On the Bilinear Hilbert Transform

Habilitationsschrift

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Introduction

A central object in the mathematical field of harmonic analysis is the Hilbert transform, a linear operator acting on functions on the real line, which is defined by

$$Hf(x) := p.v. \int_{\mathbb{R}} f(x-t)\frac{1}{t}\, dt \quad .$$

Alternatively one can write

$$H := -i\pi \mathcal{F}^{-1} M_{\text{sign}} \mathcal{F} \quad ,$$

where $\mathcal{F}$ is the Fourier transform defined by

$$\mathcal{F}f(x) := \int_{\mathbb{R}} f(x) e^{-i\xi x}\, dx \quad ,$$

and

$$M_{\text{sign}} f(x) := \text{sign(x)} f(x) \quad . \tag{1}$$

The Hilbert transform is the prototype of what is called a singular integral, a topic of interest in analysis over the past fifty years. For a general reference see [16] and its predecessors.

As written down, the Hilbert transform is certainly defined on the space of Schwartz functions and has values in the space of tempered distributions. However, it satisfies the additional regularity estimate

$$\|Hf\|_p \leq C_p \|f\|_p$$

for all $1 < p < \infty$, where C_p is a universal constant and

$$\|f\|_p := \left(\int_{\mathbb{R}} |f(x)|^p \, dx \right)^{\frac{1}{p}} \quad . \tag{2}$$

This result is due to Kolmogorov [9] and Riesz [15]. Hence the Hilbert transform extends to a bounded operator from L^p to itself, where L^p is essentially the space of measurable functions satisfying (2). This result is most easily seen for $p = 2$ from the formula (1), since $\mathcal{F}$ is an isometry of the Hilbert space L^2, and multiplication by the function sign is also an isometry of L^2.

Define the following isometries of $L^p(\mathbb{R})$ for $y, \lambda \in \mathbb{R}$ with $\lambda > 0$:

$$R_y f(x) := f(x-y) \quad ,$$

$$D^p_\lambda f(x) := \lambda^{-p} f(\lambda^{-1} x) \quad .$$

Then the Hilbert transform satisfies

$$R_y H = H R_y \quad , \tag{3}$$

$$D^p_\lambda H = H D^p_\lambda \quad . \tag{4}$$

Moreover, any bounded operator on L^p satisfying properties (3) and (4) is already a linear combination of the Hilbert transform and the identity operator. Again this is most easily seen for $p = 2$, because then any operator satisfying (3) is of the form

$$H = \mathcal{F}^{-1} M_a \mathcal{F}$$

for some bounded function a, and (4) shows that a has to be homogeneous of degree 0.

The *bilinear Hilbert transform* H_α with parameter $\alpha \in \mathbb{R}$ is a bilinear operator mapping two functions on the real line to one function on the real line. It is defined by

$$H_\alpha(f,g)(x) := p.v. \int_{\mathbb{R}} f(x-t) g(x-\alpha t) \frac{1}{t} \, dt \quad . \tag{5}$$

Again, this expression is certainly well defined if f and g are Schwartz functions.

In a series of joint papers with M. Lacey, [10], [11], [12], , and [13], we have proved that the bilinear Hilbert transform satisfies the following regularity estimates:

Theorem 0.0.1 *If $\alpha \notin \{0,1\}$, then*

$$\|H_\alpha(f_1, f_2)\|_p \le C_{p_1,p_2} \|f_1\|_{p_1} \|f_2\|_{p_2} \quad , \tag{6}$$

provided that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad , \tag{7}$$

$$\frac{2}{3} < p < \infty \quad ,$$

$$1 < p_1, p_2 \le \infty \quad .$$

In contrast to the situation for the Hilbert transform H, none of the estimates in this theorem is particularly easy to prove. For $\alpha \in \{0,1\}$ we observe

$$H_0(f,g) = H(f) \cdot g \quad ,$$

$$H_1(f,g) = H(f \cdot g) \quad .$$

In these cases one easily obtains regularity results by an application of the theorem of Kolmogorov and Riesz and by Hölder's inequality. The proof of Theorem 0.0.1 is very much in the spirit of the papers [5] and [7] on almost everywhere convergence of Fourier series.

The bilinear Hilbert transform satisfies the following invariances:

$$H_\alpha(R_y f, R_y g) = R_y H_\alpha(f,g) \quad , \tag{8}$$

$$H_\alpha(D_\lambda^{p_1} f, D_\lambda^{p_2} g) = D_\lambda^p H_\alpha(f,g) \quad , \tag{9}$$

provided p_1, p_2, and p satisfy the homogeneity condition (7). These invariances are also satisfied for the pointwise product of two functions.

A nice exposition of the theory of multilinear translation and dilation invariant operators can be found in [14]. A bilinear operator B satisfying (8) and any regularity estimate as in the theorem necessarily is of the form

$$B(f,g) := \int_{\mathbb{R}} \int_{\mathbb{R}} m(\xi,\eta) \mathcal{F} f(\xi) \mathcal{F} g(\eta) e^{i(\eta+\xi)x} \, d\xi d\eta$$

for some bounded function m. If b satisfies in addition (9), then m is homogeneous of degree 0. Hence m is determined by its values on the unit circle in $\mathbb{R}^2$. In contrast to the situation for linear operators, there is an infinite dimensional space of translation and dilation invariant bilinear operators.

The previously known conditions on the multiplier m for a translation and dilation invariant operator to be bounded on any L^p spaces implied continuity outside the lines $\xi = 0$, $\eta = 0$, and $\xi + \eta = 0$.

Define the modulation of f by ζ by

$$U_\zeta f(x) := f(x) e^{i\zeta x} = \mathcal{F}^{-1} R_\zeta \mathcal{F} f(x) \quad .$$

Then the bilinear Hilbert transform H_α satisfies

$$H_\alpha(U_{\alpha\zeta} f, U_{-\zeta} g) = U_{(\alpha-1)\zeta} H_\alpha(f,g) \quad . \tag{10}$$

If one has any bilinear operator satisfying (8), (9), (10), and any of the estimates in Theorem 0.0.1, then its multiplier m is constant on the half planes defined by $\xi + \alpha\eta > 0$ and $\xi + \alpha\eta < 0$. Hence there is a two dimensional space of such operators, which is spanned by H_α and the pointwise product.

Historically, the interest in the bilinear Hilbert transform arose from the study of the Cauchy integral or Hilbert transform on Lipschitz curves and the related first commutator of Calderon. Let

$$\gamma(x) = x + iA(x)$$

be a curve in $\mathbb{C}$, where $A' = a \in L^\infty(\mathbb{R})$. Then the Hilbert transform corresponding to the curve γ is given by

$$H_\gamma f(x) := p.v. \int \frac{f(y)(1+ia(y))\, dy}{x - y + i(A(x) - A(y))} \quad .$$

It is a celebrated result by Calderon [3] (for small norm of a) and Coifman, McIntosh, Meyer [6] (for arbitrary norm of a), that this operator is bounded from $L^2(\mathbb{R})$ to itself.

One approach to this result is to expand the nonlinear dependence on A into a Taylor series, i.e., to write the series expansion

$$\frac{1}{x - y + i(A(x) - A(y))} = \frac{1}{x-y} \sum_{k=0}^{\infty} (-i)^k \left(\frac{A(x) - A(y)}{x-y} \right)^k \quad .$$

Ignoring the factor $1 + ia(y)$ in the definition of H_γ, which is irrelevant for the question of boundedness in L^2, we are led to study the operators

$$C_k f(x) = p.v. \int \frac{(A(x) - A(y))^k}{(x-y)^{k+1}} f(y)\, dy \quad .$$

Naturally, C_0 is the Hilbert transform. For C_1, Calderon's first commutator, we write

$$p.v. \int \int_0^1 a(x + \alpha(y-x)) \frac{1}{x-y} f(y)\, d\alpha dy$$

$$= p.v. \int \int_0^1 a(x - \alpha y) f(x-y) \frac{1}{y}\, d\alpha dy$$

$$= \int_0^1 H_\alpha(f, a)(x)\, d\alpha \quad .$$

Coming from the original question concerning the Hilbert transform on γ, one seeks for fixed $a \in L^\infty$ a bound on C_1 as an operator from L^2 to itself. One of the attempts of Calderon was to prove appropriate bounds on the bilinear Hilbert transform, but he did not succeed this way and eventually proved boundedness of C_1 by different means [2].

Observe that even the result of Theorem 0.0.1 does not give boundedness of C_1, since it does not give any control over the bound for H_α as α approaches 0 or 1. This gap is filled by the following theorem, which justifies the above approach to C_1 by Calderon:

Theorem 0.0.2 *For each $N > 0$ there is a universal constant C such that for all $\alpha < N$ and all Schwartz functions f and g we have*

$$\|H_\alpha(f, g)\|_\alpha \leq C \, \|f\|_2 \, \|g\|_\infty \quad .$$

The study of the bilinear Hilbert transform can be simplified by taking into account a symmetry which is not immediately apparent from the expression (5). To exploit this symmetry, consider the trilinear form

$$\Lambda_{(\alpha)}(f_1, f_2, f_3) := \int_{\mathbb{R}} H_\alpha(f_1, f_2)(x) f_3(x)\, dx \quad .$$

This form can be generalized to the following family of forms:

$$\Lambda_\beta(f_1, f_2, f_3) := \int p.v. \int \left(\prod_{j=1}^{3} f_j(x - \beta_j t) \right) \frac{1}{t}\, dt\, dx \quad , \tag{11}$$

where $\beta = (\beta_1, \beta_2, \beta_3)$ is a vector in $\mathbb{R}^3$ which is not collinear with $\gamma := (1, 1, 1)$. By a change of variables it is no restriction to assume that β is a unit vector perpendicular to the vector γ. Hence the family Λ_β is a one parameter family which consists of all forms which are dual in the sense of (11) to $\pm H_\alpha$ with $\alpha \in \mathbb{R}$ and the forms which are dual to

$$\pm H_\infty(f_1, f_2) := f_1 \cdot H(f_2) \quad .$$

The group of permutations of $\{1,2,3\}$ is acting in an obvious way on the family Λ_β. We call β non-degenerate if $\beta_\imath \neq \beta_\jmath$ for $\imath \neq \jmath$. If β is non-degenerate, then Λ_β is dual to $\pm H_\alpha$ for some $\alpha \notin \{0,1,\infty\}$.

