2, and 1.4 have $N = 0$, and thus restrict to isomorphic *inertial* Weil-Deligne representations. They also all clearly contain the same *minimal K-type*, namely 1_K. The Weil-Deligne representations in rows 1.3 and 1.5 have

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the one in row 1.6 has

$$N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

These all clearly restrict to isomorphic inertial Weil-Deligne representations, and they also all contain the same minimal K-type, namely $\mathrm{st}_{1,2}$. And finally, the special representation $\mathrm{Sp}(3)$ in row 1.7 clearly restricts to an inertial Weil-Deligne representation that is not isomorphic to the previous two, and it is the only one whose minimal K-type is st_3. The case for rows 2.1 – 2.3 is similar. Thus we have completed the proof of Conjecture 2 for $n = 3$, conditional on the validity of Conjecture 1 for the nondegenerate Bernstein components.

Table 2.2: 3-dimensional Weil-Deligne representations

	σ	$\pi(\sigma)$	K-types in $\pi(\sigma)$	$L(s,\sigma) = L(s,\pi(\sigma))$	$\varepsilon(s,\sigma,\psi)$ $= \varepsilon(s,\pi(\sigma),\psi)$
1.1	$1 \oplus \chi_1 \oplus \chi_2$, $\chi_1,\ \chi_2,$ and $\chi_1\chi_2^{-1}$ unramified and $\neq \lvert\cdot\rvert^{\pm 1}$	$\operatorname{ind}_B^G(1 \otimes \chi_1 \otimes \chi_2)$	$1_K \oplus 2\mathrm{st}_{1,2} \oplus \mathrm{st}_3$	$(1-q^{-s})^{-1}$ $\cdot(1-\chi_1(\Phi)q^{-s})^{-1}$ $\cdot(1-\chi_2(\Phi)q^{-s})^{-1}$	$q^{3s-\frac{3}{2}}\chi_1(\Phi)^{-1}\chi_2(\Phi)^{-1}$
1.2	$\lvert\cdot\rvert^{\frac{1}{2}} \oplus \lvert\cdot\rvert^{-\frac{1}{2}} \oplus \chi$, χ unramified, $\chi \neq \lvert\cdot\rvert^{\pm\frac{3}{2}}$	$\operatorname{ind}_P^G(1_{\mathrm{GL}_2(F)} \otimes \chi)$	$1_K \oplus \mathrm{st}_{1,2}$	$(1-q^{-s-\frac{1}{2}})^{-1}$ $\cdot(1-q^{-s+\frac{1}{2}})^{-1}$ $\cdot(1-\chi(\Phi)q^{-s})^{-1}$	$q^{3s-\frac{3}{2}}\chi(\Phi)^{-1}$
1.3	$\mathrm{Sp}(2) \oplus \chi$, χ unramified, $\chi \neq \lvert\cdot\rvert^{\pm\frac{3}{2}}$	$\operatorname{ind}_P^G(\mathrm{St}_2 \otimes \chi)$	$\mathrm{st}_{1,2} \oplus \mathrm{st}_3$	$(1-q^{-s-\frac{1}{2}})^{-1}$ $\cdot(1-\chi(\Phi)q^{-s})^{-1}$	$-q^{2s-1}\chi(\Phi)^{-1}$
1.4	$\lvert\cdot\rvert \oplus 1 \oplus \lvert\cdot\rvert^{-1}$	1_G	1_K	$(1-q^{-s-1})^{-1}$ $\cdot(1-q^{-s})^{-1}$ $\cdot(1-q^{-s+1})^{-1}$	$q^{3s-\frac{3}{2}}$
1.5	$\left(\lvert\cdot\rvert^{\frac{1}{2}} \otimes \mathrm{Sp}(2)\right) \oplus \lvert\cdot\rvert^{-1}$	$\mathrm{St}_{2,1}$	$\mathrm{st}_{1,2}$	$(1-q^{-s-1})^{-1}$ $\cdot(1-q^{-s+1})^{-1}$	$-q^{2s-\frac{3}{2}}$
1.6	$\lvert\cdot\rvert \oplus \left(\lvert\cdot\rvert^{-\frac{1}{2}} \otimes \mathrm{Sp}(2)\right)$	$\mathrm{St}_{1,2}$	$\mathrm{st}_{1,2}$	$(1-q^{-s-1})^{-1}$ $\cdot(1-q^{-s})^{-1}$	$-q^{2s-\frac{1}{2}}$
1.7	$\mathrm{Sp}(3)$	St_3	st_3	$(1-q^{-s-1})^{-1}$	$q^{s-\frac{1}{2}}$

Continued on next page. . .

Table 2.2: 3-dimensional Weil-Deligne representations – continued

	σ	$\pi(\sigma)$	K-types in $\pi(\sigma)$	$L(s,\sigma)=L(s,\pi(\sigma))$	$\varepsilon(s,\sigma,\psi)$ $=\varepsilon(s,\pi(\sigma),\psi)$
2.1	$1\oplus\chi_1\oplus\chi_2$, χ_1 unramified, χ_2 ramified, $\chi_1\neq|\cdot|^{\pm1}$	$\mathrm{ind}_B^G(1\otimes\chi_1\otimes\chi_2)$	$\mathrm{ps}_{2,\chi_2}\oplus\mathrm{st}_{2,\chi_2}$	$(1-q^{-s})^{-1}$ $\cdot(1-\chi_1(\Phi)q^{-s})^{-1}$	$q^{2s-1}\chi_1(\Phi)^{-1}\varepsilon(s,\chi_2,\psi)$
2.2	$|\cdot|^{\frac{1}{2}}\oplus|\cdot|^{-\frac{1}{2}}\oplus\chi$, χ ramified	$\mathrm{ind}_P^G(1_{\mathrm{GL}_2(F)}\otimes\chi)$	$\mathrm{ps}_{2,\chi}$	$(1-q^{-s-\frac{1}{2}})^{-1}$ $\cdot(1-q^{-s+\frac{1}{2}})^{-1}$	$q^{2s-1}\varepsilon(s,\chi,\psi)$
2.3	$\mathrm{Sp}(2)\oplus\chi$, χ ramified	$\mathrm{ind}_P^G(\mathrm{St}_2\otimes\chi)$	$\mathrm{st}_{2,\chi}$	$(1-q^{-s-\frac{1}{2}})^{-1}$	$-q^{s-\frac{1}{2}}\varepsilon(s,\chi,\psi)$
3	$1\oplus\chi_1\oplus\chi_2$, χ_1, χ_2, and $\chi_1\chi_2^{-1}$ all ramified	$\mathrm{ind}_B^G(1\otimes\chi_1\otimes\chi_2)$	Not given	$(1-q^{-s})^{-1}$	$q^{s-\frac{1}{2}}\varepsilon(s,\chi_1,\psi)\varepsilon(s,\chi_2,\psi)$
4	$1\oplus\rho$, ρ 2-dimensional and irreducible	$\mathrm{ind}_P^G(\pi(\rho)\otimes1)$	Not given	$(1-q^{-s})^{-1}$	Not specified
5	Irreducible	Supercuspidal	Unique	1	Not specified

Chapter 3

Supercuspidal K-types

We now shift our focus toward an application of the theory of types to automorphic forms. This goal will be realized in Chapter 4, but in order to get there, we must build up a significant amount of local theory in this chapter. Our main focus here is on constructing and studying supercuspidal K-types. As stated in Lemma 2.2.10, these are merely induced from the types for supercuspidal representations constructed in [7]. Thus, our first task is to review that construction.

   As in the previous chapter, we assume the following notation. F will denote a nonarchimedean local field, with ring of integers $\mathfrak{o}_F$, prime ideal $\mathfrak{p}_F$ (or simply $\mathfrak{p}$ when there is no possibility of confusion), and residue field $\boldsymbol{k}_F = \mathfrak{o}_F/\mathfrak{p}_F$ of cardinality q.

3.1 Types for supercuspidal representations of $\mathrm{GL}_n(F)$

This section is a brief summary of much of the material in [7], with the ultimate goal of defining "simple types", and in particular "maximal simple types", which classify (and allow construction of) the supercuspidal representations of $\mathrm{GL}_n(F)$. This definition is not found until nearly 200 pages into [7], so we present here only the basic definitions, along with a few of the most important theorems that are necessary in the construction. We give absolutely no proofs here for any of the many claims that we make. For the proofs, or more details on any part of the exposition, see [7] and the many other sources cited therein. Note that most of the statements made here could be proven without much difficulty for the reader who has the time and inclination. Any statements requiring a much more elaborate proof are given as propositions, with a reference to where the proof may be found.

3.1.1 Strata

Definition 3.1.1. For a finite-dimensional F-vector space V, a *lattice in V* is a compact open subgroup of V, and an $\mathfrak{o}_F$-*lattice in V* is a lattice in V that is also an $\mathfrak{o}_F$-submodule of V. For a finite-dimensional F-algebra A, an $\mathfrak{o}_F$-*order in A* is an $\mathfrak{o}_F$-lattice in A that is also a subring of A (with the same 1). Finally, an $\mathfrak{o}_F$-order $\mathfrak{A}$ in A is called *(left) hereditary* if every (left) $\mathfrak{A}$-lattice is $\mathfrak{A}$-projective.

Throughout the rest of Section 3.1, the following notation will be used. We fix once and for all an integer $n > 0$ and an F-vector space V of dimension n. We let $A = \mathrm{End}_F(V)$ and $G = A^\times = \mathrm{Aut}_F(V)$, so that $G \cong \mathrm{GL}_n(F)$.

If $\mathfrak{A}$ is a hereditary $\mathfrak{o}_F$-order in A and $\mathfrak{P}$ is its Jacobson radical, then it is possible to choose a basis of V and a corresponding partition $n_1 + \cdots + n_e$ of n with respect to which

$$
\mathfrak{A} = \begin{pmatrix} \mathfrak{o}_F & \mathfrak{o}_F & \cdots & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{o}_F & \cdots & \mathfrak{o}_F \\ \vdots & & \ddots & \vdots \\ \mathfrak{p}_F & \mathfrak{p}_F & \cdots & \mathfrak{o}_F \end{pmatrix}
$$

and

$$
\mathfrak{P} = \begin{pmatrix} \mathfrak{p}_F & \mathfrak{o}_F & \cdots & \mathfrak{o}_F \\ \mathfrak{p}_F & \mathfrak{p}_F & \cdots & \mathfrak{o}_F \\ \vdots & & \ddots & \vdots \\ \mathfrak{p}_F & \mathfrak{p}_F & \cdots & \mathfrak{p}_F \end{pmatrix}.
$$

(The i, j term in each of these matrices is an $n_i \times n_j$ block. Thus, up to conjugation in G, the choice of a hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ is equivalent to just choosing an ordered partition of n.) Let $e(\mathfrak{A})$ denote the number of terms in the partition of n above. Equivalently, this is the unique integer such that $\mathfrak{p}\mathfrak{A} = \mathfrak{P}^{e(\mathfrak{A})}$, which justifies the notation. (It is also the period of the lattice chain associated to $\mathfrak{A}$; see [7] for more on lattice chains.) We say that $\mathfrak{A}$ is *principal* if $n_i = \frac{n}{e(\mathfrak{A})}$ for all i, or in other words, if all of the blocks in the block matrices above have the same size. (In this case, then, $e(\mathfrak{A})$ divides n, and up to conjugation in G, the choice of a principal hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ is equivalent to simply choosing a divisor of n.) Note that $\mathfrak{P}$ is a two-sided

ideal of $\mathfrak{A}$, and is an invertible fractional ideal of $\mathfrak{A}$ in A, with inverse

$$
\mathfrak{P}^{-1} = \begin{pmatrix} \mathfrak{o}_F & \mathfrak{p}_F^{-1} & \cdots & \mathfrak{p}_F^{-1} \\ \mathfrak{o}_F & \mathfrak{o}_F & \cdots & \mathfrak{p}_F^{-1} \\ \vdots & & \ddots & \vdots \\ \mathfrak{o}_F & \mathfrak{o}_F & \cdots & \mathfrak{o}_F \end{pmatrix}.
$$

For a much more thorough treatment of hereditary orders (and their associated lattice chains) than is found here, or than that found in [7], see [11].

Given a hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in A, we can define a discrete valuation $\nu_{\mathfrak{A}}$ on A by

$$
\nu_{\mathfrak{A}}(x) = \max \left\{ k \in \mathbb{Z} \;\middle|\; x \in \mathfrak{P}^k \right\}.
$$

Definition 3.1.2. Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in A and let $\mathfrak{P} = \mathrm{rad}(\mathfrak{A})$. Define

$$
U^0(\mathfrak{A}) = U(\mathfrak{A}) = \mathfrak{A}^{\times},
$$
$$
U^k(\mathfrak{A}) = 1 + \mathfrak{P}^k \quad \text{for } k > 0,
$$

and define the normalizer of $\mathfrak{A}$ by

$$
\mathfrak{K}(\mathfrak{A}) = \left\{ x \in G \;\middle|\; x^{-1}\mathfrak{A}x = \mathfrak{A} \right\}.
$$

Then, using the block matrix form above, $U(\mathfrak{A})$ is the standard parahoric subgroup of G (a compact open subgroup) given by the partition $n_1 + \cdots + n_e$ of n associated with $\mathfrak{A}$. Likewise, the subgroups $U^k(\mathfrak{A})$ for $k > 0$ give the standard filtration of $U(\mathfrak{A})$ by open normal subgroups. Furthermore, $\mathfrak{K}(\mathfrak{A})$ is an open, compact-mod-center subgroup of G, which is the normalizer in G of $U^k(\mathfrak{A})$ for each $k \geq 0$, and within which $U(\mathfrak{A})$ is the unique maximal compact subgroup. Note that

$$
U(\mathfrak{A})/U^1(\mathfrak{A}) \cong \prod_{i=1}^{e} \mathrm{GL}_{n_i}(\boldsymbol{k}_F).
$$

We now fix once and for all an additive character ψ of F of level 1 (i.e., such that ψ is trivial on $\mathfrak{p}_F$ but nontrivial on $\mathfrak{o}_F$), and we define an additive character ψ_A of A by $\psi_A = \psi \circ \mathrm{Tr}_{A/F}$.

Definition 3.1.3. Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in A and let $\mathfrak{P} = \mathrm{rad}(\mathfrak{A})$ as usual. For nonnegative integers m and r with $\left\lfloor \frac{m}{2} \right\rfloor \leq r < m$ and $\beta \in \mathfrak{P}^{-m}$, define

$$
\psi_{\beta}(x) = \psi_A(\beta(x-1)) \quad \text{for } x \in U^{r+1}(\mathfrak{A}).
$$

Then ψ_β is a character of $U^{r+1}(\mathfrak{A})$ that is trivial on $U^{m+1}(\mathfrak{A})$. Furthermore, for $\beta' \in \mathfrak{P}^{-m}$,

$$\psi_\beta = \psi_{\beta'} \quad \text{if and only if} \quad \beta \equiv \beta' \mod \mathfrak{P}^{-r}.$$

Thus the map $\beta + \mathfrak{P}^{-r} \mapsto \psi_\beta$ is an isomorphism between the (additive) group $\mathfrak{P}^{-m}/\mathfrak{P}^{-r}$ and the (multiplicative) group of characters of $U^{r+1}(\mathfrak{A})/U^{m+1}(\mathfrak{A})$.

Definition 3.1.4. A *stratum* in A is a 4-tuple $[\mathfrak{A}, m, r, \beta]$ where $\mathfrak{A}$ is a hereditary $\mathfrak{o}_F$-order in A, m and r are integers with $r < m$, and $\beta \in \mathfrak{P}^{-m}$.

While we are not making all the restrictions here on m and r that we made above, we will see that in all applications of strata to the representation theory of G, these restrictions will play a part. But if we assume temporarily that we are given a stratum $[\mathfrak{A}, m, r, \beta]$ satisfying

$$0 \le \left\lfloor \frac{m}{2} \right\rfloor \le r < m \tag{3.1.1}$$

then this stratum specifies a character ψ_β of the group $U^{r+1}(\mathfrak{A})$ that is trivial on $U^{m+1}(\mathfrak{A})$. Furthermore, note that ψ_β is nontrivial on $U^m(\mathfrak{A})$ if and only if $\beta \notin \mathfrak{P}^{-m+1}$, i.e., if and only if

$$\nu_{\mathfrak{A}}(\beta) = -m. \tag{3.1.2}$$

Thus, under this additional assumption, the parameter m in the stratum specifies the *level* of the character. (In general, without assuming (3.1.2), m specifies a *bound* on the level.) Since different choices of β can give us the same character, it is natural to make the following definition.

Definition 3.1.5. Two strata $[\mathfrak{A}_1, m_1, r_1, \beta_1]$ and $[\mathfrak{A}_2, m_2, r_2, \beta_2]$ are *equivalent*, written

$$[\mathfrak{A}_1, m_1, r_1, \beta_1] \sim [\mathfrak{A}_2, m_2, r_2, \beta_2]$$

if

$$\beta_1 + \mathfrak{P}_1^{-r_1} = \beta_2 + \mathfrak{P}_2^{-r_2},$$

where $\mathfrak{P}_i = \mathrm{rad}(\mathfrak{A}_i)$ for each i.

