Riemannian Embedding behind Unnormalized Optimal Transport

In this section, for readers who are interested in physical interpretations, we present the Riemannian geometry calculus behind it. In a word, the proposed metric is induced by a Riemannian embedding, from Wasserstein-2 metric in normalized density space to the one in unnormalized density space.

**Theorem 1** (Riemannian embedding). The unnormalized Wasserstein-2 metric tensor in $\mathcal{M}(\Omega)$ constrained to $\mathcal{P}(\Omega)$ is the Wasserstein-2 metric tensor.

**Proof.** We first prove that our proposed new variational problem is a geometry energy (action) function in unnormalized density space. Consider a unnormalized density path

$$
\partial_t \mu(t, x) + \nabla \cdot (\mu(t, x)v(t, x)) = f(t),
$$

with $|\Omega| = 1$ and zero flux condition $\int_{\Omega} \mu v dx = 0$. Then $f(t) = \int_{\Omega} \partial_t \mu(t, x) dx$. In addition, from the Hodge decomposition on unnormalized density $\mu(t, x)$, then

$$
v(t, x) = \nabla \Phi(t, x) + \Psi(t, x), \quad \text{where} \quad \nabla \cdot (\mu(t, x) \Psi(t, x)) = 0.
$$

Since $\mu \in \mathcal{M}_+(\Omega)$, we can identify the density path direction $\partial_t \mu$ with $\Phi$ by

$$
\Phi(t, x) = (-\Delta_{\mu})^{-1} \left( \partial_t \mu(t, x) - \int_{\Omega} \partial_t \mu(t, x) dx \right),
$$

where $\Delta_{\mu} = \nabla \cdot (\mu \nabla)$ is the elliptic operator weighted by density $\mu$. Thus

$$
\begin{align*}
&\int_0^1 \int_{\Omega} \|v(t, x)\|^2 \mu(t, x) dx dt + \frac{1}{\alpha} \int_0^1 |f(t)|^2 dt \\
= &\int_0^1 \int_{\Omega} \|\nabla \Phi(t, x)\|^2 \mu(t, x) dx dt + \frac{1}{\alpha} \int_0^1 \left( \int_{\Omega} \partial_t \mu(t, x) dx \right)^2 dt \\
\geq &\int_0^1 \int_{\Omega} \left( \Phi(t, x), (-\Delta_{\mu})^{-1} \Phi(t, x) \right) dx dt + \frac{1}{\alpha} \int_0^1 \left( \int_{\Omega} \partial_t \mu(t, x) dx \right)^2 dt \\
= &\int_0^1 \int_{\Omega} G_{UW}(\mu)(\partial_t \mu_t, \partial_t \mu_t).
\end{align*}
$$

where

$$
G_{UW}(\mu)(\partial_t \mu_t, \partial_t \mu_t) = \int_0^1 \int_{\Omega} \left( \partial_t \mu_t - \int_{\Omega} \mu_t dx \right) \left( -\Delta_{\mu} \right)^{-1} \left( \partial_t \mu_t - \int_{\Omega} \partial_t \mu dx \right) dx dt + \frac{1}{\alpha} \int_0^1 \left( \int_{\Omega} \partial_t \mu dx \right)^2 dt.
$$

Thus we have shown that

$$
U_{2}(\mu_0, \mu_1)^2 = \inf_{\mu} \left\{ \int_0^1 G_{UW}(\partial_t \mu_t, \partial_t \mu_t) dt : \mu(0, x) = \mu_0(x), \, \mu(1, x) = \mu_1(x) \right\}.
$$

The above is an action functional, with inner product $G_{UW}$.

We next prove that the inner product restricted in normalized density space is the classical $L^2$ Wasserstein inner product. If $\mu_t$ is in the normalized density space, i.e. consider $\int_{\Omega} \mu dx = 0$, then...
then

\[
G_{\text{UW}}(\mu, \tilde{\mu}) = \int_{\Omega} \left( \tilde{\mu}, (-\Delta \mu)\tilde{\mu} \right) dx
\]

\[
= \left\{ \int_{\Omega} \left( \nabla \Phi(x), \nabla \Phi(x) \right) \mu(x) dx : \tilde{\mu} = -\Delta \mu \Phi(x) \right\}.
\]

This metric tensor coincides with the one in classical optimal transport \cite{1}. This completes the proof.

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\Box
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The unnormalized Wasserstein-2 metric tensor has been proposed in \cite{1}. This is another motivation of the paper.

\section*{References}

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