The estimates of Theorem 0.0.1 with $p \geq 1$ can be stated as estimates for the form Λ_β, using the duality of L^p and $L^{p'}$ where p' denotes the conjugate exponent of p defined by $\frac{1}{p'} + \frac{1}{p} = 1$. The remaining estimates can not be stated as estimates of Λ_β. Hence we define formally the following notion of type: For each permutation σ of $\{1,2,3\}$ define the bilinear operator

$$H_\beta^\sigma : S(\mathbb{R}) \times S(\mathbb{R}) \to S'(\mathbb{R})$$

by:

$$H_\beta^\sigma(f_{\sigma(1)}, f_{\sigma(2)})(f_{\sigma(3)}) := \Lambda_\beta(f_1, f_2, f_3) \quad .$$

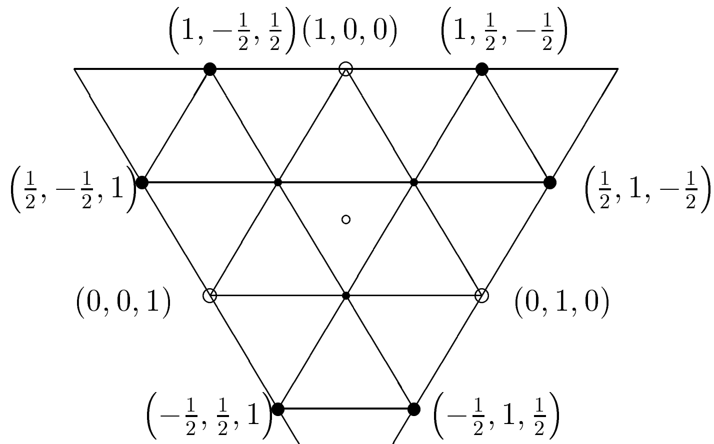
We say that Λ_β is of type (p_1, p_2, p_3) if there is a permutation σ such that

$$0 < p_{\sigma(1)}, p_{\sigma(2)}, {p_{\sigma(3)}}' \leq \infty \tag{12}$$

and the operator H_β^σ is bounded from $L^{p_{\sigma(1)}} \times L^{p_{\sigma(2)}}$ to $L^{{p_{\sigma(3)}}'}$. Then the statement of Theorem 0.0.1 is that for non-degenerate β the form Λ_β is of type (p_1, p_2, p_3) if

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1 \quad , \quad -\frac{1}{2} < \frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3} < 1 \quad . \tag{13}$$

The following diagram shows the plane of all points $(1/p_1, 1/p_2, 1/p_3)$ with $1/p_1 + 1/p_2 + 1/p_3 = 1$:



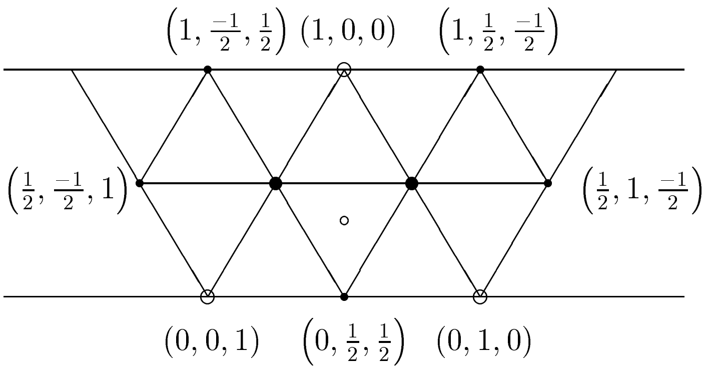
The region described by (13), which we call region A, is the open interior of the convex hull of the six large filled circles in the above diagram. The closed convex hull B of the three large empty circles is the region in which condition (12) is satisfied for all permutations σ. Moreover, each point in the intersection of A and B corresponds to an actual estimate for Λ_β itself. At each point of the remainder set $A \setminus B$ only two permutations σ satisfy (12). A further region of interest is the open convex hull of the three small filled circles. Here $2 < p_j < \infty$ for all j, hence the trilinear form acts only on functions which are locally L^2. This region has been discussed in [10].

It is not known, whether the lower bound $\frac{2}{3}$ on p in Theorem 0.0.1 can be relaxed to $\frac{1}{2}$. If this were true, we had estimates in the whole set shown by the above diagram.

In the degenerate case $\alpha = 0$, which corresponds to the trilinear form

$$\int_{\mathbb{R}} H(f_1)(x)\, f_2(x)\, f_3(x)\, dx \quad ,$$

one has estimates in the open strip bounded by the two extremal horizontal lines in the following diagram:



.

Hence on may expect uniform estimates as α tends to 0 in the intersection of this strip with the region A. In this Habilitationsschrift uniform estimates are proved in the closed convex hull of the two large filled circles. It seems plausible that one can improve this result with the ideas of [10] and [12] to obtain uniform estimates in the open interior of the convex hull of the five small filled circles in the last diagram. It would be interesting to investigate the behaviour near the points $(0, 0, 1)$ and $(0, 1, 0)$.

Finally, I would like to mention the very interesting open problem concerning the trilinear Hilbert transform:

$$H_{\alpha,\beta}(f, g, h)(x) := p.v. \int_{\mathbb{R}} f(x - y) g(x - \alpha t) h(x - \beta t) \frac{1}{t}\, dt \quad .$$

It is unknown, whether this trilinear operator satisfies any L^p estimates except in the degenerate cases $\alpha = \beta$ or $\{\alpha, \beta\} \cap \{0, 1, \infty\} \neq \emptyset$.

This Habilitationsschrift contains two chapters. These two chapters are self contained and therefore can be read independently. The first chapter proves Theorem 0.0.1, and thus summarizes the results from [10], [11], [12], [13], which are joint work with M. Lacey. The exposition contains some simplifications compared to the original papers. One of the simplifications, namely the idea to prove restricted weak type estimates for operators related to the counting function, is taken from [1]. The second chapter proves the new Theorem 0.0.2.

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Contents

Chapter 1

L^p- estimates for the bilinear Hilbert transform

1.1 The bilinear Hilbert transform

Let $\beta = (\beta_1, \beta_2, \beta_3)$ be a unit vector in $\mathbb{R}^3$ perpendicular to $\gamma := (1,1,1)$. The *Hilbert form* with parameter β is the trilinear form on $S(\mathbb{R}) \times S(\mathbb{R}) \times S(\mathbb{R})$ defined by:

$$\Lambda_\beta(f_1, f_2, f_3) := \int p.v. \int \left(\prod_{j=1}^{3} f_j(x - \beta_j t) \right) \frac{1}{t} \, dt \, dx \quad .$$

This form is continuous in each argument in the topology of the Schwartz space. For each permutation σ of the three numbers $1, 2, 3$ we define the bilinear map

$$H^\sigma_\beta : S(\mathbb{R}) \times S(\mathbb{R}) \to S'(\mathbb{R})$$

by:

$$H^\sigma_\beta(f_{\sigma(1)}, f_{\sigma(2)})(f_{\sigma(3)}) := \Lambda_\beta(f_1, f_2, f_3) \quad .$$

The bilinear operators H^σ_β are called *bilinear Hilbert transforms.* We are interested in L^p- estimates of these operators. Let $p_1, p_2, p_3 \in \mathbb{R} \cup \{\infty\}$. We say that Λ_β is of type (p_1, p_2, p_3) if there is a permutation σ such that

$$0 < p_{\sigma(1)}, p_{\sigma(2)}, {p_{\sigma(3)}}' \leq \infty \tag{1.1}$$

and there is a constant C such that

$$\left\| H^\sigma_\beta(f, g) \right\|_{p'_{\sigma(3)}} \leq C \left\| f \right\|_{p_{\sigma(1)}} \left\| g \right\|_{p_{\sigma(2)}} \tag{1.2}$$

for all Schwartz functions f and g. Here

$$\left\| f \right\|_p := \left(\int |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

and p' denotes the conjugate exponent of p defined by

$$\frac{1}{p'} + \frac{1}{p} = 1 \quad .$$

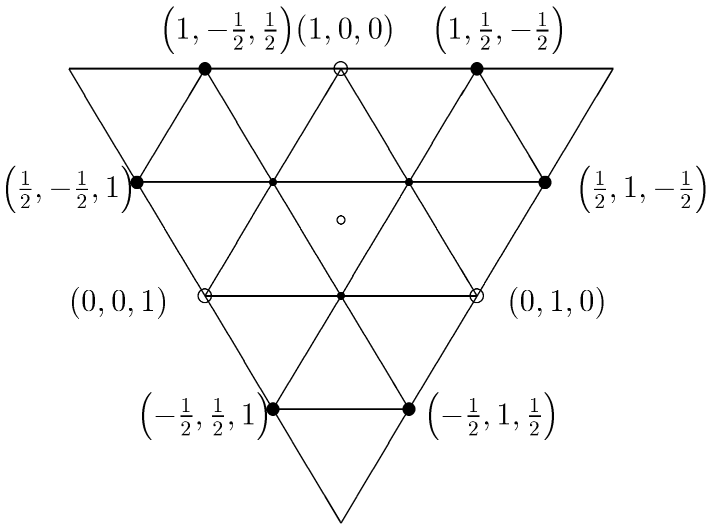
Call β non-degenerate if $\beta_\imath \neq \beta_\jmath$ for $\imath \neq \jmath$.