It is clear that this is an equivalence relation on the set of all strata in A, and it is easy to show that if $[\mathfrak{A}_1, m_1, r_1, \beta_1] \sim [\mathfrak{A}_2, m_2, r_2, \beta_2]$, then $\mathfrak{A}_1 = \mathfrak{A}_2$ and $r_1 = r_2$. (If furthermore both strata satisfy (3.1.2), then clearly $m_1 = m_2$.) Thus we could rewrite this definition as

$$[\mathfrak{A}, m_1, r, \beta_1] \sim [\mathfrak{A}, m_2, r, \beta_2] \iff \beta_1 \equiv \beta_2 \mod \mathfrak{P}^{-r}.$$

If we assume in addition that both strata satisfy (3.1.1), then this condition is equivalent to $\psi_{\beta_1} = \psi_{\beta_2}$.

In summary, an equivalence class of strata satisfying (3.1.1) is nothing more nor less than a choice of a compact open subgroup $U^{r+1}(\mathfrak{A})$ of G and a character ψ_β of this subgroup. The four terms in the tuple $[\mathfrak{A}, m, r, \beta]$ can be thought of as follows:

1. $\mathfrak{A}$ determines a parahoric subgroup $U(\mathfrak{A})$ of G.

2. r determines a compact open subgroup $U^{r+1}(\mathfrak{A})$ of G, from the standard filtration of $U(\mathfrak{A})$.

3. β determines a character ψ_β of $U^{r+1}(\mathfrak{A})$.

4. m determines a bound on the level of the character ψ_β.

3.1.2 Pure and simple strata

Definition 3.1.6. A stratum $[\mathfrak{A}, m, r, \beta]$ is called *pure* if

1. $E = F[\beta]$ is a field (i.e., the minimal polynomial of β is irreducible),

2. $E^\times \subset \mathfrak{K}(\mathfrak{A})$ (i.e., E normalizes $\mathfrak{A}$), and

3. $m = -\nu_{\mathfrak{A}}(\beta)$.

Given a pure stratum $[\mathfrak{A}, m, r, \beta]$ and letting $E = F[\beta]$, we can regard V as an E-vector space, and define

$$B = B_\beta = \operatorname{End}_E(V) = \{x \in A \mid xa = ax \quad \forall a \in E\}.$$

(Note then that B is the centralizer of E in A, or equivalently the centralizer of β in A.) We then define

$$\mathfrak{B} = \mathfrak{B}_\beta = \mathfrak{A} \cap B = \text{the centralizer of } \beta \text{ in } \mathfrak{A},$$
$$\mathfrak{Q} = \mathfrak{Q}_\beta = \mathfrak{P} \cap B = \text{the centralizer of } \beta \text{ in } \mathfrak{P}.$$

Then $\mathfrak{B}$ is a hereditary $\mathfrak{o}_E$-order in B, and $\mathfrak{Q}$ is its Jacobson radical. For $k \in \mathbb{Z}$, we define

$$\mathfrak{N}_k(\beta, \mathfrak{A}) = \left\{x \in \mathfrak{A} \;\middle|\; \beta x - x\beta \in \mathfrak{P}^k\right\}.$$

For sufficiently small k (specifically, for all $k \leq \nu_{\mathfrak{A}}(\beta)$), $\mathfrak{N}_k(\beta, \mathfrak{A}) = \mathfrak{A}$, and for sufficiently large k, $\mathfrak{N}_k(\beta, \mathfrak{A}) \subset \mathfrak{B} + \mathfrak{P}$. (Note that if β is scalar, whence $E = F$, then $\mathfrak{A} = \mathfrak{N}_k(\beta, \mathfrak{A}) = \mathfrak{B} + \mathfrak{P}$ for all k.) Hence we define

$$k_0(\beta, \mathfrak{A}) = \begin{cases} \max\{k \in \mathbb{Z} \mid \mathfrak{N}_k(\beta, \mathfrak{A}) \not\subset \mathfrak{B} + \mathfrak{P}\} & \text{if } F[\beta] \neq F \\ -\infty & \text{if } F[\beta] = F. \end{cases}$$

Note that if $F[\beta] \neq F$, then $k_0(\beta, \mathfrak{A}) \geq \nu_{\mathfrak{A}}(\beta)$.

Definition 3.1.7. A stratum $[\mathfrak{A}, m, r, \beta]$ is called *simple* if it is pure and also satisfies

$$r < -k_0(\beta, \mathfrak{A}).$$

It can be proven (see [7, (2.1.4)]) that if $[\mathfrak{A}, m, r, \beta_1]$ and $[\mathfrak{A}, m, r, \beta_2]$ are simple strata that are equivalent, then

$$\begin{aligned} k_0(\beta_1, \mathfrak{A}) &= k_0(\beta_2, \mathfrak{A}), \\ e(F[\beta_1]/F) &= e(F[\beta_2]/F), \\ f(F[\beta_1]/F) &= f(F[\beta_2]/F). \end{aligned} \tag{3.1.3}$$

This fact will be useful below.

The first examples of simple strata, which will turn out to be the foundation of the whole theory, are given by *minimal elements*:

Definition 3.1.8. Let $E = F[\beta]$ be a field, let ν_E be the normalized valuation on E, and let ϖ_F be a prime element of F. We say that β is *minimal* over F if

1. $\gcd(\nu_E(\beta), e(E/F)) = 1$, and

2. $\varpi_F^{-\nu_E(\beta)} \beta^{e(E/F)} + \mathfrak{p}_E \in \boldsymbol{k}_E$ generates the field extension $\boldsymbol{k}_E/\boldsymbol{k}_F$.

(Note that this is independent of the choice of prime element ϖ_F.)

Equivalently, if $E = F$ (i.e., $\beta \in F$), then β is always minimal over F, and if $E \neq F$, then β is minimal over F if and only if $\nu_{\mathfrak{A}}(\beta) = k_0(\beta, \mathfrak{A})$ (for any $\mathfrak{A}$ satisfying $E^\times \subset \mathfrak{K}(\mathfrak{A})$). Thus, if $E = F[\beta]$ is a field with β minimal over F, then we can choose a hereditary order $\mathfrak{A}$ in A with $E^\times \subset \mathfrak{K}(\mathfrak{A})$ (such an order will always exist in this situation) and let $m = -\nu_{\mathfrak{A}}(\beta)$. Then $[\mathfrak{A}, m, r, \beta]$ will be a simple stratum for any $r < m$. Strata of this form are referred to in [22] as *alfalfa strata*. Note that by (3.1.3), any simple stratum equivalent to an alfalfa stratum is also alfalfa.

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3.1.3 Defining sequences for simple strata

Definition 3.1.9. Let $\beta \in A$ such that $E = F[\beta]$ is a field, and let $B = B_\beta$. A *tame corestriction on A relative to E/F* is a linear map $s : A \to B$ satisfying

1. $s(b_1 a b_2) = b_1 s(a) b_2$ for all $a \in A$ and $b_1, b_2 \in B$ (i.e., s is a (B, B)-bimodule homomorphism), and

2. $s(\mathfrak{A}) = \mathfrak{A} \cap B$ for any hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ with $E^\times \subset \mathfrak{K}(\mathfrak{A})$.

For any field extension E of F contained in A, a tame corestriction s exists, and such a map is clearly unique up to multiplication by an element of $\mathfrak{o}_E^\times$. Furthermore, if $\mathfrak{A}$ is any hereditary $\mathfrak{o}_F$-order in A with $E^\times \subset \mathfrak{K}(\mathfrak{A})$, and we let $\mathfrak{P} = \mathrm{rad}(\mathfrak{A})$ as usual, then $s(\mathfrak{P}^k) = \mathfrak{P}^k \cap B$ for all $k \in \mathbb{Z}$. Thus in particular if $[\mathfrak{A}, m, r, \beta]$ is a pure stratum and $b \in \mathfrak{P}^{-r}$, then $[\mathfrak{B}_\beta, r, r-1, s(b)]$ is a stratum in B. Such a stratum is called a *derived stratum*, and it is clear by the preceding remarks that the equivalence class of this derived stratum is independent of the choice of s.

The following proposition is the content of [7, (2.4.2)], although its proof and all of the related material consumes much of Chapter 2 of [7].

Proposition 3.1.10. *Let $[\mathfrak{A}, m, r, \beta]$ be a simple stratum. Then there exists a finite sequence of strata $[\mathfrak{A}, m, r_i, \beta_i]$, $0 \leq i \leq s$, that satisfies the following properties:*

1. *$\beta = \beta_0$ and $r = r_0 < r_1 < \cdots < r_s < m$,*

2. *For each i, $F[\beta_i]$ is a field, $F[\beta_i]^\times \subset \mathfrak{K}(\mathfrak{A})$, and $\nu_\mathfrak{A}(\beta_i) = -m$,*

3. *$[\mathfrak{A}, m, r_i, \beta_{i-1}] \sim [\mathfrak{A}, m, r_i, \beta_i]$ for $1 \leq i \leq s$,*

4. *$r_i = -k_0(\beta_{i-1}, \mathfrak{A})$ for $1 \leq i \leq s$,*

5. *$k_0(\beta_s, \mathfrak{A}) = -m$ or $-\infty$ (i.e., β_s is minimal over F),*

6. *Let s_i be a tame corestriction on A relative to $F[\beta_i]/F$. The stratum $[\mathfrak{B}_{\beta_i}, r_i, r_i - 1, s_i(\beta_{i-1} - \beta_i)]$ is equivalent to a simple stratum in B_{β_i} for $1 \leq i \leq s$.*

Note that 2 is equivalent to saying that $[\mathfrak{A}, m, r_i, \beta_i]$ is a pure stratum for each i, and that 1 and 2 combined imply that $[\mathfrak{A}, m, r_i, \beta_j]$ is a pure stratum for any i, j. Also, by 3, we have $\beta_{i-1} - \beta_i \in \mathfrak{P}^{-r_i}$, and thus the tuple given in 6 is a derived stratum in B_{β_i}.

Definition 3.1.11. A sequence of strata $[\mathfrak{A}, m, r_i, \beta_i]$, $0 \leq i \leq s$, satisfying the requirements of Proposition 3.1.10 will be called a *defining sequence* for the simple stratum $[\mathfrak{A}, m, r, \beta]$.

To interpret Proposition 3.1.10 more clearly, note that a defining sequence for $[\mathfrak{A}, m, r, \beta]$ gives us the following strata, all of which are pure:

$$[\mathfrak{A}, m, r, \beta] = [\mathfrak{A}, m, r_0, \beta_0]$$

$$[\mathfrak{A}, m, r_1, \beta_0] \sim [\mathfrak{A}, m, r_1, \beta_1]$$

$$[\mathfrak{A}, m, r_2, \beta_1] \sim [\mathfrak{A}, m, r_2, \beta_2]$$

$$\vdots$$

$$[\mathfrak{A}, m, r_s, \beta_{s-1}] \sim [\mathfrak{A}, m, r_s, \beta_s]$$

(The arrows here do not represent maps, but rather are there to indicate the intended "flow" of the sequence.) In this diagram, the first term on the left is our original simple stratum, and (by parts 1 and 5 of Proposition 3.1.10) the final term on the right is an alfalfa stratum (i.e., a simple stratum given by an element that is minimal over F). In between these first and last terms, all of the terms on the right are simple strata (by parts 1 and 4), and the terms on the left are pure strata that are *not* simple, but just barely so (by part 4).

Naturally, a defining sequence for a simple stratum need not be unique. But by repeated application of (3.1.3), it is easy to see that the sequence of integers r_i (and the length s) of any defining sequence for $[\mathfrak{A}, m, r, \beta]$ is uniquely determined (in fact by the equivalence class of $[\mathfrak{A}, m, r, \beta]$), and furthermore that the equivalence classes of all the simple strata $[\mathfrak{A}, m, r_i, \beta_i]$ in the defining sequence are uniquely determined.

3.1.4 The groups $H^k(\beta, \mathfrak{A})$ and $J^k(\beta, \mathfrak{A})$

The next major step toward the definition of simple types is to define *simple characters*, which are abelian characters of a very specific kind defined on certain compact open subgroups of G. We must first define these subgroups, which occur within natural decreasing filtrations of subgroups of G, denoted $H^k(\beta, \mathfrak{A})$, $k \geq 0$. Although we will not need them until later, we will also define another closely related family of decreasing filtrations of subgroups, denoted $J^k(\beta, \mathfrak{A})$, $k \geq 0$. As the notation suggests, the requirements for β

and $\mathfrak{A}$ are familiar ones: $\beta \in A$, $\mathfrak{A}$ a hereditary $\mathfrak{o}_F$-order in A, $E = F[\beta]$ a field with $E^\times \subset \mathfrak{K}(\mathfrak{A})$, and $\nu_\mathfrak{A}(\beta)$ and $k_0(\beta, \mathfrak{A})$ both negative. In other words, we require that $[\mathfrak{A}, m, 0, \beta]$ be a simple stratum (where $m = -\nu_\mathfrak{A}(\beta)$ of course). Unfortunately, the definitions of the desired groups can't be given uniformly in terms of just β and $\mathfrak{A}$, except in the case where β is minimal over F. In the general case, we must specify these groups in terms of a defining sequence for this simple stratum.

It may be useful to keep in mind throughout this section the following motivation: simple characters are based on the character ψ_β of $U^{\lfloor \frac{m}{2} \rfloor + 1}(\mathfrak{A})$ and certain properties of it. Note that this is the character naturally associated to the (pure but not necessarily simple) stratum $[\mathfrak{A}, m, \lfloor \frac{m}{2} \rfloor, \beta]$. It will turn out that for all $k \geq 0$,

$$U^k(\mathfrak{B}_\beta) \subseteq H^k(\beta, \mathfrak{A}) \subseteq J^k(\beta, \mathfrak{A}) \subseteq U^k(\mathfrak{A}), \qquad (3.1.4)$$

and that for all $k \geq \lfloor \frac{m}{2} \rfloor + 1$, the last two of these containments are equalities. Naturally then, for such values of k, the only simple character of $H^k(\beta, \mathfrak{A})$ will be ψ_β. For smaller values of k, we will obtain simple characters by extending ψ_β from $H^{\lfloor \frac{m}{2} \rfloor + 1}(\beta, \mathfrak{A})$ to the larger group $H^k(\beta, \mathfrak{A})$. Note that for any integer m, $\lfloor \frac{m}{2} \rfloor + 1$ is the same as $\lceil \frac{m+1}{2} \rceil$. From this point forward, we will tend to favor the latter notation.

At this point, our exposition deviates slightly from that of [7]. There, the groups $H^k(\beta, \mathfrak{A})$ and $J^k(\beta, \mathfrak{A})$ are defined in terms of a pair of rings $\mathfrak{H}(\beta, \mathfrak{A})$ and $\mathfrak{J}(\beta, \mathfrak{A})$ and a filtration of ideals of each. For the sake of brevity, we have chosen here to define these groups directly, as we will have no need for the aforementioned rings and ideals. What follows is adapted from [7, (3.1.7) - (3.1.15)].

Definition 3.1.12. Let β and $\mathfrak{A}$ be such that $[\mathfrak{A}, m, 0, \beta]$ is a simple stratum (with $m = -\nu_\mathfrak{A}(\beta)$). If β is minimal over F, define

$$H^k(\beta, \mathfrak{A}) = \begin{cases} U^k(\mathfrak{B}_\beta) U^{\lceil \frac{m+1}{2} \rceil}(\mathfrak{A}) & \text{if } 0 \leq k < \lceil \frac{m+1}{2} \rceil, \\ U^k(\mathfrak{A}) & \text{if } \lceil \frac{m+1}{2} \rceil \leq k; \end{cases}$$

$$J^k(\beta, \mathfrak{A}) = \begin{cases} U^k(\mathfrak{B}_\beta) U^{\lfloor \frac{m+1}{2} \rfloor}(\mathfrak{A}) & \text{if } 0 \leq k < \lfloor \frac{m+1}{2} \rfloor, \\ U^k(\mathfrak{A}) & \text{if } \lfloor \frac{m+1}{2} \rfloor \leq k. \end{cases}$$

If β is not minimal over F, let $r = -k_0(\beta, \mathfrak{A})$ and choose a simple stratum $[\mathfrak{A}, m, r, \beta']$ that is equivalent to the (pure but not simple) stratum $[\mathfrak{A}, m, r, \beta]$, as in the construction of a defining sequence. Assuming that

$H^k(\beta', \mathfrak{A})$ and $J^k(\beta', \mathfrak{A})$ have already been defined for all $k \geq 0$, we can define

$$H^k(\beta, \mathfrak{A}) = \begin{cases} U^k(\mathfrak{B}_\beta) H^{\lceil \frac{r+1}{2} \rceil}(\beta', \mathfrak{A}) & \text{if } 0 \leq k < \lceil \frac{r+1}{2} \rceil, \\ H^k(\beta', \mathfrak{A}) & \text{if } \lceil \frac{r+1}{2} \rceil \leq k; \end{cases}$$

$$J^k(\beta, \mathfrak{A}) = \begin{cases} U^k(\mathfrak{B}_\beta) J^{\lfloor \frac{r+1}{2} \rfloor}(\beta', \mathfrak{A}) & \text{if } 0 \leq k < \lfloor \frac{r+1}{2} \rfloor, \\ J^k(\beta', \mathfrak{A}) & \text{if } \lfloor \frac{r+1}{2} \rfloor \leq k. \end{cases}$$

The existence of a defining sequence for a given simple stratum (Proposition 3.1.10) guarantees that the second part of this definition can be iterated, thereby defining the sets $H^k(\beta, \mathfrak{A})$ and $J^k(\beta, \mathfrak{A})$ for all β and $\mathfrak{A}$ as above. That these sets are actually groups follows from the fact that $U^k(\mathfrak{A})$ is normalized by $U^0(\mathfrak{B}_\beta)$, and proceeding by induction. Similarly, (3.1.4) is obvious in the first case, and follows easily by induction in the second case. However, it is certainly not clear that these groups are well-defined, due to the choice made in the second part of this definition (i.e., the fact that a defining sequence for a simple stratum is not unique). But postponing that matter for a moment, we note the following, which for small values of k (and always at least for $k = 0$ and $k = 1$) may be taken as an alternative to the above definition.