Theorem 1.1.1 *If β is non-degenerate and*

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1, \quad -\frac{1}{2} < \frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3} < 1 \ ,$$

then Λ_β is of type (p_1, p_2, p_3).

The following diagram shows the plane of all points $(1/p_1, 1/p_2, 1/p_3)$ with $1/p_1 + 1/p_2 + 1/p_3 = 1$.



Let A be the open interior of the convex hull of the six large filled circles in the above diagram. Theorem 1.1.1 states that Λ_β is of type (p_1, p_2, p_3) for all $(1/p_1, 1/p_2, 1/p_3) \in A$. The closed convex hull B of the three large empty circles is the region in which condition (1.1) is satisfied for all permutations σ and thus the type estimates (1.2) are equivalent for all six bilinear operators H^σ_β. The remainder set $A \setminus B$ splits into three connected regions $D_\jmath$ such that the exponent $p_\jmath$ is negative in the region $D_\jmath$ for $\jmath = 1, 2, 3$. In each of these regions, only two permutations σ satisfy (1.1) and thus the estimate (1.2) makes sense only for the two corresponding bilinear operators H^σ_β. A further region of interest is the open convex hull of the three small filled circles. Here $2 < p_\jmath < \infty$ for all $\jmath$, hence the trilinear form acts only on functions which are locally L^2. This region has been discussed in [10].

In this paper, C and m denote constants which may depend on β and p_1, p_2, p_3 whenever the latter are defined. The values of C and m may change from line to line.

1.2 Interpolation

The first step in the proof of Theorem 1.1.1 is the following convexity result:

Lemma 1.2.1 *Let* $(1/p_1, 1/p_2, 1/p_3)$ *and* $(1/q_2, 1/q_2, 1/q_3)$ *be two points in the region* A *and assume that* Λ_β *is of type* (p_1, p_2, p_3) *and of type* (q_1, q_2, q_3). *Then* Λ_β *is of type* (u_1, u_2, u_3) *for all* $(1/u_1, 1/u_2, 1/u_3)$ *on the line segment connecting* $(1/p_1, 1/p_2, 1/p_3)$ *and* $(1/q_2, 1/q_2, 1/q_3)$.

We prove the lemma. The conclusion of the lemma follows immediately by complex interpolation as in [4], if there exists a σ such that type (p_1, p_2, p_3) and type (q_1, q_2, q_3) can be expressed as estimates for H^σ_β. This is the case if there is a $\jmath \in \{1, 2, 3\}$ such that both $(1/p_1, 1/p_2, 1/p_3)$ and $(1/q_1, 1/q_2, 1/q_3)$ are contained in the region $B \cup D_\jmath$.

Therefore we can assume that $(1/p_1, 1/p_2, 1/p_3)$ and $(1/q_1, 1/q_2, 1/q_3)$ are in different regions $D_\jmath$. Let σ and τ be permutations such that

$$\left\| H^\sigma_\beta(f, g) \right\|_{p'_{\sigma(3)}} \leq C_p \|f\|_{p_{\sigma(1)}} \|g\|_{p_{\sigma(2)}} \quad , \tag{1.3}$$

$$\left\| H^\tau_\beta(f, g) \right\|_{q'_{\tau(3)}} \leq C_q \|f\|_{q_{\tau(1)}} \|g\|_{q_{\tau(2)}} \quad , \tag{1.4}$$

for all Schwartz functions f and g. We can assume that $\sigma(1) = \tau(1)$. Let C_p and C_q be the optimal constants in the above estimates.

Pick two different points $(1/u_1, 1/u_2, 1/u_3)$ and $(1/v_1, 1/v_2, 1/v_3)$ on the line segment connecting $(1/p_1, 1/p_2, 1/p_3)$ and $(1/q_1, 1/q_2, 1/q_3)$ such that $(1/u_1, 1/u_2, 1/u_3)$ and $(1/v_1, 1/v_2, 1/v_3)$ are both in the open interior of the region B and the distance between $(1/p_1, 1/p_2, 1/p_3)$ and $(1/u_1, 1/u_2, 1/u_3)$ is smaller than the distance between the points $(1/p_1, 1/p_2, 1/p_3)$ and $(1/v_1, 1/v_2, 1/v_3)$.

Let f be a fixed compactly supported Schwartz function. It is easy to see that there are constants C_u and C_v, possibly depending on f, such that

$$\left\| H^\sigma_\beta(f, g) \right\|_{u'_{\sigma(3)}} \leq C_u \|f\|_{u_{\sigma(1)}} \|g\|_{u_{\sigma(2)}} \quad , \tag{1.5}$$

$$\left\| H^\sigma_\beta(f, g) \right\|_{v'_{\sigma(3)}} \leq C_v \|f\|_{v_{\sigma(1)}} \|g\|_{v_{\sigma(2)}} \quad , \tag{1.6}$$

for all Schwartz functions g. Let C_u and C_v be the best constants in these inequalities. Assume to get a contradiction that C_v is larger than C_p and C_q. Then it follows by interpolation as in [4] between the estimates (1.3) and (1.6) that C_u is smaller than C_v. However, we have by duality:

$$\left\| H^\tau_\beta(f, g) \right\|_{u'_{\tau(3)}} \leq C_u \|f\|_{u_{\tau(1)}} \|g\|_{u_{\tau(2)}} \quad , \tag{1.7}$$

$$\left\| H^\tau_\beta(f, g) \right\|_{v'_{\tau(3)}} \leq C_v \|f\|_{v_{\tau(1)}} \|g\|_{v_{\tau(2)}} \quad , \tag{1.8}$$

for all Schwartz functions g, where the same constants C_u and C_v are optimal. Hence it follows by interpolation between the estimates (1.8) and (1.4) that C_v is smaller than C_u or C_q, a contradiction.

Hence C_v is smaller than C_p or C_q, which are independent of f. Hence Λ_β is of type (v_1, v_2, v_3), and now the Lemma follows by interpolation between (1.3) and (1.6), and by interpolation between (1.4) and (1.8).

This completes the proof of the lemma.

1.3 Applying Plancherel to Λ_β

Define the Fourier transform $\widehat{f}$ of a function f on $\mathbb{R}^d$ by

$$\widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-i\xi\cdot x}\, dx \quad .$$

Plancherel's theorem gives for $A \in GL(2,\mathbb{R})$, $F \in S(\mathbb{R}^2)$, $G \in S'(\mathbb{R}^2)$:

$$\int_{\mathbb{R}^2} F(Ax)G(x)\, dx = C\int_{\mathbb{R}^2} \widehat{F}(\xi)\widehat{G}(-A^T\xi)\, d\xi \quad .$$

Applying this with $F(x_1, x_2) = f_1(x_1)f_2(x_2)$ and $G(x_1, x_2) = f_3(x_1)(p.v.\frac{1}{x_2})$ gives:

$$\Lambda_\beta(f_1, f_2, f_3)$$

$$= \int\int f_1(x + (\beta_1 - \beta_3)t)f_2(x + (\beta_2 - \beta_3)t)f_3(x)\left(p.v.\frac{1}{t}\right)\, dx\, dt$$

$$= C\int\int \widehat{f_1}(\eta_1)\widehat{f_2}(\eta_2)\widehat{f_3}(-\eta_1 - \eta_2)\mathrm{sign}((\beta_3 - \beta_1)\eta_1 + (\beta_3 - \beta_2)\eta_2)\, d\eta_1\, d\eta_2$$

$$= C\int_{\eta\cdot\gamma=0} \widehat{f_1}(\eta_1)\widehat{f_2}(\eta_2)\widehat{f_3}(\eta_3)\mathrm{sign}(\eta\cdot\beta)\, d\sigma \quad .$$

Here $d\sigma$ denotes the Lebesgue measure on the hyperplane $\eta\cdot\gamma = 0$. Define the bilinear operator $\tilde{H}_\beta(f_1, f_2)$ by

$$\int \tilde{H}_\beta(f_1, f_2)(x)f_3(x)\, dx$$

$$= C\int_{\eta\cdot\gamma=0} f_1(\eta_1)f_2(\eta_2)f_3(\eta_3)1_{\mathbb{R}^+}(\eta\cdot\beta)\, d\sigma \quad .$$

1.4 The decomposition

Fix a non-degenerate β and let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a unit vector perpendicular to β and γ. Let $\epsilon := 2^{-100}\min_i\{|\alpha_i|\}$, $N := \epsilon^{-100}$. Let ϕ be a nonzero real symmetric Schwartz function such that $\widehat{\phi}$ maps into $[0,1]$ and is supported in $[-\epsilon^5, \epsilon^5]$. Define

$$D^p_\lambda f(x) = \lambda^{-p}f(\lambda^{-1}x) \quad ,$$

$$L_y f(x) := f(x - y) \quad ,$$

$$U_\eta f(x) := f(x)e^{i\eta x} \quad .$$

Then we have for all η in the hyperplane $\eta \cdot \gamma = 0$:

$$1_{\mathbb{R}^+}(\eta \cdot \beta) = C \int_{\mathbb{R}} \int_{\mathbb{R}} \prod_{\imath=1}^{3} D^{\infty}_{N^{-z}} L_{\alpha_\imath \xi + \epsilon^2 \beta_\imath} \widehat{\phi}(\eta_\imath)\, dz\, d\xi$$

with a nonvanishing constant C. Hence

$$\int\limits_{\eta\cdot\gamma=0} \widehat{f_1}(\eta_1)\widehat{f_2}(\eta_2)\widehat{f_3}(\eta_3) 1_{\mathbb{R}^+}(\eta\cdot\beta)\, d\sigma \quad .$$

$$= C \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \int\limits_{\eta\cdot\gamma=0} \prod_{\imath=1}^{3} \left(f_\imath * D^1_{N^z} U_{\alpha_\imath \xi + \epsilon^2 \beta_\imath} \phi\right)^{\widehat{}}(\eta_\imath)\, d\sigma\, d\xi\, dz$$

$$= C \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} \prod_{\imath=1}^{3} f_\imath * D^1_{N^z} U_{\alpha_\imath \xi + \epsilon^2 \beta_\imath} \phi(x)\, dx\, d\xi\, dz \quad .$$

Here we have used that the integral of a function on $\mathbb{R}^3$ over a hyperplane is up to a constant factor equal to the integral of the inverse Fourier transform over the dual line. For further reference we note that

$$\begin{aligned} & f_\imath * D^1_{N^z} U_{\alpha_\imath \xi + \epsilon^2 \beta_\imath} \phi(x) \\ &= \int_{\mathbb{R}} f_\imath(t) N^{-z} \phi(N^{-z}(x-t)) e^{iN^{-z}(\alpha_\imath \xi + \epsilon^2 \beta_\imath)(x-t)}\, dt \\ &= \left\langle f_\imath, L_x D^1_{N^z} U_{\alpha_\imath \xi + \epsilon^2 \beta_\imath} \phi \right\rangle \\ &= e^{iN^{-z}(\alpha_\imath \xi + \epsilon^2 \beta_\imath)x} L_{-\epsilon^2 \alpha_\imath^{-1} \beta_\imath} D^{\infty}_{N^z \alpha_\imath^{-1}} \left(\left(f_\imath L_x D^1_{N^z} \phi \right)^{\widehat{}} \right)(\xi) \quad . \end{aligned} \tag{1.9}$$

Here $\langle f, g \rangle$ denotes the hermitian inner product

$$\int_{\mathbb{R}} f(x)\overline{g(x)}\, dx \quad .$$

1.5 The truncated form

The nature of Theorem 1.1.1 requires to prove bounds on bilinear operators, which break the symmetry of the Hilbert forms. This breaking of the symmetry can be overcome with the aid of the following lemma, which has the same symmetry as the Hilbert form, and which allows to deduce estimates for the bilinear Hilbert transforms.

Lemma 1.5.1 *Let $1 < p_1, p_2, p_3 < 2$ with $1/p_1 + 1/p_2 + 1/p_3 < 2$. Then there is a constant $B > 1$ such that the following holds: If f_1, f_2, f_3 are Schwartz functions with $\|f_\imath\|_{p_\imath} \le B^{-1}$, and*

$$E := \bigcup_{\imath=1}^{3} \left\{ x : M_{p_\imath} f_\imath(x) > B^{-1} \right\} \quad ,$$

then:

$$\int\limits_{(x,\xi,z)\in\mathbb{R}^3 : \mathrm{dist}(x,E^c) \le N^z} \prod_{\imath=1}^{3} \left| \left\langle f_\imath, L_x D^1_{N^z} U_{\alpha_\imath \xi + \epsilon^2 \beta_\imath} \phi \right\rangle \right| dx\, d\xi\, dz < B \quad .$$

Here the maximal function $M_p f$ is defined by

$$M_p f(x) := \left(\sup_{x \in I} \frac{1}{|I|} \int_I |f(x)|^p \, dx \right)^{\frac{1}{p}} \quad .$$

Observe that the statement of the lemma becomes weaker as B becomes larger.

Lemma 1.5.1 is proved later. Now we prove that this lemma implies Theorem 1.1.1. By Lemma 1.2.1 and by symmetry it suffices to prove

$$\left\| H_\beta^{\mathrm{id}}(f_1, f_2) \right\|_p \leq C \left\| f_1 \right\|_{p_1} \left\| f_2 \right\|_{p_2} \quad .$$

for $1 < p_1, p_2 < 2$, $2/3 < p < 1$, with $1/p_1 + 1/p_2 + 1/p_3 = 1$. In our diagram this set of estimates corresponds to the open triangle spanned by two adjacent large filled circles and the adjacent small filled circle. Clearly the convex hull of this triangle and the two other triangles, which are obtained by symmetry, covers the whole region A. By an easy decomposition of the function sign and by symmetry it suffices to prove

$$\left\| \tilde{H}_\beta(f_1, f_2) \right\|_p \leq C \left\| f_1 \right\|_{p_1} \left\| f_2 \right\|_{p_2} \quad .$$

for $1 < p_1, p_2 < 2$, $2/3 < p < 1$, with $1/p_1 + 1/p_2 + 1/p_3 = 1$. For simplicity we write H for $\tilde{H}_\beta$.

Fix such p, p_1 and p_2 and observe that we can find p_3 such that p_1, p_2, p_3 satisfy the conditions in Lemma 1.5.1. Let B be as in the lemma. By homogeneity, linearity and interpolation it suffices to prove for all Schwartz functions f_1 and f_2 with $\|f_1\|_{p_1} = \|f_2\|_{p_2} = B^{-1}$ the weak type estimate

$$|\{x : \Re(H(f_1, f_2)(x)) > C\}| \leq C$$

for some constant C depending only on p_1, p_2, and β. Fix such f_1 and f_2 and define

$$E := \bigcup_{\imath=1}^{2} \left\{ x : M_{p_\imath} f_\imath(x) > B^{-1} \right\} \quad .$$

Define H_E by

$$H_E(f_1, f_2) :=$$

$$\int_{(x,\xi,z):\mathrm{dist}(x,E^c) \leq N^z} \left(L_x D^1_{N^z} U_{\alpha_3 \xi + \epsilon^2 \beta_3} \phi \right) \prod_{\imath=1}^{2} \left\langle f_\imath, L_x D^1_{N^z} U_{\alpha_\imath \xi + \epsilon^2 \beta_\imath} \phi \right\rangle \, dx \, d\xi \, dz = \quad .$$

It suffices to prove the weak type estimate for H and $H - H_E$ separately.

Let f_3 be a Schwartz function which satisfies

$$\|f_3\|_\infty \leq 1 \quad ,$$

$$\|f_3 - 1_{E_3}\|_{p_3} \leq \min\{\|H_E(f_1, f_2)\|_{p_3'}^{-1}, 1\} \quad ,$$

where

$$E_3 := \{\Re(H_E(f_1, f_2)) \geq 1\} \quad .$$

Here we have used that $H_E(f_1,f_2)$ is an $L^{p_3{}'}$ function. We have

$$|E_3| \leq \left|\int_{E_3} H_E(f_1,f_2)(x)\,dx\right| \leq |\langle f_3, H_E(f_1,f_2)\rangle| + 1$$

$$\leq B\|f_3\|_{p_3}\left|\left\langle \frac{f_3(x)}{B\|f_3\|_{p_3}}, H_E(f_1,f_2)\right\rangle\right| + 1 \quad .$$

We can assume $\|f_3\|_{p_3} \geq 1$, otherwise we have an immediate bound on E_3. Hence $M_{p_3}(f_3/B\|f_3\|_{p_3})$ is bounded by B^{-1}, and we can apply the lemma to estimate the previously displayed expression by

$$C\|f_3\|_{p_3} \leq C|E_3|^{\frac{1}{p_3}} \quad .$$

Hence $|E_3| \leq C$.

To complete the proof of the main theorem, it now suffices to show

$$\|(H - H_E)(f_1,f_2)\|_{L^1(E^c)} \leq C \quad .$$

This follows from the a priori estimate

$$\left|\int_{(x,\xi,z)\in\mathbb{R}^3:\mathrm{dist}(x,E^c)>N^z} \prod_{\imath=1}^{3}\left\langle f_\imath, L_x D^1_{N^z} U_{\alpha_\imath\xi+\epsilon^2\beta_\imath}\phi\right\rangle\,dx\,d\xi\,dz\right| \leq C \tag{1.10}$$

for all Schwartz functions f_3 supported in E^c with $\|f_3\|_\infty \leq 1$. Define $q_1 := p_1$, $q_2 := p_2$, and pick q_3 such that the sum of reciprocals of $q_\imath$ is equal to 2. Observe that $1 < q_3 < 2$. We rewrite the integrand in (1.10)with the aid of the last expression in (1.9) and prepare to apply Hölder's inequality for the integration in ξ with exponents $q_\imath{}'$. Observe that

$$\begin{aligned}
&\left\|D^\infty_{\alpha_\imath^{-1}N^z}\left(\left(f_\imath L_x D^1_{N^z}\phi\right)\hat{}\right)\right\|_{q_\imath{}'}\\
&= \left\|D^{q_\imath{}'}_{\alpha_\imath^{-1}N^z}\left((f_\imath L_x D^{q_\imath}_{N^z}\phi)\hat{}\right)\right\|_{q_\imath{}'}\\
&\leq C\left\|f_\imath L_x D^{q_\imath}_{N^z}\phi\right\|_{q_\imath}
\end{aligned} \tag{1.11}$$