Corollary 3.1.13. *Let* $[\mathfrak{A}, m, r_i, \beta_i]$, $0 \leq i \leq s$, *be a defining sequence for the simple stratum* $[\mathfrak{A}, m, 0, \beta]$, *and let* $r = -k_0(\beta, \mathfrak{A})$. *Then for* $0 \leq k \leq \lceil \frac{r+1}{2} \rceil$,

$$H^k(\beta, \mathfrak{A}) = U^k(\mathfrak{B}_{\beta_0}) U^{\lceil \frac{r_1+1}{2} \rceil}(\mathfrak{B}_{\beta_1}) \cdots U^{\lceil \frac{r_s+1}{2} \rceil}(\mathfrak{B}_{\beta_s}) U^{\lceil \frac{m+1}{2} \rceil}(\mathfrak{A}),$$

and for $0 \leq k \leq \lfloor \frac{r+1}{2} \rfloor$,

$$J^k(\beta, \mathfrak{A}) = U^k(\mathfrak{B}_{\beta_0}) U^{\lfloor \frac{r_1+1}{2} \rfloor}(\mathfrak{B}_{\beta_1}) \cdots U^{\lfloor \frac{r_s+1}{2} \rfloor}(\mathfrak{B}_{\beta_s}) U^{\lfloor \frac{m+1}{2} \rfloor}(\mathfrak{A}).$$

We now record some of the properties of these groups that will be most important to us in what follows. The following proposition is adapted from [7, (3.1.15)].

Proposition 3.1.14. *Let* β *and* $\mathfrak{A}$ *be such that* $[\mathfrak{A}, m, 0, \beta]$ *is a simple stratum (with* $m = -\nu_\mathfrak{A}(\beta)$*).*

1. *For all* $k \geq 0$, $H^k(\beta, \mathfrak{A})$ *and* $J^k(\beta, \mathfrak{A})$ *are well-defined independently of the choices made in the definition above. Furthermore, they depend only on the equivalence class of the simple stratum* $[\mathfrak{A}, m, 0, \beta]$.

2. $H^0(\beta, \mathfrak{A})$ and $J^0(\beta, \mathfrak{A})$ are compact open subgroups of G, and the $H^k(\beta, \mathfrak{A})$ for $k > 0$ form a decreasing filtration of $H^0(\beta, \mathfrak{A})$ by open normal subgroups (and likewise for $J^k(\beta, \mathfrak{A})$).

3. For all $k \geq 0$, $H^k(\beta, \mathfrak{A})$ and $J^k(\beta, \mathfrak{A})$ are normalized by $\mathfrak{K}(\mathfrak{B}_\beta)$.

4. For all $k > 0$, $H^k(\beta, \mathfrak{A})$ is a normal subgroup of $J^0(\beta, \mathfrak{A})$, and $J^k(\beta, \mathfrak{A})\big/H^k(\beta, \mathfrak{A})$ is a finite elementary abelian p-group (where p is the characteristic of $\boldsymbol{k}_F$).

5. For all $l, k > 0$, the commutator group $[J^l(\beta, \mathfrak{A}), J^k(\beta, \mathfrak{A})]$ is contained in $H^{l+k}(\beta, \mathfrak{A})$.

6. Let $r = -k_0(\beta, \mathfrak{A})$. Then for $0 \leq l < k \leq \left\lceil \frac{r+1}{2} \right\rceil$, we have a natural isomorphism

$$H^l(\beta, \mathfrak{A})\big/H^k(\beta, \mathfrak{A}) \cong U^l(\mathfrak{B}_\beta)\big/U^k(\mathfrak{B}_\beta) .$$

Similarly for $0 \leq l < k \leq \left\lfloor \frac{r+1}{2} \right\rfloor$, we have a natural isomorphism

$$J^l(\beta, \mathfrak{A})\big/J^k(\beta, \mathfrak{A}) \cong U^l(\mathfrak{B}_\beta)\big/U^k(\mathfrak{B}_\beta) .$$

It will turn out that the three groups

$$H^1(\beta, \mathfrak{A}) \lhd J^1(\beta, \mathfrak{A}) \lhd J^0(\beta, \mathfrak{A})$$

will be the ones needed in the definition of simple types. In this context, we note the natural isomorphism

$$J^0(\beta, \mathfrak{A})/J^1(\beta, \mathfrak{A}) \cong U(\mathfrak{B}_\beta)/U^1(\mathfrak{B}_\beta)$$

given above, which will play a significant role later.

3.1.5 Simple characters

We are now ready to define simple characters. Once again, the definition of, as well as the computation of (and many of the properties of), these characters, are relatively straightforward in the case where β is minimal over F, but in the general case must be done in terms of a defining sequence for the associated simple stratum. It is well worth pointing out that the notation we have chosen for sets of simple characters differs slightly from that of [7]. There, the set of simple characters of the group $H^k(\beta, \mathfrak{A})$ for

$k > 0$ is denoted $\mathcal{C}(\mathfrak{A}, k-1, \beta)$ (note the difference in the index k). We have chosen to refer to the same set as $\mathcal{C}(\mathfrak{A}, k, \beta)$, as this seems to lead to somewhat prettier notation. We hope that this will not cause the reader any confusion.

To begin with, let $[\mathfrak{A}, m, 0, \beta]$ be a simple stratum, and let $\det_{B_\beta}$ denote the determinant map from B_β to $F[\beta]$. The starting point of the definition of simple characters is the observation of the following two facts about ψ_β:

1. the restriction of ψ_β to $U^{\left\lceil \frac{m+1}{2} \right\rceil}(\mathfrak{A}) \cap B_\beta^\times$ factors through $\det_{B_\beta}$;

2. $\psi_\beta(x^{-1}ax) = \psi_\beta(a)$ for all $x \in \mathfrak{K}(\mathfrak{B}_\beta)$ and all $a \in U^{\left\lceil \frac{m+1}{2} \right\rceil}(\mathfrak{A})$ (i.e., $\mathfrak{K}(\mathfrak{B}_\beta)$ normalizes ψ_β).

The first of these facts requires some effort to prove, but does not merit stating as a proposition. The second is clear.

Definition 3.1.15. Let β and $\mathfrak{A}$ be such that $[\mathfrak{A}, m, 0, \beta]$ is a simple stratum (with $m = -\nu_{\mathfrak{A}}(\beta)$), and let $1 \leq k \leq m$. *Assume that β is minimal over F.* If $\left\lceil \frac{m+1}{2} \right\rceil \leq k \leq m$, define $\mathcal{C}(\mathfrak{A}, k, \beta) = \{\psi_\beta\}$. If $1 \leq k < \left\lceil \frac{m+1}{2} \right\rceil$, then define $\mathcal{C}(\mathfrak{A}, k, \beta)$ to be the set of all characters θ of $H^k(\beta, \mathfrak{A})$ which satisfy the following:

1. The restriction of θ to $H^{\left\lceil \frac{m+1}{2} \right\rceil}(\beta, \mathfrak{A}) = U^{\left\lceil \frac{m+1}{2} \right\rceil}(\mathfrak{A})$ is equal to ψ_β.

2. The restriction of θ to $H^k(\beta, \mathfrak{A}) \cap B_\beta^\times$ factors through $\det_{B_\beta}$.

It follows immediately from this definition and the second observation above that every $\theta \in \mathcal{C}(\mathfrak{A}, k, \beta)$ is normalized by $\mathfrak{K}(\mathfrak{B}_\beta)$. We now proceed with the general case.

Definition 3.1.16. Let β and $\mathfrak{A}$ be such that $[\mathfrak{A}, m, 0, \beta]$ is a simple stratum (with $m = -\nu_{\mathfrak{A}}(\beta)$), and let $1 \leq k \leq m$. *Assume now that β is not minimal over F.* Let $r = -k_0(\beta, \mathfrak{A})$, and choose a simple stratum $[\mathfrak{A}, m, r, \beta']$ that is equivalent to the (pure but not simple) stratum $[\mathfrak{A}, m, r, \beta]$, as in the construction of a defining sequence. We will assume that $\mathcal{C}(\mathfrak{A}, k', \beta')$ has already been defined for all k'. If $r < k \leq m$, define $\mathcal{C}(\mathfrak{A}, k, \beta) = \mathcal{C}(\mathfrak{A}, k, \beta')$. Otherwise, let $\mathcal{C}(\mathfrak{A}, k, \beta)$ be the set of all characters θ of $H^k(\beta, \mathfrak{A})$ which satisfy the following:

1. (a) If $\left\lceil \frac{r+1}{2} \right\rceil \leq k \leq r$, then $\theta = \theta_0 \cdot \psi_{\beta-\beta'}$ for some $\theta_0 \in \mathcal{C}(\mathfrak{A}, k, \beta')$.

   (b) If $1 \leq k < \left\lceil \frac{r+1}{2} \right\rceil$, then the restriction of θ to $H^{\left\lceil \frac{r+1}{2} \right\rceil}(\beta, \mathfrak{A})$ is equal to $\theta_0 \cdot \psi_{\beta-\beta'}$ for some $\theta_0 \in \mathcal{C}(\mathfrak{A}, \left\lceil \frac{r+1}{2} \right\rceil, \beta')$.

2. The restriction of θ to $H^k(\beta, \mathfrak{A}) \cap B_\beta^\times$ factors through $\det_{B_\beta}$.

3. $\mathfrak{K}(\mathfrak{B}_\beta)$ normalizes θ.

A few remarks are in order. First, recall that for $k' \geq \lceil \frac{r+1}{2} \rceil$, $H^{k'}(\beta, \mathfrak{A}) = H^{k'}(\beta', \mathfrak{A})$. Thus defining the elements of $\mathcal{C}(\mathfrak{A}, k, \beta)$ in terms of those of $\mathcal{C}(\mathfrak{A}, k', \beta')$ makes sense as long as $k' \geq \lceil \frac{r+1}{2} \rceil$, which is true in all cases of the definition above. Second, note that $\beta - \beta' \in \mathfrak{P}^{-r}$, so in part 1, $\psi_{\beta - \beta'}$ is a character of $U^{\lceil \frac{r+1}{2} \rceil}(\mathfrak{A})$ that is trivial on $U^{r+1}(\mathfrak{A})$. To show that these two definitions are actually much more uniform than they may at first appear, we note the following, which may be taken as an alternative to the pair of definitions above.

Corollary 3.1.17. *Let $[\mathfrak{A}, m, r_i, \beta_i]$, $0 \leq i \leq s$, be a defining sequence for the simple stratum $[\mathfrak{A}, m, 0, \beta]$. Then for $1 \leq k \leq m$ and $0 \leq i \leq s$, $\mathcal{C}(\mathfrak{A}, k, \beta_i)$ is the set of all characters θ of $H^k(\beta_i, \mathfrak{A})$ satisfying the following criteria:*

1. *The restriction of θ to $H^k(\beta, \mathfrak{A}) \cap B_\beta^\times$ factors through $\det_{B_\beta}$.*

2. *$\mathfrak{K}(\mathfrak{B}_\beta)$ normalizes θ.*

3. *(a) If $i = s$, let $k' = \max\{k, \lceil \frac{m+1}{2} \rceil\}$. Then the restriction of θ to $H^{k'}(\beta_s, \mathfrak{A})$ is equal to ψ_{β_s}.*

   (b) If $i < s$, let $k' = \max\{k, \lceil \frac{r_{i+1}+1}{2} \rceil\}$. Then the restriction of θ to $H^{k'}(\beta_i, \mathfrak{A})$ is equal to $\theta_0 \cdot \psi_{\beta_i - \beta_{i+1}}$ for some $\theta_0 \in \mathcal{C}(\mathfrak{A}, k', \beta_{i+1})$.

Once again, it is not clear that the sets $\mathcal{C}(\mathfrak{A}, k, \beta)$ are well-defined in general, since defining sequences for simple strata are not unique. In this case, establishing this takes a significant effort. Furthermore, it is not obvious except in certain cases that these sets are nonempty. However, as we hinted at above, the basic idea in the definition of simple characters is to start with ψ_β on $H^{\lceil \frac{m+1}{2} \rceil}(\beta, \mathfrak{A})$, and extend to larger subgroups. Thus we may hope that, for $k' > k$, restriction of characters from $H^k(\beta, \mathfrak{A})$ to $H^{k'}(\beta, \mathfrak{A})$ will map elements of $\mathcal{C}(\mathfrak{A}, k, \beta)$ to $\mathcal{C}(\mathfrak{A}, k', \beta)$. This is indeed the case, and in fact these maps are always surjective and their fibers can be described explicitly. The following proposition is a summary of the culminating results of [7, (3.2)–(3.3)].

Proposition 3.1.18. *Let $[\mathfrak{A}, m, 0, \beta]$ be a simple stratum, and let $1 \leq k \leq m$. The set $\mathcal{C}(\mathfrak{A}, k, \beta)$ is well-defined and nonempty, and depends only on the*

equivalence class of the (pure but not necessarily simple) stratum $[\mathfrak{A}, m, k - 1, \beta]$. More specifically, let $[\mathfrak{A}, m, r_i, \beta_i]$, $0 \leq i \leq s$ be a defining sequence for $[\mathfrak{A}, m, 0, \beta]$. We can compute all of the sets $\mathcal{C}(\mathfrak{A}, k, \beta)$ inductively as follows.

1. (a) *If $\left\lceil \frac{m+1}{2} \right\rceil \leq k \leq m$, then*

$$\mathcal{C}(\mathfrak{A}, k, \beta_s) = \{\psi_{\beta_s}\}.$$

   (b) *If $1 \leq k < \left\lceil \frac{m+1}{2} \right\rceil$, then all characters in $\mathcal{C}(\mathfrak{A}, k, \beta_s)$ are extensions of ψ_{β_s}. Furthermore, if $\theta \in \mathcal{C}(\mathfrak{A}, k, \beta_s)$, then*

$$\mathcal{C}(\mathfrak{A}, k, \beta_s) = \{\theta \cdot \chi \mid \chi \in X\},$$

   where X is the group of characters of $U^k(\mathfrak{B}_{\beta_s})$ that are trivial on $U^{\left\lceil \frac{m+1}{2} \right\rceil}(\mathfrak{B}_{\beta_s})$ and factor through $\det_{B_{\beta_s}}$.

2. *Let $0 \leq i < s$ and abbreviate $r = r_{i+1}$.*

   (a) *If $r < k \leq m$, then*

$$\mathcal{C}(\mathfrak{A}, k, \beta_i) = \mathcal{C}(\mathfrak{A}, k, \beta_{i+1}).$$

   (b) *If $\left\lceil \frac{r+1}{2} \right\rceil \leq k \leq r$, then*

$$\mathcal{C}(\mathfrak{A}, k, \beta_i) = \left\{ \theta \cdot \psi_{\beta_i - \beta_{i+1}} \mid \theta \in \mathcal{C}(\mathfrak{A}, k, \beta_{i+1}) \right\}.$$

   (c) *If $1 \leq k < \left\lceil \frac{r+1}{2} \right\rceil$, then all characters in $\mathcal{C}(\mathfrak{A}, k, \beta_i)$ are extensions of characters in $\mathcal{C}(\mathfrak{A}, \left\lceil \frac{r+1}{2} \right\rceil, \beta_i)$, i.e., restriction of characters defines a surjective map from $\mathcal{C}(\mathfrak{A}, k, \beta_i)$ to $\mathcal{C}(\mathfrak{A}, \left\lceil \frac{r+1}{2} \right\rceil, \beta_i)$. Furthermore, if $\theta \in \mathcal{C}(\mathfrak{A}, k, \beta_i)$ restricts to $\overline{\theta} \in \mathcal{C}(\mathfrak{A}, \left\lceil \frac{r+1}{2} \right\rceil, \beta_i)$, then the fiber of $\overline{\theta}$ under this restriction map is*

$$\{\theta \cdot \chi \mid \chi \in X\},$$

   where X is the group of characters of $U^k(\mathfrak{B}_{\beta_i})$ that are trivial on $U^{\left\lceil \frac{r+1}{2} \right\rceil}(\mathfrak{B}_{\beta_i})$ and factor through $\det_{B_{\beta_i}}$.