Here we have used the Hausdorff Young inequality. For $\imath = 1,2$ we estimate this by

$$C\left(\sup_{x\in I:|I|\geq N^z}\frac{1}{I}\int_I |f_\imath(x)|^{p_\imath}\,dx\right)^{\frac{1}{p_\imath}}$$

$$\leq C\left(1 + N^{-z}\mathrm{dist}(x,E_c)\right)\inf_{y\in E^c} M_{\mathbf{p}_\imath} f_\imath(y)$$

$$\leq C\left(1 + N^{-z}\mathrm{dist}(x,E_c)\right) \quad .$$

For $\imath = 3$ we can estimate (1.11) by:

$$C\left(1 + N^{-z}\mathrm{dist}(x,E_c)\right)^{-3} \quad ,$$

because f_3 is supported in E^c. Hence we can estimate the left hand side of (1.10) by:

$$\int\limits_{(x,z)\in\mathbb{R}^2:\mathrm{dist}(x,E^c)>N^z} C\left(1+N^{-z}\mathrm{dist}(x,E^c)\right)^{-1}\,dz\,dx$$

$$=\int_E\int_{-\infty}^{\log_N \mathrm{dist}(x,E^c)} C\left(1+N^{-z}\mathrm{dist}(x,E^c)\right)^{-1}\,dz\,dx$$

$$\leq\int_E\int_{-\infty}^{0} C\left(1+N^{-z}\right)^{-1}\,dz\,dx\leq C|E|\leq C \quad .$$

This completes the reduction of Theorem 1.1.1 to Lemma 1.5.1

1.6 Discretization of the trilinear forms

Fix p_1, p_2, p_3,f_1, f_2, f_3, and E as in Lemma 1.5.1. We discretize the truncated trilinear form that appears in this lemma as follows:

$$\int\limits_{(x,\xi,z)\in\mathbb{R}^3:\mathrm{dist}(x,E^c)\leq N^z} \prod_{\imath=1}^{3}\left|\left\langle f_\imath, L_x D^1_{N^z} U_{\alpha_\imath\xi+\epsilon^2\beta_\imath}\phi\right\rangle\right|\,dx\,d\xi\,dz$$

$$=\int\limits_{(x,\xi,z)\in\mathbb{R}^3:\mathrm{dist}(N^z x,E^c)\leq N^z} N^{-\frac{z}{2}}\prod_{\imath=1}^{3}\left|\left\langle f_\imath, D^2_{N^z} L_x U_{\alpha_\imath\xi+\epsilon^2\beta_\imath}\phi\right\rangle\right|\,dx\,d\xi\,dz$$

$$=\int\limits_{(x,\xi,z)\in[0,1]^3}\Lambda_{x,\xi,z}(f_1,f_2,f_3)\,dx\,d\xi\,dz \quad ,$$

where $\Lambda_{x,\xi,z}(f_1,f_2,f_3)$ is defined as

$$\sum_{(k,n,l)\in\mathbf{P}(x,z,E)} N^{-\frac{k+z}{2}}\prod_{\imath=1}^{3}\left|\left\langle f_\imath, D^2_{N^{k+z}} L_{n+x} U_{\alpha_\imath(l+\xi)+\epsilon^2\beta_\imath}\phi\right\rangle\right| \quad ,$$

and $\mathbf{P}(x,z,E)$ is the set of all $(k,n,l)\in\mathbb{Z}^3$ such that

$$\mathrm{dist}\left(N^{k+z}(n+x),E^c\right)\leq N^{k+z} \quad .$$

It suffices to prove

$$|\Lambda_{x,\xi,z}(f_1,f_2,f_3)|\leq C$$

with a constant independent of $(x,\xi,z)\in[0,1]^3$.

1.7 The setup and the geometric interpretation

Fix $(x,\xi,z)\in[0,1]^3$. Let $\mathbf{P}$ be a finite subset of $\mathbf{P}_{x,z,E}$. We write P, P' for triples (k,n,l) and (k',n',l') in $\mathbf{P}$. Define for $\imath\in\{1,2,3\}$ and $P\in\mathbf{P}$:

$$\phi_{\imath,P}=D^2_{N^{k+z}}L_{n+x}U_{\alpha_\imath(l+\xi)+\epsilon^2\beta_\imath}\phi \quad ,$$

$$a_\imath(P)=|\langle f_\imath,\phi_{\imath,P}\rangle| \quad ,$$

$$I_P=[N^{k+z}n,N^{k+z}(n+1)) \quad .$$

We have to prove a bound on

$$\sum_{P\in\mathbf{P}}|I_P|^{-\frac{1}{2}}\prod_{\imath=1}^{3}a_\imath(P) \quad ,$$

independently of the choice of $\mathbf{P}\subset\mathbf{P}_{x,z,E}$. If I is an interval and A a positive number, we write AI for the interval of length $A|I|$ which has the same center as I. Observe that for all $P\in\mathbf{P}$ and all $\imath\in\{1,2,3\}$ we have

$$3I_P\not\subset E \quad ,$$

$$|\phi_{\imath,P}(x)|\le C_m\left(1+\frac{|x-c(I_P)|}{|I_P|}\right)^{-m} \quad .$$

Pick closed intervals $\omega_{\imath,P}=\omega_{\imath,k,l}$ such that

1. The center of $\omega_{\imath,k,l}$ satisfies

$$\left|c(\omega_{\imath,k,l})-N^{-k-z}(l+\xi+\epsilon^2\alpha_\imath^{-1}\beta_\imath)\right|\le\epsilon^5N^{-k-z}$$

2. The length of $\omega_{\imath,k,l}$ satisfies

$$\epsilon^3N^{-k-z}\le|\omega_{\imath,k,l}|\le 2\epsilon^3N^{-k-z}$$

3. If two intervals $\omega_{\imath,k,l}$ and $\omega_{\imath',k',l'}$ are not disjoint, then one is contained in the other,

4. If $\omega_{\imath,k,l}$ strictly contains $\omega_{\imath',k',l'}$, then $\omega_{\imath,k,l}$ strictly contains $\omega_{\imath'',k',l'}$ for all $\imath''\in\{1,2,3\}$.

Observe that for such a choice of intervals we have for all $\imath$ and P:

$$\mathrm{supp}(D^\infty_{\alpha_\imath^{-1}}\widehat{\phi}_{\imath,P})\subset\omega_{\imath,P} \quad .$$

The distance between $\omega_{\imath,P}$ and $\omega_{\imath',P}$ for $\imath\neq\imath'$ is less than ϵN^{-k-z} and larger than ϵ^2N^{-k-z}, because

$$2^{-1}\epsilon\ge\left|\epsilon^2\alpha_\imath^{-1}\beta_\imath-\epsilon^2\alpha_{\imath'}^{-1}\beta_{\imath'}\right|=\epsilon^2\left|\alpha_\imath^{-1}\alpha_{\imath'}^{-1}\right|\ge 2\epsilon^2 \quad .$$

In particular the intervals $\omega_{\imath,P}$ and $\omega_{\imath',P}$ are disjoint. By symmetry we can assume that $\omega_{1,P}<\omega_{2,P}<\omega_{3,P}$ for each P, in the sense that $\xi_1<\xi_2<\xi_3$ for all $\xi_\imath\in\omega_{\imath,P}$.

We prove that such a choice of intervals $\omega_{\imath,P}$ exists. Define open intervals $\tilde{\omega}_{\imath,k,l}$ by

$$c(\tilde{\omega}_{\imath,k,l}) = N^{-k-z}(l + \xi + \epsilon^2 \alpha_\imath^{-1} \beta_\imath) \quad ,$$

$$|\tilde{\omega}_{\imath,k,l}| = \epsilon^3 N^{-k-z} \quad .$$

Let $\tilde{\omega}_{k,l}$ denote the double of the convex hull of $\tilde{\omega}_{1,k,l}$, $\tilde{\omega}_{2,k,l}$, and $\tilde{\omega}_{3,k,l}$. For each $\imath, k, l$, we will prove that there is a point in each connected component of

$$\Delta(\tilde{\omega}_{\imath,k,l}) := (1 + 2\epsilon^6)\tilde{\omega}_{\imath,k,l} \setminus (1 + \epsilon^6)\tilde{\omega}_{\imath,k,l}$$

which is not contained in any $\tilde{\omega}_{k',l'}$ with $k' > k$. Pick such a point in each of the two connected components, and define $\omega_{\imath,k,l}$ to be the convex hull of those two points. Then it is obvious that the $\omega_{\imath,k,l}$ satisfy all of the above properties.

We prove the existence of the above two points. For any $k' \in \mathbb{Z}$ let $\Omega_{k'}$ be the complement of the union of all intervals $\tilde{\omega}_{k',\tilde{l}}$ with $\tilde{l} \in \mathbb{Z}$. Observe that $\Omega_{\tilde{l}}$ is the union of intervals length at least $\epsilon N^{-k'-z}$, and any interval of length $N^{-k'-z}$ has nonempty intersection with $\Omega_{k'}$. Now fix $\imath, k, l$ and let I_k be a connected component of $\Delta(\tilde{\omega}_{\imath,k,l})$. Choose inductively $I_{k'}$ for $k' > k$ to be a connected component of $\Omega_{k'}$ which intersects the interval $\frac{1}{2}I_{k'-1}$. Obviously we have $I_{k'} \subset I_{k'-1} \subset I_k$. The intersection of all $I_{k'}$ with $k' > k$ gives the point we are looking for.

1.8 Trees

We will decompose the finite set $\mathbf{P} \subset \mathbf{P}_{x,z,E}$ into subsets T, for which we do have good estimates on the partial sums

$$\sum_{P \in T} |I_P|^{-\frac{1}{2}} \prod_{\imath=1}^{3} a_\imath(P) \quad .$$

In addition we will have good control over the number of subsets, which will allow to sum up the estimates for the subsets to an estimate for the whole set $\mathbf{P}$.