3.1.6 Extending simple characters

As mentioned above, the groups that will be used to define simple types are precisely the groups $H^1(\beta, \mathfrak{A}) \subseteq J^1(\beta, \mathfrak{A}) \subseteq J^0(\beta, \mathfrak{A})$. Note that $\mathcal{C}(\mathfrak{A}, 1, \beta)$ is a set of characters defined on the smallest of these three groups. Thus the last remaining task before we can define simple types will be to further

extend these characters to the groups $J^1(\beta, \mathfrak{A})$ and $J^0(\beta, \mathfrak{A})$. The first of these steps is relatively straightforward, but to explain it requires a bit of new notation and a small proposition.

Let $[\mathfrak{A}, m, 0, \beta]$ be a simple stratum in A, let $r = -k_0(\beta, \mathfrak{A})$, and suppose $1 \leq k \leq r$. Let $\theta \in \mathcal{C}(\mathfrak{A}, k, \beta)$, and let $W = J^k(\beta, \mathfrak{A})/H^k(\beta, \mathfrak{A})$. By Proposition 3.1.14, we may view W as an $\mathbb{F}_p$-vector space, where p is the characteristic of $\mathbf{k}_F$. (In fact, it turns out that W is always a $\mathbf{k}_F$-vector space of even dimension, but this need not concern us at the moment.) For $x, y \in G$, we will write $[x, y]$ to denote the commutator $xyx^{-1}y^{-1}$. By the same proposition just mentioned,

$$[J^k(\beta, \mathfrak{A}), J^k(\beta, \mathfrak{A})] \subset H^k(\beta, \mathfrak{A}).$$

Also, it follows from the definitions of simple characters and of the groups $J^*(\beta, \mathfrak{A})$ that θ is normalized by $J^0(\beta, \mathfrak{A})$ (cf. [7, (3.3.1)]). From all this, it is straightforward to show that the map $(x, y) \mapsto \theta[x, y]$ induces a well-defined alternating bilinear form on W, which we will denote h_θ. The following proposition is [7, (3.4.1)].

Proposition 3.1.19. *Assume the notation above. The bilinear form h_θ : $W \times W \to \mathbb{C}^\times$ defined above is nondegenerate.*

In this situation, it is more or less well known that there exists a unique irreducible representation η of $J^k(\beta, \mathfrak{A})$ whose restriction to $H^k(\beta, \mathfrak{A})$ contains θ, that
$$(\dim \eta)^2 = \#W = [J^k(\beta, \mathfrak{A}) : H^k(\beta, \mathfrak{A})],$$
and that the restriction of η to $H^k(\beta, \mathfrak{A})$ is in fact a direct sum of $\dim \eta$ copies of θ. We will sketch the construction of η here; see (for example) [2, (8.3)] for details. Let W' be a maximal totally isotropic subspace of W under h_θ. Since h_θ is alternating and nondegenerate, the dimension of W' is half that of W. Let J' be the inverse image of W' in $J^k(\beta, \mathfrak{A})$. Then $J'/\text{Ker}\,\theta$ is a maximal abelian subgroup of $J^k(\beta, \mathfrak{A})/\text{Ker}\,\theta$, so we may extend θ to a character θ' of J'. Define

$$\eta = \text{Ind}_{J'}^{J^k(\beta, \mathfrak{A})}(\theta').$$

The claims about η made above (as well its independence of the choices of W' and θ') now follow with relative ease by applying the Frobenius formula for the character of an induced representation.

Now let $k = 1$, and consider a simple character θ of $H^1(\beta, \mathfrak{A})$. Let η be the unique irreducible representation of $J^1(\beta, \mathfrak{A})$ whose restriction to

$H^1(\beta, \mathfrak{A})$ contains θ, as above. Unfortunately, the issue of extending η to $J^0(\beta, \mathfrak{A})$ is much more subtle. Many extensions may exist that are not suitable for defining simple types, so we must choose only certain extensions of a specific kind. Furthermore, we would still like to be able to say something about how many such extensions exist. We begin by recalling a standard set of definitions.

Definition 3.1.20. Temporarily let G be any group, let H_1 and H_2 be subgroups of G, and let ρ_1 and ρ_2 be irreducible representations of H_1 and H_2, respectively. Let $g \in G$. Let $H_1^g = g^{-1}H_1 g$ and denote by ρ_1^g the irreducible representation of H_1^g given by $h \mapsto \rho_1(ghg^{-1})$. We say that g intertwines ρ_1 with ρ_2 if

$$\mathrm{Hom}_{H_1^g \cap H_2}(\rho_1^g, \rho_2) \neq 0.$$

We define the *G-intertwining of ρ_1 with ρ_2* by

$$I_G(\rho_1|H_1, \rho_2|H_2) = \{g \in G \mid g \text{ intertwines } \rho_1 \text{ with } \rho_2\}.$$

(When there is no ambiguity, we will often drop the subgroups from this notation and just write $I_G(\rho_1, \rho_2)$.) Finally, we say that *ρ_1 and ρ_2 intertwine in G* if $I_G(\rho_1, \rho_2) \neq \emptyset$.

We note the following basic facts:

1. $I_G(\rho_1|H_1, \rho_2|H_2)$ is always a union of double-cosets $H_1 g H_2$.

2. The identity element always intertwines ρ with itself.

3. g intertwines ρ_1 with ρ_2 if and only if g^{-1} intertwines ρ_2 with ρ_1.

4. By the previous two points, the relation defined by intertwining in G is reflexive and symmetric; in general it is not transitive.

If we have a single subgroup H of G and an irreducible representation ρ of H, we simply say *g intertwines ρ* if g intertwines ρ with itself. Similarly, in place of $I_G(\rho|H, \rho|H)$, we simply write $I_G(\rho|H)$ (or just $I_G(\rho)$), and we refer to this set as the *G-intertwining of ρ*.

There are many ways to motivate the concept of intertwining (as it has many useful applications), but perhaps one of the simplest is this: in the case that G is a compact group and H_1 and H_2 are closed subgroups, $\langle \mathrm{Ind}_{H_1}^G(\rho_1), \mathrm{Ind}_{H_2}^G(\rho_2) \rangle_G$ is equal to the number of distinct double-cosets

H_1gH_2 in $I_G(\rho_1, \rho_2)$. This follows from an easy application of Mackey theory, and hence this statement has generalizations to many other settings in which Mackey theory is valid.

We return to the problem at hand. In particular, we resume using our previous notation, in which $G = A^\times \cong \mathrm{GL}_n(F)$.

Definition 3.1.21. Let $[\mathfrak{A}, m, 0, \beta]$ be a simple stratum in A, let $\theta \in \mathcal{C}(\mathfrak{A}, 1, \beta)$, and let η be the unique irreducible representation of $J^1(\beta, \mathfrak{A})$ whose restriction to $H^1(\beta, \mathfrak{A})$ contains θ. A β-*extension* of η is an irreducible representation κ of $J^0(\beta, \mathfrak{A})$ such that

1. The restriction of κ to $J^1(\beta, \mathfrak{A})$ is equal to η, and

2. $B_\beta^\times \subseteq I_G(\kappa)$, i.e., κ is intertwined by every element of $B_\beta^\times$.

Let $E = F[\beta]$, $B = B_\beta$, and $\mathfrak{B} = \mathfrak{B}_\beta$. For any character χ of $\boldsymbol{k}_E^\times$, we can view χ as a character of $\mathfrak{o}_E^\times$ that is trivial on $1 + \mathfrak{p}_E$, and then $\chi \circ \det_B$ defines a character of $U(\mathfrak{B})$ that is trivial on $U^1(\mathfrak{B})$. Recalling the canonical isomorphism $U(\mathfrak{B})/U^1(\mathfrak{B}) \cong J^0(\beta, \mathfrak{A})/J^1(\beta, \mathfrak{A})$, we can thus view $\chi \circ \det_B$ as a character of $J^0(\beta, \mathfrak{A})$ that is trivial on $J^1(\beta, \mathfrak{A})$. The following proposition is [7, (5.2.2)].

Proposition 3.1.22. *Let all notation be as above.*

1. *A β-extension of η exists.*

2. *If κ is a β-extension of η, then all other β-extensions of η are of the form $\kappa \otimes (\chi \circ \det_B)$ for some character χ of $\boldsymbol{k}_E^\times$ as above.*

3. *Distinct characters χ of $\boldsymbol{k}_E^\times$ yield distinct (nonisomorphic) representations $\kappa \otimes (\chi \circ \det_B)$. Thus the number of distinct β-extensions of η is equal to $q_E - 1$.*

3.1.7 Simple types

We are finally ready to define simple types.

Definition 3.1.23. A *simple type in G* is a pair (J, λ) consisting of a compact open subgroup J of G and an irreducible representation λ of J, of one of the following two forms.

1. $J = J^0(\beta, \mathfrak{A})$ and $\lambda = \kappa \otimes \sigma$, where

    (a) $\mathfrak{A}$ is a *principal* hereditary $\mathfrak{o}_F$-order in A and $\beta \in A$ such that $[\mathfrak{A}, m, 0, \beta]$ is a simple stratum (with $m = -\nu_{\mathfrak{A}}(\beta)$ of course);

(b) for some $\theta \in \mathcal{C}(\mathfrak{A}, 1, \beta)$, κ is a β-extension of the unique irreducible representation η of $J^1(\beta, \mathfrak{A})$ whose restriction to $H^1(\beta, \mathfrak{A})$ contains θ;

(c) if we let $E = F[\beta]$, $\mathfrak{B} = \mathfrak{B}_\beta$, $e = e(\mathfrak{B})$, and $d = \frac{n}{[E:F]}$, so that we have isomorphisms

$$J^0(\beta, \mathfrak{A})/J^1(\beta, \mathfrak{A}) \cong U(\mathfrak{B})/U^1(\mathfrak{B}) \cong \prod_{i=1}^e \mathrm{GL}_d(\boldsymbol{k}_E),$$

then σ is the inflation to $J^0(\beta, \mathfrak{A})$ of the e-fold tensor product $\sigma_0 \otimes \cdots \otimes \sigma_0$ for some irreducible cuspidal representation σ_0 of $\mathrm{GL}_d(\boldsymbol{k}_E)$.

2. $J = U(\mathfrak{A})$, where $\mathfrak{A}$ is a *principal* hereditary $\mathfrak{o}_F$-order in A, and if we let $e = e(\mathfrak{A})$ and $d = \frac{n}{e}$ so that we have an isomorphism

$$U(\mathfrak{A})/U^1(\mathfrak{A}) \cong \prod_{i=1}^e \mathrm{GL}_d(\boldsymbol{k}_F),$$

then λ is the inflation to $U(\mathfrak{A})$ of the e-fold tensor product $\sigma_0 \otimes \cdots \otimes \sigma_0$ for some irreducible cuspidal representation σ_0 of $\mathrm{GL}_d(\boldsymbol{k}_F)$.

Note that, with the terminology above, if $\beta \in F$, then $E = F$, $\mathfrak{B} = \mathfrak{A}$, and $J^0(\beta, \mathfrak{A}) = U(\mathfrak{A})$. Thus the second form in this definition is almost a special case of the first form, in which $\beta \in F$, *but κ is the trivial character of $J^0(\beta, \mathfrak{A})$*. (Note that the trivial character of $H^1(\beta, \mathfrak{A})$ is never a simple character, and thus as this definition is written, κ cannot be trivial. This is why we must treat the second form as a separate case.) Thus, in either case, there is a simple stratum $[\mathfrak{A}, m, 0, \beta]$ associated to the type (J, λ), such that $J = J^0(\beta, \mathfrak{A})$.

Definition 3.1.24. Let (J, λ) be a simple type, and assume all the notation of the previous definition. We say that (J, λ) is a *maximal simple type* if $e(\mathfrak{B}) = 1$. (Naturally, in the second case of the previous definition, this means $e(\mathfrak{A}) = 1$.)

Note that the condition in this definition is equivalent to requiring that $e(E/F) = e(\mathfrak{A})$, since

$$e(\mathfrak{B}) = \frac{e(\mathfrak{A})}{e(E/F)}.$$

Also, using the block matrix form of hereditary orders, it is equivalent to requiring that $\mathfrak{B} \cong M_d(\mathfrak{o}_E)$, where $d = \frac{n}{[E:F]}$. The following is a summary

of the culminating results of Chapter 6 of [7] (part of which is not proved until the end of Chapter 8 of loc. cit.).

Proposition 3.1.25.

1. Let (J, λ) be a maximal simple type. If (π, V) is an irreducible smooth representation of G that contains λ (i.e., $\langle \pi, \lambda \rangle_J > 0$), then π is supercuspidal. Furthermore, if (π', V') is any other smooth representation of G that contains λ, then π' is inertially equivalent to π (i.e., there exists an unramified character χ of $F^\times$ such that $\pi' \cong (\chi \circ \det) \otimes \pi$).

2. Conversely, let (π, V) be an irreducible supercuspidal representation of G. Then there exists a maximal simple type (J, λ) such that π contains λ. Furthermore, if (J', λ') is any other simple type, then π contains λ if and only if (J', λ') is conjugate to (J, λ) (i.e., there exists some $g \in G$ such that $J' = J^g$ and $\lambda' \cong \lambda^g$). Thus in particular any simple type contained in π is maximal. Also, any simple type (J, λ) contained in π occurs in π with multiplicity 1 (i.e., $\langle \pi, \lambda \rangle_J = 1$).

3. Let (J, λ) be a maximal simple type, with an associated simple stratum $[\mathfrak{A}, m, 0, \beta]$, and let $E = F[\beta]$ as usual. If Λ is any extension of λ from J to $E^\times J$, then the representation

$$\pi = \mathrm{Ind}_{E^\times J}^G(\Lambda)$$

is irreducible and supercuspidal.

4. Conversely, let (π, V) be an irreducible supercuspidal representation of G that contains a maximal simple type (J, λ), with an associated simple stratum $[\mathfrak{A}, m, 0, \beta]$, and let $E = F[\beta]$. Then there exists a unique extension Λ of λ from J to $E^\times J$ such that

$$\pi \cong \mathrm{Ind}_{E^\times J}^G(\Lambda).$$

3.2 A bound on the characters of supercuspidal K-types

Our goal now is to establish a bound on the traces of K-types for a large class of suitably regular elements of $K = \mathrm{GL}_n(\mathfrak{o}_F)$. In Chapter 4, we will utilize the main result of this section to prove an existence theorem for global automorphic representations. While the technical requirements of that theorem have provided the specific hypotheses we use for the class of

elements of K considered here, a basic guiding principle is as follows: for a large class (indeed a large open set) of elements g suitably far from the center of K, the character of any K-type evaluated at g should be uniformly bounded, and this bound should not depend on q (the cardinality of the residue field of F). We prove here that this is true for supercuspidal K-types when n is prime. Furthermore, we prove that for all non-central elements g, there is a bound on $\operatorname{Tr}\tau(g)$ as τ varies over all such K-types, though this latter bound will in general depend on g (and thus on q).

3.2.1 A little linear algebra

We begin with a very general and simple lemma, which is likely a known result. The proof is easy enough to be a problem on an algebra qualifying exam, but as we will apply this in a few different contexts below, it seems reasonable to state it explicitly here and include a complete proof.

Let K be any field, n any positive integer, and L/K a field extension of degree n. For any intermediate field M, define

$$X_M = \big\{ [u_0 : \ldots : u_{m-1}] \in \mathbb{P}^{m-1}(L) \mid \{u_0, \ldots, u_{m-1}\} \text{ is}$$
$$\text{linearly independent over } M \big\},$$

where $m = [L : M]$. Note that $X_L = \mathbb{P}^0(L)$ is a set with one element. In the lemma below, our primary interest is the set X_K, so to simplify notation, we let $X = X_K$. Note that the set X is invariant under the natural action of the group $\operatorname{GL}_n(K)$ on $\mathbb{P}^{n-1}(L)$.

Lemma 3.2.1. *Assume the notation above, and fix $g \in \operatorname{GL}_n(K)$. Then g has a fixed point in X if and only if the minimal polynomial of g is irreducible over K and has a root in L. In this case, the fixed points are in one-to-one correspondence with the set of pairs (γ, b_γ), where γ is an eigenvalue of g in L and $b_\gamma \in X_{K(\gamma)}$. In particular, if the characteristic polynomial of g is irreducible, the number of fixed points is equal to the number of eigenvalues of g in L.*

Proof. First we note that for any $h \in \operatorname{GL}_n(K)$, there is an obvious bijection between the fixed points of g and those of hgh^{-1}. Thus we are free to work with a conjugate of g. For a polynomial $p(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in$

$K[x]$, let C_p be the corresponding $d \times d$ companion matrix:

$$
\begin{pmatrix}
0 & 1 & & \\
& \ddots & \ddots & \\
& & 0 & 1 \\
-a_0 & -a_1 & \cdots & -a_{n-1}
\end{pmatrix}.
$$

Let $p_1, \ldots, p_r$ be the invariant factors of g, ordered so that $p_i \mid p_{i+1}$ for each $i < r$. By conjugating g as necessary within $\mathrm{GL}_n(K)$, we may assume that g is a block-diagonal matrix with blocks $C_{p_1}, \ldots, C_{p_r}$ along the diagonal. Let $d_i = \deg(p_i)$ for each i, and let

$$
e_j = \sum_{i=1}^{j} d_i \qquad \text{for } 0 \le j \le r
$$

(so that $e_0 = 0$ and $e_r = n$). Let $x = [u_0 : \ldots : u_{n-1}] \in X$, and let $\gamma = \frac{u_1}{u_0}$. A simple computation now shows that in order for x to be a fixed point of g, we must have $p_1(\gamma) = 0$ and

$$
u_{e_j + i} = u_{e_j} \gamma^i \text{ for all } 0 \le j < r \text{ and } 0 \le i < d_{j+1}. \tag{3.2.1}
$$

If p_1 is not irreducible over K, or if $d_i > d_1$ for any $i > 1$, then these coordinates clearly are not K-linearly independent, contradicting that $x \in X$. On the other hand, if this is not the case, then p_1 must be irreducible and $p_1 = \cdots = p_r$. Let $p = p_1$ and $d = d_1$, and note that $r = \frac{n}{d}$. Since γ must be a root of p in L, any choice of $u_0, u_d, \ldots, u_{(r-1)d} \in L$ that is linearly independent over the field $K(\gamma)$, combined with (3.2.1), will yield a fixed point of g in X. Modulo scalar multiplication (by scalars in $L^\times$), these fixed points are all distinct, and clearly they correspond with points in $X_{K(\gamma)}$. $\square$

As a special case of this lemma, note that if n is prime and g is not scalar, then the number of fixed points of g in X is the number of eigenvalues of g in L.

3.2.2 Maximal simple types when n is prime

We return now to the setting of the previous sections, in which F is a nonarchimedean local field, with all of its associated notation. As we will be dealing with the constructions of Section 3.1 in a special case, but will frequently make use of explicit matrix computations, we fix $V = F^n$, and we fix a basis of V so that we may identify $A = \mathrm{End}_F(V)$ with $M_n(F)$ and

$G = A^\times$ with $\mathrm{GL}_n(F)$. Throughout this entire section, *we will assume that n is prime.*

Let (J, λ) be a maximal simple type for G (cf. Section 3.1.7). Let $K = \mathrm{GL}_n(\mathfrak{o}_F)$, and assume (after conjugating J and λ as necessary) that $J \subset K$. Let $[\mathfrak{A}, m, 0, \beta]$ be a simple stratum underlying this simple type. Since n is prime, we may in fact assume (cf. [22]) that this stratum is alfalfa (i.e., that $\beta \in A$ is minimal over F), and thus that the defining sequence for $[\mathfrak{A}, m, 0, \beta]$ has length 1. In this setting, the definitions of the groups $H^k(\beta, \mathfrak{A})$ and $J^k(\beta, \mathfrak{A})$ become greatly simplified. Indeed, Corollary 3.1.13 gives

$$J^0(\beta, \mathfrak{A}) = \mathfrak{B}_\beta^\times U^{\lfloor \frac{m+1}{2} \rfloor}(\mathfrak{A}),$$
$$J^1(\beta, \mathfrak{A}) = U^1(\mathfrak{B}_\beta) U^{\lfloor \frac{m+1}{2} \rfloor}(\mathfrak{A}), \text{ and} \qquad (3.2.2)$$
$$H^1(\beta, \mathfrak{A}) = U^1(\mathfrak{B}_\beta) U^{\lceil \frac{m+1}{2} \rceil}(\mathfrak{A}).$$

Recall that $J = J^0(\beta, \mathfrak{A})$; throughout this section we will also abbreviate $J^1 = J^1(\beta, \mathfrak{A})$ and $H^1 = H^1(\beta, \mathfrak{A})$. The expressions above simplify even further after subdividing into a few cases. Let $\mathfrak{P}$ be the Jacobson radical of $\mathfrak{A}$, and let $E = F[\beta]$ be the field extension associated to the stratum $[\mathfrak{A}, m, 0, \beta]$ (cf. Section 3.1.2). Since n is prime, either $E = F$ or $[E : F] = n$, and E/F is either unramified or totally ramified. Furthermore, since the simple type (J, λ) is maximal (cf. Definition 3.1.24), we have $e(\mathfrak{B}_\beta) = 1$, or equivalently $e(\mathfrak{A}) = e(E/F)$. Thus (cf. Section 3.1.1) when E/F is unramified, $\mathfrak{A} = M_n(\mathfrak{o}_F)$, whereas when E/F is totally ramified, $\mathfrak{A}$ is the subring of $M_n(\mathfrak{o}_F)$ of matrices that are upper-triangular modulo $\mathfrak{p}$. Unraveling the definitions further, if $E = F$, then $\mathfrak{B}_\beta = \mathfrak{A}$, whence $J = \mathfrak{A}^\times = K$ and $H^1 = J^1 = U^1(\mathfrak{A}) = 1 + \mathfrak{p} M_n(\mathfrak{o}_F)$. On the other hand, if $[E : F] = n$, then $\mathfrak{B}_\beta = \mathfrak{o}_E$, and thus $U^1(\mathfrak{B}_\beta) = 1 + \mathfrak{p}_E$. Thus we have completely described the group J in all cases, along with the subgroups J^1 and H^1.

We can also explicitly describe the representation λ in just a few cases. Suppose first that $[E : F] = n$. Then we must be in the first case of Definition 3.1.23. Furthermore, in the notation of that definition, we must have $d = e = 1$, whence σ is merely a character of $J/J^1 \cong \boldsymbol{k}_E^\times$. So in fact λ is merely a β-extension of the representation η of J^1, because by Proposition 3.1.22, the β-extensions of η already account for twisting by characters of $\boldsymbol{k}_E^\times$. The representation η is the unique irreducible representation of J^1 whose restriction to H^1 contains a certain simple character θ (which is the unique simple character of H^1 contained in λ). Furthermore, it is clear from Definition 3.1.15 that in this setting, any extension of the character ψ_β from $U^{\lceil \frac{m+1}{2} \rceil}(\mathfrak{A})$ to H^1 is a simple character.

Now suppose that $E = F$. In this setting, we may be in either of the two cases of Definition 3.1.23, but either way we have $d = n$ and $e = 1$. Thus in the second case of that definition, λ is just the inflation to K of an irreducible cuspidal representation of $\mathrm{GL}_n(\boldsymbol{k}_F)$, whereas in the first case, λ is the tensor product of such a representation with an irreducible representation (in fact a character) κ of K. After Unraveling the definitions, it is not hard to see that in the latter case, $\kappa = \chi \circ \det$, where χ may ultimately be any character of $\mathfrak{o}_F^\times$ that is nontrivial on $1 + \mathfrak{p}_F$ (i.e., that has level at least 1). Taking both cases together, we see that when $E = F$, λ is the twist of an irreducible cuspidal representation of $\mathrm{GL}_n(\boldsymbol{k}_F)$ by an arbitrary character of $\mathfrak{o}_F^\times$.

To summarize all of the above, we obtain all maximal simple types (up to conjugacy) from the following three cases, which are distinguished from each other simply by the field extension E/F:

1. **The depth zero case:** In this case, $E = F$, $\mathfrak{A} = M_n(\mathfrak{o}_F)$, and $J = \mathfrak{A}^\times = K$. Here $\lambda = (\chi \circ \det) \otimes \sigma$, where χ is a character of $\mathfrak{o}_F^\times$ and σ is the inflation to K of an irreducible cuspidal representation of $\mathrm{GL}_n(\boldsymbol{k}_F)$.

2. **The unramified case:** In this case, E is an unramified extension of F of degree n, $\mathfrak{A} = M_n(\mathfrak{o}_F)$, and $\mathfrak{P}^k = M_n(\mathfrak{p}_F^k)$ for all integers k. The definitions of J and λ in this case will be further described below.

3. **The ramified case:** Here, E is a totally ramified extension of F of degree n, and $\mathfrak{A}$ is the ring of matrices in $M_n(\mathfrak{o}_F)$ for which the reduction modulo $\mathfrak{p}_F$ is upper triangular. In this case, $\mathfrak{A}^\times$ is the usual Iwahori subgroup of K, and $\mathfrak{P}$ is the ideal of matrices for which the reduction modulo $\mathfrak{p}_F$ is nilpotent upper triangular. Here again, the definitions of J and λ will be described below.

In the last two cases above, (3.2.2) simplifies to

$$J = \mathfrak{o}_E^\times(1 + \mathfrak{P}^{\lfloor \frac{m+1}{2} \rfloor}),$$
$$J^1 = (1 + \mathfrak{p}_E)(1 + \mathfrak{P}^{\lfloor \frac{m+1}{2} \rfloor}), \text{ and}$$
$$H^1 = (1 + \mathfrak{p}_E)(1 + \mathfrak{P}^{\lceil \frac{m+1}{2} \rceil}).$$

Clearly if m is odd then $H^1 = J^1$, whereas if m is even, then H^1 is a proper subgroup of J^1. Note also that we have an exact sequence

$$1 \to J^1 \to J \to \boldsymbol{k}_E^\times \to 1, \tag{3.2.3}$$

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which in this case splits since $\boldsymbol{k}_E^\times \cong \mu_E$, the group of roots of unity of order prime to p in E. Thus $J = \boldsymbol{k}_E^\times \ltimes J^1$, where the action of $\boldsymbol{k}_E^\times$ on J^1 is by conjugation.

We now elaborate slightly on the description of λ in the last two cases above. We begin with the character ψ_β of the group $1 + \mathfrak{P}^{\lceil \frac{m+1}{2} \rceil}$ (defined from the stratum $[\mathfrak{A}, m, 0, \beta]$ using our fixed character ψ of F). Let θ be an extension of ψ_β from this group to H^1 (as mentioned above, in this setting, any such extension is a simple character). Let η be the unique irreducible representation of J^1 whose restriction to H^1 contains θ. Obviously if m is odd, this is just θ itself, whereas if m is even, then η has dimension $[J^1 : H^1]^{\frac{1}{2}}$, which is a power of q (cf. Section 3.1.6). By Proposition 3.1.22, there are exactly $\#\boldsymbol{k}_E^\times$ isomorphism classes of β-extensions of η, each of which is a twist of a single one by a character of $\boldsymbol{k}_E^\times$. But by Theorem 2 of [17], this accounts for all extensions of η to J. We may thus take λ to be any extension of η from J^1 to J.

Our first task is to more carefully analyze the representation λ in one particular case, namely when E is unramified and λ is *not* 1-dimensional (i.e., when m is even). The result that we derive here is probably well known to the experts, but the exact statement that we require does not seem to appear in the literature. At any rate, the details are quite technical, so we collect them here. It is likely that a very similar statement holds more generally, but the lemma below is sufficient for our needs. The proof of this lemma is very similar to others found in the literature (see for example [4, 4.1 - 4.2], [3, 4.1]), but adapted to the current setting.

Lemma 3.2.2. *Let $[\mathfrak{A}, m, 0, \beta]$ be an alfalfa stratum with $\mathfrak{A} = M_n(\mathfrak{o}_F)$ and $E = F[\beta]$ an unramified extension of F of degree n. Assume that m is even, and let H^1, J^1, J, θ, η, and λ be as above. Then*

$$|\mathrm{Tr}\, \lambda(a(1+x))| = 1$$

for any $x \in \mathfrak{P}^{\lfloor \frac{m+1}{2} \rfloor}$ and any $a \in \mathfrak{o}_E^\times$ whose reduction modulo $\mathfrak{p}_E$ is not in $\boldsymbol{k}_F^\times$.

Proof. For convenience, let $k = \lfloor \frac{m+1}{2} \rfloor = \frac{m}{2}$, so that

$$\begin{aligned} H^1 &= (1 + \mathfrak{p}_E)(1 + \mathfrak{P}^{k+1}), \\ J^1 &= (1 + \mathfrak{p}_E)(1 + \mathfrak{P}^k), \text{ and} \\ J &= \mathfrak{o}_E^\times (1 + \mathfrak{P}^k). \end{aligned}$$

Recall from Section 3.1.6 that θ is fixed under conjugation by J. Thus $\operatorname{Ker}\theta \lhd J$, so we let

$$\overline{H^1} = H^1/\operatorname{Ker}\theta\,, \quad \overline{J^1} = J^1/\operatorname{Ker}\theta\,, \quad \overline{J} = J/\operatorname{Ker}\theta\,,$$

and let $\overline{\theta}$ (resp. $\overline{\eta}$, $\overline{\lambda}$) be the composition of θ (resp. η, λ) with the quotient map. Thus $\overline{\eta}$ is the unique irreducible representation of $\overline{J^1}$ whose restriction to $\overline{H^1}$ contains $\overline{\theta}$, and $\overline{\lambda}$ is an extension of $\overline{\eta}$ to $\overline{J}$.

As in Section 3.1.6, let $W = \overline{J^1}\big/\overline{H^1} \cong J^1/H^1$, and define the alternating bilinear form $h_\theta : W \times W \to \mathbb{C}^\times$ by

$$(\overline{x}, \overline{y}) \mapsto \theta[x, y].$$

By Proposition 3.1.19, h_θ is nondegenerate, from which it follows that $\overline{H^1}$ is the center of $\overline{J^1}$ (and hence also of $\overline{J}$).

Although we will not need this result, note that in this setting

$$W \cong (1 + \mathfrak{P}^k)\big/(1 + \mathfrak{p}_E^k)(1 + \mathfrak{P}^{k+1}) \cong \mathfrak{P}^k\big/\mathfrak{p}_E^k + \mathfrak{P}^{k+1}$$

is a $\boldsymbol{k}_F$-vector space of dimension $n^2 - n$. Thus the representation λ will have dimension $q^{\frac{n^2-n}{2}}$.

The split exact sequence (3.2.3) reduces to

$$1 \to \overline{J^1} \to \overline{J} \to \boldsymbol{k}_E^\times \to 1,$$

which still splits. We regard $\boldsymbol{k}_E^\times$ as a group of automorphisms of $\overline{J^1}$, acting by conjugation. Fix $a \in \boldsymbol{k}_E^\times \smallsetminus \boldsymbol{k}_F^\times$. Note that the commutator map $V \to V$ defined by $v \mapsto a^{-1}vav^{-1}$ is an isomorphism. Thus if $g \in \overline{J^1}$, we can choose $g_0 \in \overline{J^1}$ such that $a^{-1}g_0 a g_0^{-1} = gh^{-1}$ for some $h \in \overline{H^1}$, whence $g_0^{-1}(ag)g_0 = ah$. Thus every element of $\overline{J}$ of the form ag, $g \in \overline{J^1}$, is conjugate to an element of the form ah, with $h \in \overline{H^1}$. So we will be finished if we can prove that $\big|\operatorname{Tr}\overline{\lambda}(ah)\big| = 1$ for all $h \in \overline{H^1}$.

Let $A = \langle a \rangle \subset \boldsymbol{k}_E^\times$. Note that $\overline{J^1}$ is a finite p-group (where p is the characteristic of $\boldsymbol{k}_F$), so its order is relatively prime to that of A. Since $\overline{\theta}$ is fixed by the action of A, the isomorphism class of $\overline{\eta}$ is as well. Under these circumstances, in [17], Glauberman gives a one-to-one correspondence between isomorphism classes of irreducible representations of $\overline{J^1}$ fixed by A and those of $\overline{J^1}^A = \overline{H^1}$. This correspondence maps $\overline{\eta}$ to $\overline{\theta}$ (by Theorem 5(d) of [17], for example). By Theorem 2 of [17], there exists a certain canonical extension of $\overline{\eta}$ to $\overline{J}$, and $\overline{\lambda}$ is a twist of it by a uniquely determined character

χ of $\boldsymbol{k}_E^\times$. Thus by Theorem 3 of [17], there exists a constant $\epsilon = \pm 1$ such that

$$\operatorname{Tr} \overline{\lambda}(ah) = \epsilon \, \chi(a)\overline{\theta}(h)$$

for all $h \in \overline{H^1}$. (Note that the constant ϵ depends on a and on η, but this need not concern us here.) The result now follows. $\qquad\square$

3.2.3 Main theorem

We now come to the main theorem of this chapter.

Theorem 3.2.3. *Let n be a prime integer, let $g \in K = \operatorname{GL}_n(\mathfrak{o}_F)$, and assume g is not in the center of K. There exists a constant C_g such that for all supercuspidal K-types τ,*

$$|\operatorname{Tr}(\tau(g))| \le C_g.$$

Let $\overline{g} \in \operatorname{GL}_n(\boldsymbol{k}_F)$ be the reduction of g modulo $\mathfrak{p}_F$. Then if the characteristic polynomial of $\overline{g}$ is irreducible, we may take $C_g = n$. Otherwise, if $\overline{g}$ has at least two distinct eigenvalues, then we may take $C_g = 0$.