A *tree* is a quadruple $(T, \imath_T, I_T, \xi_T)$ where T is a nonempty subset of $\mathbf{P}$, $\imath \in \{1, 2, 3\}$, I_T is an interval and ξ_T a number, such that $I_P \subset I_T$ and $\xi_P \in \omega_{\imath,P}$ for all $P \in T$, and there is a $P_T \in T$ with $I_{P_T} = I_T$. The number $\imath_T$ is called the type, I_T the top interval, ξ_T the top frequency, and P_T, which is unique, the top element of the tree. We will call the set T itself a tree, with the understanding that the data $\imath_T$, I_T and ξ_T are associated to T.

Pick $p'_\imath < s_\imath < \infty$ such that

$$\sum_{\imath=1}^{3} \frac{1}{s_\imath} > 1 \quad .$$

The following two lemmata give the main analytic estimates for the trees. As we show in the next sections, these lemmata give enough information to prove Lemma 1.5.1 by mainly conbinatorial arguments. The proofs of these two lemmata themselves is postponed until afterwards.

Lemma 1.8.1 *If $T \subset \mathbf{P}$ is a tree and $\jmath \neq \imath_T$, then:*

$$\left\| \left(\sum_{P \in T} \frac{a_\jmath(P)^2}{|I_P|} 1_{I_P} \right)^{\frac{1}{2}} \right\|_1 \leq C\,|I_T| \quad .$$

Lemma 1.8.2 *If $\imath, \jmath \in \{1,2,3\}$, $\imath \neq \jmath$, $\mu \in \mathbb{Z}$ and $\mathbf{P}' \subset \mathbf{P}$ is a disjoint union of trees of type $\imath$:*

$$\mathbf{P}' := \dot{\bigcup_{T \in \mathrm{T}}} T \quad ,$$

such that the set $\{I_P \times \omega_{\jmath,P} : P \in \mathbf{P}'\}$ is a set of pairwise disjoint rectangles and

$$\left\| \left(\sum_{P \in T} \frac{a_\jmath(P)^2}{|I_P|} 1_{I_P} \right)^{\frac{1}{2}} \right\|_1 \geq 2^{\frac{\mu}{s_\jmath}}\,|I_T|$$

for each $T \in \mathrm{T}$, then:

$$\sum_{T \in \mathrm{T}} |I_T| \leq C 2^{-\mu} \quad .$$

1.9 Selection of the trees

We partition $\mathbf{P}$ into trees $T_0, \ldots, T_N$ as follows. Let $n \in \mathbb{N}_0$ and assume by induction that $T_{n'}$ is already chosen for all n' with $n' \leq n$. Define

$$\mathbf{P}_n := \mathbf{P} \setminus \bigcup_{n' \leq n} T_{n'} \quad .$$

We can assume that $\mathbf{P}_n$ is not empty. Let μ be the maximal integer for which there exists a tree $T \subset \mathbf{P}_n$ such that

$$a_\jmath(P_T) \geq 2^{\frac{\mu}{s_\jmath}} |I_T|^{\frac{1}{2}} \tag{1.12}$$

for the top element $P_T \in T$ and $\jmath = \imath_T$, or

$$\left\| \left(\sum_{P \in T} \frac{a_\jmath(P)^2}{|I_P|} 1_{I_P} \right)^{\frac{1}{2}} \right\|_1 \geq 2^{\frac{\mu}{s_\jmath}+3} |I_T| \tag{1.13}$$

for some $\jmath \neq \imath_T$. If there is a tree satisfying (1.12), pick such a tree T such that $|I_T|$ is maximal, otherwise pick a $\jmath$ and a tree T such that (1.13) is satisfied and $\xi_T(\jmath - \imath_T)$ is maximal. Obviously the effect of the last condition is to maximize ξ_T if $\jmath > \imath_T$ and to minimize ξ_T if $\jmath < \imath_T$. Define $T_{n+1} \subset \mathbf{P}_n$ to be the maximal tree with respect to set inclusion which has the same type, top element, and top frequency as T.

Since $\mathbf{P}$ is finite and we can assume that $a_1(P) \neq 0$ for all $P \in \mathbf{P}$, this algorithm yields a partition of $\mathbf{P}$ into trees. Let $\mathrm{T}_{\mu,\imath,\jmath}$ be the set of trees in this partition which are selected based on the parameters μ, $\imath$, and $\jmath$.

1.10 Estimates on the selected trees

Lemma 1.10.1 *If $T \in \mathrm{T}_{\mu,\imath,\jmath}$, then*

$$\mu \le C \quad , \tag{1.14}$$

$$a_\imath(P) \le 2^{\frac{\mu+1}{s_\imath}} |I_P|^{\frac{1}{2}} \quad , \tag{1.15}$$

for all $P \in T$, and, for $\imath \neq \imath'$

$$\left(\sum_{P \in T} a'_\jmath(P)^2 \right)^{\frac{1}{2}} \le C 2^{\frac{\mu}{s'_\jmath}} |I_T|^{\frac{1}{2}} \quad . \tag{1.16}$$

We prove the lemma. If $\imath \neq \jmath$, then (1.14 follows immediately from Lemma 1.8.1. If $\imath = \jmath$, then we pick $\imath' \neq \imath$ and observe that each element $P \in \mathbf{P}$ forms a tree $\{P\}$ of type $\imath'$. This allows us to apply Lemma 1.8.1 again to conclude the bound on μ.

We turn to inequality (1.15). If there was a $P \in T$ which did not satisfy this inequality, then $\{P\}$ would be a tree which satisfied (1.12) with $\mu + 1$ instead of μ, a contradiction to the choice of μ. This proves inequality (1.15).

The left hand side of (1.16) is equal to the L^2- norm of the function

$$f := \left(\sum_{P \in T} \frac{a_{\jmath'}(P)^2}{|I_P|} 1_{I_P} \right)^{\frac{1}{2}} \quad .$$

By choice of μ we have

$$\|f\|_1 \le 2^{\frac{\mu+1}{\sigma'_\jmath}+3} |I_T| \tag{1.17}$$

Let $\mathbf{I}$ be the set of all intervals of the type $[N^{k+z}n, N^{k+z}(n+1))$ with $k, n \in \mathbb{Z}$. We call an interval in $\mathbf{I}$ a *grid interval.* In particular all intervals I_P with $P \in \mathbf{P}$ are grid intervals. By interpolation it suffices to prove the appropriate estimate for the BMO- norm of f with respect to the grid $\mathbf{I}$, i.e.,

$$\sup_{J \in \mathbf{I}} \left(\inf_c \frac{1}{|J|} \int_J |f(x) - c| \, dx \right) \le C 2^{\frac{\mu}{s_{\jmath'}}} \quad .$$

Pick a grid interval J. We have

$$\inf_c \int_J |f(x) - c| \, dx \le \inf_c \int_J \left| \sum_{P \in T} \frac{a_{\jmath'}(P)^2}{|I_P|} 1_{I_P}(x) - c^2 \right|^{\frac{1}{2}} dx$$

$$\le \sum_{I \in \mathbf{I}} \int_I \left(\sum_{P \in T : I_P \subset I} \frac{a_{\jmath'}(P)^2}{|I_P|} 1_{I_P}(x) \right)^{\frac{1}{2}} dx$$

Here $\mathbf{I}_J$ is the collection of maximal intervals $I \in \mathbf{I}$ which satisfy $I \subset J$ and $I = I_P$ for some $P \in T$. In the last line of the previous calculation we have used that two grid intervals are either disjoint or one is contained in the other. It is easy to see that

$$(\{P \in T : I_P \subset I\}, \imath, I, \xi_T)$$

is a tree for each $I \in \mathbf{I}_J$. This tree also satisfies the analogue of (1.17). Hence we can estimate the previously displayed expression by

$$\sum_{I \in \mathbf{I}_J} C 2^{\frac{\mu}{s_{\jmath'}}} |I| \le C 2^{\frac{\mu}{s_{\jmath'}}} |J| \quad ,$$

because the intervals in $\mathbf{I}_J$ are pairwise disjoint and contained in J. This completes the proof of Lemma 1.10.1.

1.11 Counting the selected trees

Lemma 1.11.1 *There is a constant C such that for all μ, $\imath$, $\jmath$ we have*

$$\sum_{T \in \mathrm{T}_{\mu,\imath,\jmath}} |I_T| \le C 2^{-\mu} \quad .$$

We prove the lemma. First consider the case $\imath = \jmath$. If $P_T, P_{T'}$ are the top elements of two different trees $T, T' \in \mathrm{T}_{\mu,\imath,\jmath}$, then $I_T \times \omega_{\imath,P_T} \cap I_{T'} \times \omega_{\imath,P_{T'}} \cap = \emptyset$. To see this assume the contrary. We can assume that $I_{T'} \subset I_T$, $\xi_T \in \omega_{\imath,P_{T'}}$, and that T has been selected first. Then $(T \cup T', \imath, I_T, \xi_T)$ is a tree, a contradiction to the maximality of T. Now the conclusion of Lemma 1.11.1 follows from Lemma 1.8.2, applied to the set $\{P_T : T \in \mathrm{T}_{\mu,\imath,\jmath}\}$, where each $\{P_T\}$ is considered as a tree of type $\imath'$ with $\imath' \neq \imath$.

Now assume $\imath < \jmath$. For a tree T define T^{red} to be the set of $P \in T$ such that I_P is not minimal in $\{I_{P'} : P' \in T\}$ with respect to set inclusion. If T^{red} is nonempty, it is again a tree. For $T \in \mathrm{T}_{\mu,\imath,\jmath}$ we have

$$\left\| \left(\sum_{P \in T^{\mathrm{red}}} \frac{a_\jmath(P)^2}{|I_P|} 1_{I_P} \right)^{\frac{1}{2}} \right\|_1$$

$$\ge \left\| \left(\sum_{P \in T} \frac{a_\jmath(P)^2}{|I_P|} 1_{I_P} \right)^{\frac{1}{2}} \right\|_1 - \sum_{P \in T \setminus T^{\mathrm{red}}} a_\jmath(P) |I_P|^{\frac{1}{2}}$$

$$\ge 2^{\frac{\mu}{s_\jmath}+3} |I_T| - 2^{\frac{\mu+1}{s_\jmath}} \sum_{P \in T \setminus T^{\mathrm{red}}} |I_P| \ge 2^{\frac{\mu}{s_\jmath}} |I_T| \quad .$$

Here we have used that the intervals I_P with $P \in T \setminus T^{\mathrm{red}}$ are pairwise disjoint.