Proof. Recall (from Lemma 2.2.10 for example) that all supercuspidal K-types are of the form $\tau = \operatorname{Ind}_J^K(\lambda)$, for some maximal simple type (J, λ) with $J \subset K$. Thus we will of course prove this theorem in three cases, corresponding to the three cases in the construction of maximal simple types described above. As we will be computing the characters of induced representations, much of the proof will be based on the well known Frobenius formula:

$$\operatorname{Tr}\operatorname{Ind}_H^G(\pi)(g) = \sum_{\substack{x \in G/H \\ x^{-1}gx \in H}} \operatorname{Tr}\pi(x^{-1}gx), \qquad (3.2.4)$$

for any smooth representation π of an open subgroup H of a profinite group G. Note that in this formula, the condition $x^{-1}gx \in H$ is equivalent to $gxH = xH$, or in other words that the coset of x in G/H is fixed under the natural left action of G.

The depth zero case:

If τ is a "depth zero" K-type, it is merely the twist by a character of $\mathfrak{o}_F^\times$ of the inflation to K of an irreducible cuspidal representation of $\operatorname{GL}_n(\boldsymbol{k}_F)$. Since this group is finite, and twisting clearly has no effect on the absolute value of the trace, the first claim is clear for this case. Thus we need only consider the character of τ when $\overline{g}$ has at least two distinct eigenvalues.

Though the cuspidal representations of $\mathrm{GL}_n(\mathbb{F}_q)$ were not constructed in general until much later (see in particular [16]), their characters were first computed in [18]. In that paper, these particular characters are referred to by the notation[1] $(g^{\{1\}})$, which is defined to be equal to $(-1)^{n-1}I_n^k[1]$. To compute these characters, we fix a *regular* character θ of $\mathbb{F}_{q^n}^\times$ (i.e., one for which $\theta, \theta^q, \ldots, \theta^{q^{n-1}}$ are all distinct, or in other words one that maps a generator of $\mathbb{F}_{q^n}^\times$ to a root of unity of order $q^n - 1$). Then for an integer k as described above, θ^k will also be a regular character. From Section 5, Example (ii) of loc. cit., using the fact that n is prime, the following is clear:

- If the characteristic polynomial of $\bar{g}$ is not irreducible (but $\bar{g}$ does have at least two distinct eigenvalues), then $(-1)^{n-1}I_n^k[1](\bar{g}) = 0$.

- If the characteristic polynomial of $\bar{g}$ is irreducible, then

$$(-1)^{n-1}I_n^k[1](\bar{g}) = (-1)^{n-1}\sum_{\gamma}\theta^k(\gamma),$$

  where the sum is over the n distinct eigenvalues of $\bar{g}$.

The same example referenced above also shows that if $\bar{g}$ has a single eigenvalue γ of multiplicity n, then

$$(-1)^{n-1}I_n^k[1](\bar{g}) = (1-q)(1-q^2)\cdots(1-q^{r-1})\theta(\gamma),$$

where r is the number of Jordan blocks in the Jordan canonical form of $\bar{g}$. In particular, the dimension of any irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{F}_q)$ (and thus of any depth zero K-type) is $(q-1)(q^2-1)\cdots(q^{n-1}-1)$.

The unramified case:

Temporarily, let (J, λ) be a maximal simple type from the "unramified case", with associated alfalfa stratum $[\mathfrak{A}, m, 0, \beta]$, and let $\tau = \mathrm{Ind}_J^K(\lambda)$ be the corresponding K-type. In this case, $E = F[\beta]$ is an unramified extension of F of degree n, and $\mathfrak{A}^\times = K$. (Recall that $\beta \in M_n(F)$, and thus that we regard E as being explicitly embedded in the F-algebra $M_n(F)$.) To apply

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[1] Note that the g in this notation is not at all the same thing as our group element g. The fact that these characters are the cuspidal ones is clear from Theorem 13 of [18]. Elsewhere in that paper, the notations $J_n(k)$ and $I_g[\{1\}]$ are also used to mean exactly the same thing as $I_n^k[1]$. Here k is an integer parameter such that $k, kq, \ldots, kq^{n-1}$ are distinct modulo $q^n - 1$. In the other notations, g essentially refers to this entire set of n integers.

the Frobenius formula, we must first define a model of the coset space K/J that is equipped with the same left action of K, then determine the points fixed by the element g in this space. To simplify notation, we let $k = \lfloor \frac{m+1}{2} \rfloor$, so that $J = \mathfrak{o}_E^\times(1 + \mathfrak{P}^k)$.

Much like in Lemma 3.2.1, we consider the natural action of K on $\mathbb{P}^{n-1}(\mathfrak{o}_E)$, and we define X as the set of all points in $\mathbb{P}^{n-1}(\mathfrak{o}_E)$ with homogeneous coordinates

$$[u_0 : \ldots : u_{n-1}]$$

such that $\{u_0, \ldots, u_{n-1}\}$ is an $\mathfrak{o}_F$-basis of $\mathfrak{o}_E$. It is clear that K acts transitively on X, and that $\mathfrak{o}_E^\times$ is the stabilizer of some point $x \in X$. A straightforward computation shows that the normal subgroup $1 + \mathfrak{P}^k$ induces the equivalence relation of congruence modulo $\mathfrak{p}_E^k$ on the coordinates u_i of points in X. If we let X_k denote the quotient of X under this equivalence (which we may think of as a subset of $\mathbb{P}^{n-1}(\mathfrak{o}_E/\mathfrak{p}_E^k)$), and let x_k denote the class of x, then we have a K-equivariant bijection

$$K/J \to X_k$$

defined by $aJ \mapsto a \cdot x_k$.

Note that the choice of the point x will depend on the embedding of E into $M_n(F)$, and hence on the element $\beta \in M_n(F)$. Thus the actual bijection established here will vary for different types (J, λ), even for different ones having the same value of k. However, the action of K on X_k is the same in all cases, and it will turn out that this will be all that matters for our purpose. For example, since λ has dimension either 1 or $q^{\frac{n^2-n}{2}}$ (depending on whether m is odd or even, respectively), the character of any simple type (J, λ) evaluated at any element of J is clearly bounded by the latter value. Thus by the Frobenius formula, the first claim of the theorem will be proved for all K-types of the "unramified case" once we can show that the number of fixed points of g in X_k is bounded as $k \to \infty$. Since this has nothing to do with the choice of a fixed type, we now forget about J and λ (and β, m, etc.) until near the end of this section, and work only with the sets X_k, for *all $k > 0$*.

Note that for each $k' < k$, we get a K-equivariant surjection $X_k \to X_{k'}$. These form a projective system

$$X_1 \leftarrow X_2 \leftarrow \cdots$$

of K-sets (in which all the sets, and hence the fibers of each map, are finite), and $X = \varprojlim X_k$. Clearly if g has a fixed point in X_k, then the image of this

point in $X_{k'}$ must be a fixed point of g as well. We may also consider the subset $\widetilde{X}$ of $\mathbb{P}^{n-1}(E)$ consisting of points whose homogeneous coordinates are F-linearly independent; this is precisely the set considered in Lemma 3.2.1 applied to the extension E/F. Note that the natural map $X \to \widetilde{X}$ is K-equivariant and injective. Of course we once again see that any fixed point of g in X must map to a fixed point of g in $\widetilde{X}$. But by Lemma 3.2.1, since n is prime and g is not scalar, there are at most n fixed points of g in $\widetilde{X}$, hence there are at most n in X as well. If the number of fixed points in X_k were unbounded as $k \to \infty$, we would be able to find infinitely many sequences of fixed points

$$(x^{(i)}), \qquad x^{(i)} \in X_i$$

that are compatible with the maps above, or in other words infinitely many fixed points of g in $\varprojlim X_k$, which is just X. As this is a contradiction, the number of fixed points of g in X_k must be bounded as $k \to \infty$, which proves the first claim in this case.

To establish the last two claims of the theorem in this case, note that for any k, the action of $K = \mathrm{GL}_n(\mathfrak{o}_F)$ on X_k is the same as the natural action of $\mathrm{GL}_n(\mathfrak{o}_F)/1 + \mathfrak{P}^k \cong \mathrm{GL}_n(\mathfrak{o}_F/\mathfrak{p}_F^k)$ on X_k, composed with the quotient map. In particular, X_1 is precisely the set considered in Lemma 3.2.1 applied to the field extension $\boldsymbol{k}_E/\boldsymbol{k}_F$. From that lemma, using once again the fact that n is prime, we see that if $\overline{g}$ has at least two distinct eigenvalues, then there are only two possibilities:

- If the characteristic polynomial of $\overline{g}$ is not irreducible, then $\overline{g}$ (and thus also g) has no fixed points in X_1. But then g can have no fixed points in X_k for any $k > 0$.

- If the characteristic polynomial of $\overline{g}$ is irreducible, then $\overline{g}$ (and hence also g) has exactly n fixed points in X_1, corresponding to the n distinct eigenvalues of $\overline{g}$ in $\boldsymbol{k}_E$.

The first of these situations clearly proves the last claim of the theorem in the "unramified case". We now elaborate on the second situation.

Let p be the characteristic polynomial of g and $\overline{p}$ that of $\overline{g}$, and assume that $\overline{p}$ is irreducible over $\boldsymbol{k}_F$. Let $C_p \in \mathrm{GL}_n(\mathfrak{o}_F)$ be the companion matrix of p as in the proof of Lemma 3.2.1, and likewise for $C_{\overline{p}} \in \mathrm{GL}_n(\boldsymbol{k}_F)$. Since p must be irreducible, we know that g is conjugate in $\mathrm{GL}_n(F)$ to C_p, but since $\overline{p}$ is irreducible, we can say more. Indeed, there exists a $\overline{g}$-cyclic vector $\overline{v} \in \boldsymbol{k}_F^n$, i.e., a vector for which

$$\{\overline{v}, \overline{g}\overline{v}, \ldots, \overline{g}^{n-1}\overline{v}\}$$

is a basis of $\boldsymbol{k}_F^n$, and the matrix of $\bar{g}$ with respect to this basis is $C_{\bar{p}}$. It is easy to see that any lift v of $\bar{v}$ from $\boldsymbol{k}_F^n$ to $\mathfrak{o}_F^n$ must be g-cyclic, and such a vector yields a basis of F^n consisting of vectors in $\mathfrak{o}_F^n$. With respect to this basis, the matrix of g is of course C_p. Thus in fact g is conjugate in K to C_p, so for the purposes of computing characters, we may assume $g = C_p$.

Now, the same computation as in the proof of Lemma 3.2.1 shows that the fixed points of g in X_k correspond precisely to the roots of p in $(\mathfrak{o}_E/\mathfrak{p}_E^k)^\times$. As mentioned above, if $k = 1$, there are clearly n of these. But now since they are all distinct, we can apply Hensel's lemma to conclude that there are exactly n such roots in $(\mathfrak{o}_E/\mathfrak{p}_E^k)^\times$ for all $k > 1$ as well.

Thus we have exactly n fixed points of g in X_k for all $k > 0$. We now return to the context of the beginning of this section of the proof, and let (J, λ) be a simple type, with underlying alfalfa stratum $[\mathfrak{A}, m, 0, \beta]$, $E = F[\beta]$ an unramified extension of degree n, and $k = \left\lfloor \frac{m+1}{2} \right\rfloor$. Let $\tau = \mathrm{Ind}_J^K(\lambda)$ be the corresponding K-type. Now each of the n fixed points of g in X_k corresponds to an $x \in K/J$ such that $x^{-1}gx \in J$. Recalling that $J = \mathfrak{o}_E^\times(1 + \mathfrak{P}^k)$, we see that the reduction modulo $\mathfrak{p}$ of $x^{-1}gx$ will be an element of $(\mathfrak{o}_E/\mathfrak{p}_E)^\times$. Since its minimal polynomial is irreducible of degree n (because it is a conjugate of $\bar{g}$) it must in fact be in $\boldsymbol{k}_E^\times \smallsetminus \boldsymbol{k}_F^\times$. Thus, if m is even, Lemma 3.2.2 implies that $\left| \mathrm{Tr}\, \lambda(x^{-1}gx) \right| \le 1$. On the other hand, if m is odd, then λ is one-dimensional, so the same is clearly true. Thus either way, the Frobenius formula implies

$$|\mathrm{Tr}\, \tau(g)| \le n.$$

The ramified case:

As in the previous section of the proof, we temporarily let (J, λ) be a maximal simple type from the "ramified case", with associated alfalfa stratum $[\mathfrak{A}, m, 0, \beta]$. In this case, $E = F[\beta]$ is a ramified extension of F of degree n, and $\mathfrak{A}^\times$ is the Iwahori subgroup of K. Once again, let $k = \left\lfloor \frac{m+1}{2} \right\rfloor$, so that $J = \mathfrak{o}_E^\times(1 + \mathfrak{P}^k)$. Let $\rho = \mathrm{Ind}_J^{\mathfrak{A}^\times}(\lambda)$, and let

$$\tau = \mathrm{Ind}_{\mathfrak{A}^\times}^K(\rho) = \mathrm{Ind}_J^K(\lambda)$$

be the corresponding K-type. We will use the same strategy here as in the previous section of the proof, except that we will deal primarily with the induction to $\mathfrak{A}^\times$, which in this case is a proper subgroup of K.

Let ϖ be a uniformizer of E, and define $X \subset \mathbb{P}^{n-1}(\mathfrak{o}_E)$ to be the set of all points with homogeneous coordinates

$$[u_0 : u_1\varpi : \ldots : u_{n-1}\varpi^{n-1}], \quad u_i \in \mathfrak{o}_E^\times.$$

(Note that this is equivalent to saying the coordinates form an $\mathfrak{o}_F$-basis of $\mathfrak{o}_E$, with strictly increasing E-valuations.) Again it is easy to see that $\mathfrak{A}^\times$ acts transitively on X, and that $\mathfrak{o}_E^\times$ is the stabilizer of some point $x \in X$. A tedious but straightforward computation shows that in this case, the normal subgroup $1 + \mathfrak{P}^k$ induces the equivalence relation of congruence modulo $\mathfrak{p}_E^k$ on the units u_i appearing in the coordinates of points in X:

$$[u_0 : u_1\varpi : \ldots : u_{n-1}\varpi^{n-1}] \sim [u_0' : u_1'\varpi : \ldots : u_{n-1}'\varpi^{n-1}]$$

$$\text{if and only if}$$

$$u_i \equiv u_i' \mod \mathfrak{p}_E^k \text{ for each } i.$$

If we once again let X_k denote the quotient of X under this equivalence, and let x_k denote the class of x, then $aJ \mapsto a \cdot x_k$ again defines an $\mathfrak{A}^\times$-equivariant bijection

$$\mathfrak{A}^\times / J \to X_k.$$

The same comments apply as before: the actual bijection given above will be different for subgroups J coming from different types, but the action of $\mathfrak{A}^\times$ on the set X_k will be the same regardless; and since the dimension of λ is bounded by a fixed value, we may now forget all about the specific type, and deal only with counting fixed points of g in the sets X_k, for all $k > 0$.

We first dispense easily with the last two claims of the theorem. If $g \in K$ is not K-conjugate to any element of $\mathfrak{A}^\times$, then it clearly cannot be conjugate to any element of J for *any* simple type (J, λ). Thus for such an element g, the Frobenius formula implies that $\text{Tr}\,\tau(g) = 0$ for all K-types τ of the "ramified case". On the other hand, if g is conjugate to an element of $\mathfrak{A}^\times$, then for the purpose of computing traces, we may assume $g \in \mathfrak{A}^\times$. Of course, we then see that $\bar{g} \in \text{GL}_n(\boldsymbol{k}_F)$ is upper-triangular. A simple computation shows that such a matrix $\bar{g}$ can have a fixed point in X_1 only if all its diagonal entries (its eigenvalues in $\boldsymbol{k}_F^\times$) are the same, in which case every point of X_1 is a fixed point. Just as in the previous section of the proof, we have a projective system

$$X_1 \leftarrow X_2 \leftarrow \cdots$$

of $\mathfrak{A}^\times$-sets, with $X = \varprojlim X_k$, and any fixed point of g in X_k must map to a fixed point in $X_{k'}$ for $k' < k$. Thus, if $\bar{g}$ has at least two distinct eigenvalues, it has no fixed point in X_1, and thus has no fixed point in X_k for all $k > 0$. This proves that $\text{Tr}\,\rho(g) = 0$ for all types in this case. But since the condition on g here depends only on its conjugacy class in K, applying the Frobenius formula to $\tau = \text{Ind}_{\mathfrak{A}^\times}^K(\rho)$ yields $\text{Tr}\,\tau(g) = 0$ for all K-types τ in this case

as well. Note that in this section of the proof, the trace bound of n in the second claim of the theorem does not arise at all.

Finally, we deal with the first claim of the theorem. The only remaining possibility for g is that it be a non-scalar element of K whose reduction modulo $\mathfrak{p}$ is upper-triangular with one eigenvalue of multiplicity n. We once again consider the subset $\widetilde{X}$ of $\mathbb{P}^{n-1}(E)$ consisting of points whose homogeneous coordinates are F-linearly independent; this is again exactly the set considered in Lemma 3.2.1 applied to the extension E/F. We again have a natural map $X \to \widetilde{X}$ that is K-equivariant and injective, and so any fixed point of g in X must map to a fixed point of g in $\widetilde{X}$. But by Lemma 3.2.1, since n is prime and g is not scalar, there are at most n fixed points of g in $\widetilde{X}$, hence there are at most n in X as well. By exactly the same argument as in the previous section of the proof, we see that the number of fixed points of g in X_k must be bounded as $k \to \infty$, which proves the first claim. This completes the final case of the proof. $\qquad\square$

Remark 1. The first case of the proof above stands in contrast to the rest of the proof, because it is the only part that does not involve counting fixed points of some endomorphism of a variety. A proof along those lines might be found as follows. In [16], Deligne and Lusztig construct a certain variety for each n, equipped with an action of the group $\mathrm{GL}_n(\mathbb{F}_q)$, and show that the irreducible representations of the group arise as subspaces of the ℓ-adic cohomology of the variety (for any $\ell \neq p$). The Lefschetz fixed point theorem then implies that the trace of such a representation evaluated at g can be computed by counting the number of fixed points of g in the variety. As the Deligne-Lusztig theory applies to a vastly more general class of groups, this approach might have the same advantage. The author would like to thank Jared Weinstein for suggesting this alternative line of proof.

Remark 2. In the second and third cases of the proof above, the guiding principle that the character of a supercuspidal K-type should be 0 "almost everywhere", and bounded by n "almost everywhere else" (outside the center), is clearly motivated by Lemma 3.2.1. At first glance, one might like to conclude from this lemma that in the one case where our bound is not explicit (when $\bar{g}$ has a single eigenvalue of multiplicity n), the bound is given by the number of fixed points of g in X_1. But of course, this bound can fail remarkably, essentially due to the fact that in this situation, Hensel's Lemma cannot guarantee uniqueness of roots of a polynomial as we go from $k = 1$ to $k = 2$, $k = 3$, $\ldots$. However, it should be possible (indeed easy, with enough computation) to find an explicit bound for any fixed g, even when there is no "nice" canonical form for the conjugacy class of g in K.

A simple matrix computation gives a system of $n-1$ polynomials in $n-1$ variables (defining an affine variety over F) for the fixed points of g. By the multivariable version of Hensel's Lemma, the failure of the bound to be just the number of fixed points in X_k for some k will be reflected in the fact that the Jacobian determinant of this system, evaluated at some given fixed point from X_k, vanishes modulo $\mathfrak{p}^k$. But as long as g is not scalar, this Jacobian determinant will not vanish in $\mathfrak{o}_F$, which is why the number of fixed points of g in X_k is ultimately bounded as $k \to \infty$. In short, one can compute this Jacobian determinant, and some expression involving its F-valuation should provide the desired bound.

Remark 3. A few final comments are in order about the requirement here that n be prime. First, to remove the restriction that n be prime and still deal with the full generality of supercuspidal K-types, it would become necessary to deal with the complete definition of maximal simple types from Section 3.1, rather than just the three relatively simple cases used here. This seems a daunting task. But even if one were to restrict one's attention to the case of "essentially tame" supercuspidals, so as to only require the same three cases as were used here, trouble seems to arise. For example, it is already clear just from the "depth zero" case of the proof above that when n is not prime, this theorem will completely fail as it is currently stated. Furthermore, in both of the other two cases of the proof, significant use was made, *in the exact same context*, of the fact that n was prime. Namely, in all cases, the proof relied on the fact that a polynomial, irreducible over one field but with a root in a degree n extension, must (if n is prime) have degree either 1 or n. To generalize this theorem to composite values of n, the hypotheses would have to be changed considerably to account for the failure of this fact.

Chapter 4

Automorphic representations of unitary groups with prescribed local types

In this chapter, we will state and prove a theorem guaranteeing the existence, "almost always", of automorphic representations with prescribed local types on certain kinds of unitary groups. Much of the preliminary material consists of the definitions and observations needed to make such a statement precise. This theorem will make very significant use of Theorem 3.2.3.

4.1 Representations of $\mathrm{U}(n)$

Our focus in most of this chapter will be on unitary groups, defined over a global field F, which we will require to be compact at all infinite places of F. Thus it will be useful to set down some preliminaries on the compact Lie group $\mathrm{U}(n)$:

$$\mathrm{U}(n) = \left\{ g \in M_n(\mathbb{C}) \mid g\, {}^{\mathrm{t}}\overline{g} = 1 \right\}.$$

Let $\mathfrak{g} = \mathfrak{u}(n)$ be the (real) Lie algebra of $\mathrm{U}(n)$:

$$\mathfrak{u}(n) = \left\{ X \in M_n(\mathbb{C}) \mid X + {}^{\mathrm{t}}\overline{X} = 0 \right\}.$$

This is of course the algebra of skew-Hermitian matrices in $M_n(\mathbb{C})$. Let $\mathfrak{g}_\mathbb{R} = i\mathfrak{g}$, the algebra of Hermitian matrices in $M_n(\mathbb{C})$, and let

$$\mathfrak{g}^\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C} = \mathfrak{g}_\mathbb{R} \oplus i\mathfrak{g}_\mathbb{R} = \mathfrak{gl}(n, \mathbb{C}).$$

We could at this point choose any maximal torus T in $\mathrm{U}(n)$ and corresponding Cartan subalgebra of $\mathfrak{g}$, but to make everything quite clear, we will fix a specific choice of maximal torus and work relative to a fixed basis. Thus, let T be the maximal torus in $\mathrm{U}(n)$ consisting of diagonal matrices, and let $\mathfrak{h}$ be the corresponding Cartan subalgebra of $\mathfrak{u}(n)$:

$$\mathfrak{h} = \left\{ \begin{pmatrix} ia_1 & & \\ & \ddots & \\ & & ia_n \end{pmatrix} \;\middle|\; a_i \in \mathbb{R} \right\}.$$

Again, let $\mathfrak{h}_{\mathbb{R}} = i\mathfrak{h}$, and let $\mathfrak{h}^{\mathbb{C}} = \mathfrak{h} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{h}_{\mathbb{R}} \oplus i\mathfrak{h}_{\mathbb{R}}$. Finally, let $\mathfrak{h}_{\mathbb{R}}^*$ and $(\mathfrak{h}^{\mathbb{C}})^*$ denote the dual spaces of $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}^{\mathbb{C}}$, respectively.

For bases, we choose the "standard" bases of $\mathfrak{g}^{\mathbb{C}}$, $\mathfrak{h}_{\mathbb{R}}$, and $\mathfrak{h}^{\mathbb{C}}$: Let E_{ij} be the $n \times n$ matrix with a 1 in the i, j position and zeros elsewhere, and let $e_i = E_{ii}$ for each i. We also let e_i^* denote the functionals of the corresponding dual basis, so $e_i^*(e_j) = \delta_{ij}$ for each i, j. Thus any linear functional $\lambda \in (\mathfrak{h}^{\mathbb{C}})^*$ can be written uniquely as $\lambda = \sum_{i=1}^{n} a_i e_i^*$ ($a_i \in \mathbb{C}$), and such a λ will be analytically integral if and only if $a_i \in \mathbb{Z}$ for all i. (These are precisely the functionals that arise as the weights of representations of $\mathrm{U}(n)$.) For such $\lambda \in (\mathfrak{h}^{\mathbb{C}})^*$, we define a character ϕ_λ of T by

$$\phi_\lambda(\exp(H)) = e^{\lambda(H)}.$$

Note that if we write $\lambda = \sum a_i e_i^*$ with $a_i \in \mathbb{Z}$, then ϕ_λ is given explicitly by

$$\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} \mapsto z_1^{a_1} \cdots z_n^{a_n}.$$

In this setting then, the set of roots Δ of $\mathrm{U}(n)$ is

$$\Delta = \left\{ \lambda_{ij} = e_i^* - e_j^* \;\middle|\; 1 \leq i \neq j \leq n \right\}.$$

With respect to our chosen basis of $(\mathfrak{h}^{\mathbb{C}})^*$, the sets of positive and simple roots are, respectively,

$$\Delta^+ = \left\{ \lambda_{ij} \mid 1 \leq i < j \leq n \right\} \text{ and}$$
$$\Pi = \left\{ \lambda_{i,i+1} = e_i^* - e_{i+1}^* \;\middle|\; 1 \leq i < n \right\}.$$

With these choices, we find that a weight $\lambda = \sum a_i e_i^*$ is dominant if and only if $a_i \geq a_j$ for all $i < j$. By the theorem of the highest weight, the irreducible

representations of $\mathrm{U}(n)$ are in one-to-one correspondence with the set Λ of dominant, analytically integral functionals on $\mathfrak{h}^{\mathbb{C}}$:

$$\Lambda = \left\{ \lambda = \sum_{i=1}^{n} a_i e_i^* \;\middle|\; a_i \in \mathbb{Z} \quad \forall i, \text{ and } a_1 \geq \cdots \geq a_n \right\}.$$

For $\lambda \in \Lambda$, we will denote by ξ_λ the corresponding representation of $\mathrm{U}(n)$.

We define a bilinear form $B_0 : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ by

$$B_0(X, Y) = \mathrm{Tr}\, XY.$$

Note that for any $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$, there is a unique $H_\lambda \in H$ such that $\lambda(H) = B_0(H, H_\lambda)$ for all $H \in \mathfrak{h}_{\mathbb{R}}$. We define an inner product on $\mathfrak{h}_{\mathbb{R}}^*$ by

$$\langle \lambda_1, \lambda_2 \rangle = B_0(H_{\lambda_1}, H_{\lambda_2}).$$

Note that if we express λ_1 and λ_2 in terms of the basis chosen above, this inner product amounts to merely taking the dot product of the corresponding coefficient vectors. Let δ be half the sum of the positive roots:

$$\delta = (\tfrac{n-1}{2})e_1^* + (\tfrac{n-3}{2})e_2^* + \cdots + (\tfrac{3-n}{2})e_{n-1}^* + (\tfrac{1-n}{2})e_n^*.$$

The Weyl dimension formula now gives

$$\begin{aligned}
\dim(\xi_\lambda) &= \prod_{\alpha \in \Delta^+} \frac{\langle \lambda + \delta, \alpha \rangle}{\langle \delta, \alpha \rangle} \\
&= \prod_{1 \leq i < j \leq n} \frac{a_i - a_j + j - i}{j - i} \\
&= \frac{\prod\limits_{i<j}(a_i - a_j + j - i)}{\prod\limits_{k=1}^{n-1} k!}
\end{aligned}$$

for any $\lambda = \sum a_i e_i^* \in \Lambda$. Note that the above expression is a polynomial of degree $\frac{n^2-n}{2}$ in the n variables $a_1, \ldots, a_n$. Throughout this chapter, we will refer to this polynomial as the Weyl polynomial for $\mathrm{U}(n)$. The following proposition is adapted slightly from [15, Prop. 1.9].

Proposition 4.1.1 (Chenevier-Clozel)**.** *Let $g \in \mathrm{U}(n)$, and assume g is not central. There exists a polynomial in n variables $P_g(X_1, \ldots, X_n)$, of degree strictly less than that of the Weyl polynomial, such that for all $\lambda = \sum a_i e_i^* \in \Lambda$,*

$$|\mathrm{Tr}\, \xi_\lambda(g)| \leq P_g(a_1, \ldots, a_n).$$

It will be convenient to abuse notation slightly and refer to the Weyl polynomial and the polynomial P_g above as polynomials on $\mathfrak{h}_{\mathbb{R}}^*$, with the understanding that when $\lambda = \sum a_i e_i^*$, $P(\lambda)$ means $P(a_1, \ldots, a_n)$. Note that the degree of such a polynomial is well-defined independently of our choice of basis for $\mathfrak{h}_{\mathbb{R}}^*$.

4.2 Global types on a unitary group

4.2.1 The unitary group G

Let F be a totally real number field and E a totally imaginary quadratic extension of F. Let n be prime, and let M be a central simple algebra of dimension n^2 over E, and denote by $x \mapsto x^*$ an involution of the second kind of M, i.e., an F-algebra anti-automorphism of M of order 2 whose restriction to E (the center of M) is the non-trivial element of $\mathrm{Gal}(E/F)$. Note that for any F-algebra R, there is an obvious natural extension of * to an R-algebra anti-automorphism of $M \otimes_F R$ that is nontrivial on $E \otimes_F R$, which we will also denote by *, as this will cause no confusion. Let G be the unitary group defined (over F) by M and *. Explicitly, this is given by

$$G(R) = \{g \in M \otimes_F R \mid gg^* = 1\} \quad \text{for every } F\text{-algebra } R.$$

Let M_0 be a maximal $\mathfrak{o}_E$-order in M, i.e., a maximal subring of M (with the same 1) that is the $\mathfrak{o}_E$-span of an E-basis of M. With this choice, we can extend the definition of the functor G from the category of F-algebras to the category of $\mathfrak{o}_F$-algebras, as follows:

$$G(R) = \{g \in M_0 \otimes_{\mathfrak{o}_F} R \mid gg^* = 1\} \quad \text{for every } \mathfrak{o}_F\text{-algebra } R,$$

where again * denotes the obvious natural extension of the involution on M_0. The natural isomorphism $M_0 \otimes_{\mathfrak{o}_F} R \to M \otimes_F R$ when R is an F-algebra assures us that this definition agrees with the one above. This allows us to define the group $G(\mathfrak{o}_{F_v})$ when v is a finite place of F. Note that $G(\mathfrak{o}_{F_v})$ is a maximal compact subgroup of $G(F_v)$.

In all that follows, we will fix a choice of M and * for which $G(F_v) \cong \mathrm{U}(n)$ for each infinite place v of F. For each such v, we fix an isomorphism

$$\iota_v : G(F_v) \to \mathrm{U}(n).$$

Let S be the set of places of F which split in E. (By the assumptions above, these are all finite.) For each $v \in S$, we will fix an isomorphism

$$\iota_v : G(F_v) \to \mathrm{GL}_n(F_v).$$

Note that the restriction of this map to $G(\mathfrak{o}_{F_v})$ is an isomorphism $G(\mathfrak{o}_{F_v}) \cong \mathrm{GL}_n(\mathfrak{o}_{F_v})$, which we will also refer to as ι_v, as this should cause no confusion. For the finite places v of F that do not split in E, we will use v to also denote the unique place of E lying over v, so that $E_v = E \otimes_F F_v$.

Let Z be the center of G, which is the unitary group of rank 1 defined over F using the extension E/F. This is given explicitly by

$$Z(R) = \{x \in E \otimes_F R \mid xx^* = 1\} \quad \text{for every } F\text{-algebra } R.$$

Since the center of M_0 is $\mathfrak{o}_E$, we can again extend this functor to the category of $\mathfrak{o}_F$-algebras in the same way as above, in order to define the groups $Z(\mathfrak{o}_{F_v})$. Note that $Z(F)$ is just $E^1 = \{x \in E \mid N_{E/F}(x) = 1\}$. For each infinite place v of F, $Z(F_v)$ is isomorphic (via the restriction of ι_v) to the center of $\mathrm{U}(n)$, which we identify with the circle group S^1. Similarly, for each finite place v that splits in E, $Z(F_v)$ is isomorphic (via the restriction of ι_v) to $F_v^\times$, and $Z(\mathfrak{o}_{F_v})$ is isomorphic to $\mathfrak{o}_{F_v}^\times$. For the finite places v that do not split in E, we have $Z(F_v) = Z(\mathfrak{o}_{F_v}) = E_v^1$, the group of units of norm 1 in E_v.

4.2.2 Automorphic forms on G

Let $\mathbb{A}$ be the ring of adeles of F, and $\mathbb{A}_E$ that of E. Recall that there is a natural embedding of $\mathbb{A}$ into $\mathbb{A}_E$ and a norm map $N_{E/F} : \mathbb{A}_E \to \mathbb{A}$. With this notation, the adelic points of the center of G are given by

$$Z(\mathbb{A}) = \{x \in \mathbb{A}_E^\times \mid N_{E/F}(x) = 1\}.$$

To simplify notation, we will let

$$\begin{aligned}
G_0 &= G(\widehat{\mathfrak{o}_F}) \times G(F \otimes_{\mathbb{Q}} \mathbb{R}) \\
&= \prod_{v \nmid \infty} G(\mathfrak{o}_{F_v}) \times \prod_{v \mid \infty} G(F_v), \text{ and} \\
Z_0 &= Z(G_0) = \prod_{v \nmid \infty} Z(\mathfrak{o}_{F_v}) \times \prod_{v \mid \infty} Z(F_v).
\end{aligned}$$

Note that, since G was chosen to be compact at all the infinite places of F, G_0 (resp. Z_0) is actually a maximal compact open subgroup of $G(\mathbb{A})$ (resp. $Z(\mathbb{A})$). Note that the subgroup of rational points of Z_0 is just the group $\mathfrak{o}_E^1$ of units of norm 1 in $\mathfrak{o}_E$, which is simply the finite group μ_E of roots of unity in E.