The desired estimate now follows from Lemma (1.8.2) as soon as we prove that for any $T, T' \in \mathrm{T}_{\mu,\imath,\jmath}$ and any $P \in T^{\mathrm{red}}$, $P' \in T'^{\mathrm{red}}$ with $P \neq P'$ we have $I_P \times \omega_{\jmath,P} \cap I_{P'} \times \omega_{\jmath,P'} = \emptyset$. To see this assume to the contrary that $I_{P'} \subset I_P$, $\omega_{\jmath,P} \subset \omega_{\jmath,P'}$. Then it is easy to see that $P \neq P'$ implies $\omega_{\jmath,P} \neq \omega_{\jmath,P'}$. Hence we have also $\omega_{\imath,P} \subset \omega_{\jmath,P'}$. Since $\imath < \jmath$, this implies $\xi_{T'} < \xi_T$. In particular $T \neq T'$ and T has been selected first. Pick $P'' \in T' \setminus T'^{\mathrm{red}}$ such that $I_{P''} \subset I_{P'}$, $I_{P''} \neq I_{P'}$. Then we have

$$\xi_T \in \omega_{\jmath,P'} \subset \omega_{\imath,P''} \quad .$$

Hence P'' qualifies to be in the tree T, a contradiction to the maximality of T. This finishes the proof in the case $\imath < \jmath$.

The case $\imath > \jmath$ is done similarly. This completes the proof of Lemma 1.11.1.

1.12 The proof of Lemma 1.5.1

We have with Hölder's inequality, Lemma 1.10.1, and Lemma 1.11.1:

$$\sum_{P\in\mathbf{P}} |I_P|^{-\frac{1}{2}} \prod_{\imath=1}^{3} a_\imath(P)$$

$$\leq \sum_{\mu\leq C} \sum_{\imath,\jmath=1}^{3} \sum_{T\in \mathrm{T}_{\mu,\imath,\jmath}} \left(\sup_{P\in T} a_\imath(P)|I_P|^{-\frac{1}{2}}\right) \prod_{\jmath'\neq\imath} \left(\sum_{P\in T} a_{\jmath'}(P)^2\right)^{\frac{1}{2}}$$

$$\leq \sum_{\mu\leq C} \sum_{\imath,\jmath=1}^{3} \sum_{T\in \mathrm{T}_{\mu,\imath,\jmath}} 2^{\left(\frac{1}{s_1}+\frac{1}{s_2}+\frac{1}{s_3}\right)\mu} |I_T|$$

$$\leq \sum_{m\leq C} \sum_{\imath,\jmath=1}^{3} C 2^{\left(\frac{1}{s_1}+\frac{1}{s_2}+\frac{1}{s_3}-1\right)\mu} \quad .$$

This is a convergent geometric series by choice of s_1, s_2, s_3, which completes the proof of Lemma 1.5.1. It remains to prove Lemmata 1.8.1 and 1.8.2.

1.13 Proof of Lemma 1.8.1

In order to prove Lemma 1.8.1, fix a tree T and $\jmath \neq \imath_T$. Let $g \in S(\mathbb{R})$ and $1 < t \leq 2$. For each $k \in \mathbb{Z}$ there exists at most one $l \in \mathbb{Z}$ such that $\xi_T \in \omega_{\imath,k,l}$. If such an l exists, define g_k by

$$\widehat{D^{\infty}_{\alpha_\jmath^{-1}} g_k} = \widehat{D^{\infty}_{\alpha_\jmath^{-1}} g}\ 1_{\omega_{\jmath,k,l}} \quad .$$

Then we have

$$\left\| \left(\sum_{P\in T} \frac{|\langle g, \phi_{\jmath,P}\rangle|^2}{|I_P|} 1_{I_P} \right)^{\frac{1}{2}} \right\|_t = \left\| \left(\sum_k \sum_{P\in T: |I_P| = N^k} \frac{|\langle g_k, \phi_{\jmath,P}\rangle|^2}{|I_P|} 1_{I_P} \right)^{\frac{1}{2}} \right\|_t$$

It is easy to see that the intervals I_P with $P \in T$ and $|I_P| = N^k$ are pairwise disjoint. Hence we can estimate the previously displayed expression by

$$C \left\| \left(\sum_k (M g_k)^2 \right)^{\frac{1}{2}} \right\|_t \leq C \left\| \left(\sum_k g_k^2 \right)^{\frac{1}{2}} \right\|_t \leq C\|g\|_t \quad .$$

Here we have used the Fefferman Stein maximal inequality and the fact that g_k are part of a Littlewood Paley decomposition of g since the intervals $\omega_{\jmath,k,l}$ form a lacunary family of disjoint intervals.

Applying this calculation to $f_\jmath 1_{3I_T}$ and $t = p_\jmath$ gives with Hölder's inequality:

$$\left\| \left(\sum_{P\in T} \frac{|\langle f_\jmath 1_{3I_T}, \phi_{\jmath,P}\rangle|^2}{|I_P|} 1_{I_P} \right)^{\frac{1}{2}} \right\|_1 \leq C|I_T| \inf_{x\in 3I_T} M_{p_\jmath} f_\jmath(x) \leq C|I_T| \quad ,$$

since $3I_T$ is not contained in E. Finally, we have

$$\left\| \left(\sum_k \sum_{P\in T:|I_P|=N^k} \left| \left\langle f_\jmath 1_{(3I_T)^c}, \phi_{\jmath,P} \right\rangle \right|^2 |I_P|^{-1} 1_{I_P} \right)^{\frac{1}{2}} \right\|_1$$

$$\leq C \left\| \left(\sum_k N^k |I_T|^{-1} M(f 1_{(3I_T)^c}) \right) \right\|_{L^1(I_T)} \leq C|I_T| \inf_{x\in I_T} M_{p_\jmath} f_\jmath(x) \leq C|I_T| \quad .$$

This completes the proof of Lemma 1.8.1. For further reference note that a slight modification of the above calculation proves for any interval I which contains I_T and for large m the estimate

$$\left\| \left(\sum_{P\in T} \frac{|\langle g, \phi_{\jmath,P}\rangle|^2}{|I_P|} 1_{I_P} \right)^{\frac{1}{2}} \right\|_t \leq C|I_T|^{\frac{1}{t}} \inf_{x\in I_T} M_t(g(M1_I)^m)(x) \quad . \tag{1.18}$$

1.14 A good lambda inequality

We prove Lemma 1.8.2. We have to estimate the L^1 norm of the *counting function*

$$\mathbf{N} := \sum_{I\in \mathrm{T}} 1_{I_T} \quad .$$

Fix $\lambda \geq 0$ and consider the set:

$$E_\lambda := \left\{ x \in \mathbb{R} : M_{p_\jmath - \delta} f_\jmath(x) \leq \gamma \lambda^{\frac{1}{p_\jmath}}, \mathbf{N}(x) \geq 5\lambda \right\} \quad ,$$

for some small positive constants $\gamma = \gamma(\mu, p_\jmath)$ and $\delta = \delta(p_\jmath)$ to be specified later. The set E_λ is clearly contained in the set

$$F_\lambda := \{x \in \mathbb{R} : \mathbf{N}(x) \geq \lambda\} \quad .$$

Let I be a maximal grid interval contained in F_λ and assume $I \cap E_\lambda \neq \emptyset$. Define

$$\mathbf{N}_I = \sum_{T\in \mathrm{T}: I_T \subset I} 1_{I_T} \quad .$$

Then $\mathbf{N} - \mathbf{N}_I$ is constant on I and bounded by λ on I, since otherwise the smallest grid interval which strictly containes I was also contained in F_λ, a contradiction to the maximality of I.

We assume the following a priori inequality, which we will prove later:

$$\left\| \mathbf{N}_I^{\frac{1}{s_\jmath}} \right\|_{p_\jmath - \delta} \leq C 2^{-\frac{\mu}{s_\jmath}} |I|^{\frac{1}{p_\jmath - \delta}} \inf_{x\in I} M_{p_\jmath - \delta} f_\jmath \quad . \tag{1.19}$$

With this inequality we have

$$|\{\mathbf{N}_I(x) \geq \lambda\}| \leq C\,|I| \left(2^{-\frac{\mu}{s_\jmath}} \lambda^{-\frac{1}{s_\jmath}} \inf_{x\in I} M_{p_\jmath - \delta} f_\jmath(x) \right)^{p_\jmath - \delta} \quad .$$

Since $I \cap E_\lambda \neq \emptyset$, the infimum on the right hand side is bounded by $\gamma\lambda^{\frac{1}{p_J}}$. Moreover it is bounded by 1, since otherwise I was contained in the set where the maximal function $M_{p_J} f$ is larger than 1, and hence $\mathbf{N}_I = 0$, which is impossible.