For a character ω of $Z(\mathbb{A})$ that is trivial on $Z(F)$, we let $\mathcal{A}(G(F)\backslash G(\mathbb{A}), \omega)$ be the space of automorphic forms on $G(\mathbb{A})$ with "central character" ω. This

is the space of smooth complex-valued functions on $G(\mathbb{A})$ that are invariant under left translation by elements of $G(F)$, and transform by ω under left translation by elements of $Z(\mathbb{A})$. Automorphic forms are in general also required to be G_0-finite, $\mathcal{Z}$-finite (where $\mathcal{Z}$ is the center of the universal enveloping algebra of the Lie algebra of G), and of moderate growth. However, in this setting, since G is compact at all the infinite places of F, any smooth, left $G(F)$-invariant function on $G(\mathbb{A})$ will automatically satisfy these remaining three properties. The group $G(\mathbb{A})$ acts on $\mathcal{A}(G(F)\backslash G(\mathbb{A}),\omega)$ by right translation, and the resulting representation is smooth. Furthermore, it decomposes as a direct sum of irreducible subspaces, each of which occurs with finite multiplicity. Clearly any irreducible representation of $G(\mathbb{A})$ occurring in $\mathcal{A}(G(F)\backslash G(\mathbb{A}),\omega)$ must have central character ω (justifying the terminology for automorphic forms). These irreducible representations, as ω ranges over all characters of $Z(F)\backslash Z(\mathbb{A})$, are the automorphic representations of $G(\mathbb{A})$. For an automorphic representation π with central character ω, we will write $m(\pi)$ for its multiplicity in $\mathcal{A}(G(F)\backslash G(\mathbb{A}),\omega)$.

Since $Z(F)\backslash Z(\mathbb{A})$ is compact, its spectrum (i.e., its Pontryagin dual, the group of characters ω considered above) is discrete. Thus we may consider the space of *all* automorphic forms on $G(\mathbb{A})$, which is simply a discrete direct sum:

$$\mathcal{A}(G(F)\backslash G(\mathbb{A})) = \bigoplus_{\omega} \mathcal{A}(G(F)\backslash G(\mathbb{A}),\omega).$$

Note that this is simply the space of smooth functions on $G(\mathbb{A})$ that are left $G(F)$-invariant. Clearly each automorphic representation π of $G(\mathbb{A})$ occurs in $\mathcal{A}(G(F)\backslash G(\mathbb{A}))$ with multiplicity $m(\pi)$. As a final note, we recall that, by a well known theorem, any automorphic representation π of $G(\mathbb{A})$ can be written as a "restricted" tensor product

$$\pi = \bigotimes_{v}' \pi_v,$$

where for each place v of F, π_v is an irreducible smooth representation of $G(F_v)$, and π_v is unramified for almost all v.

4.2.3   Global types for G

If π is an automorphic representation of $G(\mathbb{A})$, its central character ω_π is by definition trivial on $Z(F)$, so the restriction of ω_π to Z_0 must be trivial on $Z_0 \cap Z(F) = \mathfrak{o}_E^1$. This motivates the following definition.

Definition 4.2.1. A *global type for G* is an irreducible representation $\tau = \bigotimes_v \tau_v$ of G_0 satisfying the following:

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1. For each infinite place v of F, τ_v is an irreducible representation of $G(F_v)$. Thus (using the notation of Section 4.1) $\tau_v = \xi_\lambda \circ \iota_v$ for some $\lambda \in \Lambda$.

2. For all $v \notin S$ and almost all $v \in S$, τ_v is the trivial representation of $G(\mathfrak{o}_{F_v})$.

3. For all $v \in S$ for which τ_v is not 1-dimensional, $\tau_v = \tau_v' \circ \iota_v$ for some supercuspidal K-type τ_v' (where $K = \mathrm{GL}_n(\mathfrak{o}_{F_v})$).

4. If ω_v is the central character of τ_v for each place v, then the character $\omega_\tau = \prod \omega_v$ of Z_0 is trivial on $\mathfrak{o}_E^1$.

Note that the infinite tensor product in this definition makes sense since τ_v is trivial for almost all places v. For an infinite place v, we will denote by $\lambda_v(\tau)$ the unique $\lambda \in \Lambda$ such that $\tau_v = \xi_\lambda \circ \iota_v$. For any global type τ, let $S(\tau)$ be the set of finite places v for which $\dim \tau_v > 1$. Before going any further, we record a crucial lemma, which is an immediate consequence of Theorem 3.2.3 and Proposition 4.1.1.

Lemma 4.2.2. *Let* $x \in G_0 \smallsetminus Z(\mathbb{A})$, *and assume* x *is semisimple. Then there exists a constant* C_x, *and for each infinite place* v *of* F *a polynomial* $P_{x,v}$ *on* $\mathfrak{h}_{\mathbb{R}}^*$, *such that for all global types* τ,

$$|\mathrm{Tr}\,\tau(x)| \le C_x \cdot n^{\#S(\tau)} \cdot \prod_{v|\infty} P_{x,v}(\lambda_v(\tau)).$$

Each of the polynomials $P_{x,v}$ *has degree strictly less than that of the Weyl polynomial of* $\mathrm{U}(n)$.

Proof. Since x is semisimple and not in the center, it must have at least two distinct eigenvalues. Thus there are at most finitely many places $v \in S$ at which the reduction of x_v modulo $\mathfrak{p}_{F_v}$ has a single eigenvalue of multiplicity n. For each of these places, Theorem 3.2.3 gives us a constant C_{x_v} such that

$$|\mathrm{Tr}\,\tau_v(x_v)| \le C_{x_v}$$

for every supercuspidal K-type τ_v, where $K = \mathrm{GL}_n(\mathfrak{o}_{F_v})$. Let C_x be the product of these constants C_{x_v}. At all other finite places $v \in S$, we have by the same theorem

$$|\mathrm{Tr}\,\tau_v(x_v)| \le n$$

for every supercuspidal K-type τ_v. For each infinite place, let $P_{x,v}$ be the polynomial given by Proposition 4.1.1 applied to $\iota_v(x_v) \in \mathrm{U}(n)$. Then for

any global type $\tau = \bigotimes_v \tau_v$, since τ_v is 1-dimensional outside of ∞ and $S(\tau)$, we have

$$|\mathrm{Tr}\,\tau(x)| = \prod_{v \in S(\tau)} |\mathrm{Tr}\,\tau_v(x_v)| \cdot \prod_{v|\infty} |\mathrm{Tr}\,\tau_v(x_v)|,$$

and the result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Now let $\pi = \bigotimes'\pi_v$ be an automorphic representation of $G(\mathbb{A})$ for which π_v is either supercuspidal or a twist of an unramified representation at each finite place v of F, and is unramified for each $v \notin S$. For each place v, define a representation $\tau_v(\pi)$ as follows:

1. If v is an infinite place of F, let $\tau_v(\pi) = \pi_v$. This is an irreducible representation of $G(F_v) \cong \mathrm{U}(n)$.

2. If v is a finite place not in S, let $\tau_v(\pi)$ be the trivial representation of $G(\mathfrak{o}_{F_v})$.

3. If $v \in S$, regard π_v as a representation of $\mathrm{GL}_n(F_v)$ via ι_v, and let $\tau_v(\pi) = \tau \circ \iota_v$, where τ is the unique minimal K-type in the restriction of π_v to $K = \mathrm{GL}_n(\mathfrak{o}_{F_v})$ (cf. Lemma 2.2.10).

Define $\tau(\pi) = \bigotimes_v \tau_v(\pi)$. Then $\tau(\pi)$ is a global type for G, and it is clear from Lemma 2.2.10 that $\tau(\pi)$ is the unique global type that occurs in the restriction of π to G_0, and that it occurs in π with multiplicity 1. We will call this the global type corresponding to π, or more succinctly, the type of π. For a global type τ, let $\mathcal{R}(\tau)$ denote the set of distinct isomorphism classes of automorphic representations of $G(\mathbb{A})$ of type τ.

Since a global type τ is an irreducible representation of the compact group G_0, it admits a unitary central character ω_τ. By definition ω_τ is a character of the group Z_0, trivial on the finite subgroup $\mathfrak{o}_E^1$. If π is an automorphic representation of $G(\mathbb{A})$ of type τ, then its central character ω_π must be an extension of ω_τ from Z_0 to $Z(\mathbb{A})$ that is trivial on E^1. Note that there are a finite number of such extensions, determined by the characters of the finite group $Z(F)Z_0\backslash Z(\mathbb{A})$ (which is easily seen to be isomorphic to a certain subgroup of the ideal class group of F).

If we choose for each finite place v of F an abelian character θ_v of $G(\mathfrak{o}_{F_v})$, and for each infinite place v an abelian character θ_v of $G(F_v)$, such that $\theta_v = 1$ for almost all finite places v, then we can form a character $\theta = \prod \theta_v$ of G_0. For such a character θ and a global type τ, we can form $\theta\tau$, the twist of τ by θ, in the obvious way:

$$(\theta\tau)_v = \theta_v \otimes \tau_v.$$

Note that the central character of $\theta\tau$ is $\theta^n\omega_\tau$. Thus in order for $\theta\tau$ to be a global type as well, it is clearly necessary to impose the additional requirements that $\theta_v = 1$ for all $v \notin S$, and that

$$\theta^n|_{\mathfrak{o}_E^1} = 1.$$

From now on, when we refer to twisting a global type, or likewise the twist class of a global type, it will be understood that we mean twisting by characters satisfying all the properties specified here.

A character of G_0 as described above can always be extended to a unitary character $\chi = \prod\chi_v$ of $G(\mathbb{A})$, for which χ_v will be unramified for almost all $v \in S$ and all finite $v \notin S$, and for which $\chi^n|_{E^1} = 1$. Conversely, given such a character χ of $G(\mathbb{A})$, its restriction θ to G_0 will satisfy all the requirements of the previous paragraph. If π is an automorphic representation of $G(\mathbb{A})$ of type τ, then we can twist π by the character χ to obtain an automorphic representation $\chi\pi$, and clearly it will have type $\theta\tau$. Thus for the purposes of counting automorphic representations of a given type, it will suffice to deal with global types only up to twisting. In this context, we note that both $\dim(\tau)$ and the set $S(\tau)$ are invariant under twisting.

4.3   Main theorem

We now come to the main theorem of this chapter, which gives us an idea of the number of automorphic representations of $G(\mathbb{A})$ of type τ. We will count the number of such representations by computing $m(\tau)$, the multiplicity of τ in the restriction of $\mathcal{A}(G(F)\backslash G(\mathbb{A}))$ to G_0. Note that, since the type τ of an automorphic representation π always occurs in π with multiplicity one, $m(\tau) = \#\mathcal{R}(\tau)$ for all global types τ if and only if the multiplicity one theorem holds for G. We will not assume this theorem[1] here, so we cannot conclude that $m(\tau)$ is equal to the number of distinct automorphic representations of type τ. But in any case $m(\tau)$ is the sum of the multiplicities of these automorphic representations:

$$m(\tau) = \sum_{\pi \in \mathcal{R}(\tau)} m(\pi),$$

and in general, $m(\tau) \geq \#\mathcal{R}(\tau)$. And we can certainly conclude that $\mathcal{R}(\tau) \neq \emptyset \iff m(\tau) > 0$. In other words, there exist automorphic representations of type τ if and only if $m(\tau)$ is not zero.

---

[1] The author is unaware of the current status of the multiplicity one theorem for unitary groups, an active area of very recent research. It is known at least for $n = 2$ and $n = 3$, cf. for example [25, Section 13.3].

Theorem 4.3.1. *There exist constants C_1 and C_2, and for each infinite place v of F a polynomial P_v on $\mathfrak{h}_{\mathbb{R}}^*$, all depending only on the group G, such that for all global types τ,*

$$m(\tau) \geq C_1 \dim(\tau) - C_2 \cdot n^{\#S(\tau)} \cdot \prod_{v \mid \infty} P_v(\lambda_v(\tau)).$$

Each of the polynomials P_v has degree strictly less than that of the Weyl polynomial of $\mathrm{U}(n)$.

Proof. As mentioned above, $\mathcal{A}(G(F)\backslash G(\mathbb{A}))$ is simply the space of smooth functions on $G(\mathbb{A})$ that are invariant under left translation by elements of $G(F)$. Note that this is merely the induced representation:

$$\mathcal{A}(G(F)\backslash G(\mathbb{A})) = \mathrm{Ind}_{G(F)}^{G(\mathbb{A})}(1). \tag{4.3.1}$$

To deal with the restriction of this representation to G_0, we apply Proposition 2.2.1. Let R be a set of double coset representatives for

$$G(F)\backslash G(\mathbb{A})/G_0 \ .$$

Note that R is finite, for example by [26, 8.7]. To simplify notation, we let $K_g = G(F)^g \cap G_0$ for $g \in R$. Since G_0 is a compact subgroup of $G(\mathbb{A})$ and $G(F)^g$ a discrete subgroup, K_g is finite. Applying Proposition 2.2.1 to (4.3.1) now yields

$$\mathrm{Res}_{G_0}^{G(\mathbb{A})} \mathrm{Ind}_{G(F)}^{G(\mathbb{A})}(1) = \bigoplus_{g \in R} \mathrm{Ind}_{K_g}^{G_0} \mathrm{Res}_{K_g}^{G(F)^g}(1).$$

Now if τ is any global type for G, then $m(\tau)$ is merely the multiplicity of τ in the representation above. So we have (relaxing our notation somewhat, as the restriction functors are implied)

$$
\begin{aligned}
m(\tau) &= \left\langle \tau, \bigoplus_{g \in R} \mathrm{Ind}_{K_g}^{G_0}(1) \right\rangle_{G_0} \\
&= \sum_{g \in R} \left\langle \tau, \mathrm{Ind}_{K_g}^{G_0}(1) \right\rangle_{G_0} \\
&= \sum_{g \in R} \left\langle \tau, 1 \right\rangle_{K_g} \\
&= \sum_{g \in R} \frac{1}{\#K_g} \sum_{x \in K_g} \mathrm{Tr}\,\tau(x)
\end{aligned}
$$

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for all global types τ. Note that $K_g \cap Z(\mathbb{A}) = G(F) \cap Z_0 = \mathfrak{o}_E^1 = \mu_E$, and by definition a global type τ is assumed to be trivial on this subgroup. Thus in the last sum above, the terms for which x is central all satisfy $\operatorname{Tr} \tau(x) = \dim(\tau)$. Letting $C_1 = \#\mu_E \cdot \left(\sum_{g \in R} \frac{1}{\#K_g} \right)$, we have, for all global types τ,

$$m(\tau) = C_1 \dim(\tau) + \sum_{g \in R} \frac{1}{\#K_g} \sum_{\substack{x \in K_g \\ x \notin Z(\mathbb{A})}} \operatorname{Tr} \tau(x).$$

Now let $g \in R$, and let $x \in K_g \smallsetminus Z(\mathbb{A})$. Then x is of finite order (as it belongs to the finite group K_g) and thus is semisimple. Hence we can apply Lemma 4.2.2 to x, to get a constant C_x and polynomials $P_{x,v}$ for $v \mid \infty$, such that

$$|\operatorname{Tr} \tau(x)| \le C_x \cdot n^{\#S(\tau)} \cdot \prod_{v|\infty} P_{x,v}(\lambda_v(\tau))$$

for all global types τ. As there are only finitely many such x to consider, we may sum the constants C_x and the polynomials $P_{x,v}$, and the result follows. $\qquad\square$

Corollary 4.3.2. *For all but a finite number of twist classes of global types τ of G, there exist automorphic representations of $G(\mathbb{A})$ of type τ.*

Proof. For $v \in S$, the smallest possible dimension of a supercuspidal K-type for $\operatorname{GL}_n(F_v)$ is $(q_v - 1)(q_v^2 - 1) \cdots (q_v^{n-1} - 1)$, where q_v is the cardinality of the residue field of F_v. Clearly this is greater than n for almost all v. Let P denote the Weyl polynomial for $\mathrm{U}(n)$. Then for any infinite place v of F, $\dim(\tau_v) = P(\lambda_v(\tau))$, so

$$\dim(\tau) \ge \prod_{v \in S(\tau)} \left((q_v - 1) \cdots (q_v^{n-1} - 1) \right) \cdot \prod_{v|\infty} P(\lambda_v(\tau))$$

for all global types τ. Thus, assuming the notation of the Theorem, if we enumerate the twist classes of global types τ, it is clear that

$$\frac{\dim(\tau)}{n^{\#S(\tau)} \cdot \prod_{v|\infty} P_v(\lambda_v(\tau))}$$

grows without bound. The result now follows. $\qquad\square$

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