Maximizing the expression on the right hand side of the previous inequality over λ gives

$$|\{\mathbf{N}_I(x) \geq \lambda\}| \leq C\,|I| \left(2^{-\frac{\mu}{s_J}}\gamma^{\frac{p_J}{s_J}}\right)^{p_J-\delta} \quad .$$

Now we pick γ equal to an appropriate small constant times $2^{\frac{\mu}{p_J}}$, then we have

$$|\{\mathbf{N}_I(x) \geq \lambda\}| \leq 0.1\,|I| \quad .$$

Taking unions we obtain $|E_\lambda| \leq 0.1\ |F_\lambda|$. Now we obtain the desired bound for $\mathbf{N}$ by the standard “good lambda”- technique:

$$\|\mathbf{N}\|_1 = 5\int_{\lambda=0}^{\infty} |\{\mathbf{N}(x) \geq 5\lambda\}|\ d\lambda$$

$$\leq 5\int \left|\left\{M_{p_J-\delta} f_J(x) \geq \gamma\lambda^{\frac{1}{p_J}}\right\}\right|\ d\lambda + 5\int |E_\lambda|\ d\lambda$$

$$\leq \gamma^{-p_J}\left\|M_{p_J-\delta} f_J\right\|_{p_J}^{p_J} + 0.5\ \int |\{\mathbf{N}(x) \geq \lambda\}|\ d\lambda$$

$$\leq C2^{-\mu}\|f_J\|_{p_J}^{p_J} + 0.5\ \|\mathbf{N}\|_1 \quad .$$

Since $\|f_J\|_{p_J} = 1$, this proves Lemma 1.8.2. It remains to prove inequality (1.19).

1.15 An interpolation of iterated Banach spaces

We prove (1.19). Define

$$\mathrm{T}_I := \{T \in \mathrm{T} : I_T \subset I\}\ ,\quad \mathbf{P}_I := \bigcup_{T\in\mathrm{T}_I} T \quad .$$

For each $T \in \mathrm{T}_I$ let $B_T = L^1(I_T, l^2(T))$ be the Banach space of integrable functions on I_T with values in the sequence space $l^2(T)$. Here the measure of I_T is assumed to be normalized to 1. Then we have

$$|I_T|^{-1}\left\|\left(\sum_{P\in T}\frac{a_J(P)^2}{|I_P|}1_{I_P}\right)^{\frac{1}{2}}\right\|_1 = \|F_T(f_J)\|_B \quad ,$$

where the linear operator F_T is defined by

$$(F_T(f))(x,P) = \frac{\langle f, \phi_{J,P}\rangle}{|I_P|^{\frac{1}{2}}}1_{I_P}(x) \quad .$$

Let B be the l^2- direct sum of the Banach spaces B_T with $T \in \mathrm{T}_I$, and extend F_T canonically to a map into B by defining its image to be in the component B_T.

Now consider the Banachspace $L^q(\mathbb{R}, l^r(\mathrm{T}_I, B))$ of q- integrable functions on $\mathbb{R}$ with values in $l^r(\mathrm{T}_I, B)$, the r- summable sequences of elements of B indexed by T_I. Define S to be the linear operator from functions on $\mathbb{R}$ to functions in $L^q(\mathbb{R}, l^r(\mathrm{T}, B))$ defined by

$$Sf(x,T) = F_T(f) 1_{I_T}(x) \quad .$$

With (1.18) we observe for small δ:

$$\|Sf\|_{L^{1+2\delta}(\mathbb{R}, l^\infty(\mathrm{T}_I, B))} \leq \left\| \sup_{T \in \mathrm{T}_I} \|F_T(f)\|_B \, 1_{I_T} \right\|_{1+2\delta}$$

$$\leq \|M_{1+\delta}(f(M1_I)^m)\|_{1+2\delta} \leq \|f(M1_I)^m\|_{1+2\delta}$$

We will prove in the next sections for small $\delta > 0$ the estimate

$$\|Sf\|_{L^{2+\delta}(\mathbb{R}, l^{2+\delta}(\mathrm{T}, B))} \leq C\|f\|_{2+\delta} \tag{1.20}$$

Complex interpolation gives for a different δ and a different m:

$$\|Sf\|_{L^{p_j - \delta}(\mathbb{R}, l^{s_j}(\mathrm{T}, B))} \leq \|f(M1_I)^m\|_{p_j - \delta} \quad .$$

By assumption we have a lower bound on $\|F_T(f_j)\|_B$, which gives

$$2^{\frac{\mu}{s_j}} \left\| \left(\sum_{T \in \mathrm{T}_I} 1_{I_T} \right)^{\frac{1}{s_j}} \right\|_{p_j - \delta} \leq C\|f_j(M1_I)^m\|_{p_j - \delta} \quad .$$

This clearly proves (1.19). It remains to prove inequality (1.20).

1.16 The interpolation endpoint near 2

We prove (1.20). Consider the space T_I with weighted counting measure, each tree T having the weight $|I_T|$. Consider the sublinear operator mapping functions on $\mathbb{R}$ to functions on T_I defined by

$$Af(T) = \left(|I_T|^{-1} \sum_{P \in T} |\langle f, \phi_{j,P} \rangle|^2 \right)^{\frac{1}{2}} \quad .$$

We will show that A is of strong type $(2+\delta, 2+\delta)$. Since we have

$$\|Sf\|_{L^{2+\delta}(\mathbb{R}, l^{2+\delta}(\mathrm{T}_I, B))}$$

$$\leq \left(\int_{\mathbb{R}} \sum_{T \in T_I} 1_T(y) \left(\frac{1}{|I_T|} \int_{I_T} \sum_{P \in T} \frac{|\langle f, \phi_{j,P} \rangle|^2}{|I_P|} 1_P(x) \, dx \right)^{\frac{2+\delta}{2}} \right)^{\frac{1}{2+\delta}}$$

$$= \left(\sum_{T \in T_I} |I_T| \left(|I_T|^{-1} \sum_{P \in T} |\langle f, \phi_{\jmath,P} \rangle|^2 \right)^{\frac{2+\delta}{2}} \right)^{\frac{1}{2+\delta}} = \|Af\|_{2+\delta} \quad ,$$

this will prove inequality (1.20).

By interpolation it suffices to prove restricted weak type $(2+\delta, 2+\delta)$ for A. Let f be the characteristic function of a set and $\lambda > 0$. Since f is bounded by 1, inequality (1.18) with $t = 2$ shows that the set

$$\mathrm{T}_\lambda := \{ T \in \mathrm{T}_I : Af(T) > \lambda \}$$

is empty if λ is greater than a certain constant. Therefore we can assume $\lambda < C$. Define

$$\mathrm{T}^1_\lambda := \left\{ T \in \mathrm{T}_I : \left(|I_T|^{-1} \sum_{P \in T : \left| \left\langle f, \phi_{\jmath,P} \right\rangle \right| \leq \lambda |I_P|^{\frac{1}{2}}} |\langle f, \phi_{\jmath,P} \rangle|^2 \right)^{\frac{1}{2}} \geq \frac{\lambda}{2} \right\} .$$

Then we have

$$|\mathrm{T}_\lambda| \leq \sum_{T \in \mathrm{T}^1_\lambda} |I_T| + \sum_{T \in \mathrm{T}_\lambda \setminus \mathrm{T}^1_\lambda} |I_T|$$

$$\leq \frac{C}{\lambda^2} \sum_{T \in \mathrm{T}^1_\lambda} \sum_{P \in T : \left| \left\langle f, \phi_{\jmath,P} \right\rangle \right| \leq \lambda |I_P|^{\frac{1}{2}}} |\langle f, \phi_{\jmath,P} \rangle|^2$$

$$+ \frac{C}{\lambda^2} \sum_{\log_2 \lambda \leq \kappa \leq C} \sum_{P \in \mathbf{P}_I : 2^\kappa \leq \left| \left\langle f, \phi_{\jmath,P} \right\rangle \right| |I_P|^{-\frac{1}{2}} \leq 2^{\kappa+1}} |\langle f, \phi_{\jmath,P} \rangle|^2$$

We will prove in the next section that

$$\sum_{T \in \mathrm{T}^1_\lambda} \sum_{P \in T : \left| \left\langle f, \phi_{\jmath,P} \right\rangle \right| \leq \lambda |I_P|^{\frac{1}{2}}} |\langle f, \phi_{\jmath,P} \rangle|^2 \leq C \|f\|_2^2 \quad , \tag{1.21}$$

and, for each κ,

$$\sum_{P \in \mathbf{P}_I : 2^\kappa \leq \left| \left\langle f, \phi_{\jmath,P} \right\rangle \right| |I_P|^{-\frac{1}{2}} \leq 2^{\kappa+1}} |\langle f, \phi_{\jmath,P} \rangle|^2 \leq C \|f\|_2^2 \quad , \tag{1.22}$$

This gives

$$|\mathrm{T}_\lambda| \leq C \lambda^{-2} \log \lambda \|f\|_2^2 \leq C \left(\lambda^{-1} \|f\|_{2+\delta} \right)^{2+\delta} \quad ,$$

which proves the desired restricted weak type inequality and finishes the proof of (1.20). It remains to prove (1.21) and (1.22).

1.17 Almost orthogonality

We prove (1.22). Define $\mathbf{P}^\kappa := \{P \in \mathbf{P}_I : 2^\kappa \le |\langle f, \phi_{\jmath,P}\rangle|\,|I_P|^{-\frac{1}{2}} \le 2^{\kappa+1}\}$. We have

$$\left(\sum_{P\in\mathbf{P}^\kappa} |\langle f, \phi_{\jmath,P}\rangle|^2\right)^2 \le \|f\|_2^2 \left\|\sum_{P\in\mathbf{P}^\kappa} \langle f, \phi_{\jmath,P}\rangle\, \phi_{\jmath,P}\right\|_2^2$$

$$\le \|f\|_2^2\, 2 \sum_{P,P'\in\mathbf{P}^\kappa:\omega_{\jmath,P}\subset\omega_{\jmath,P'}} |\langle f, \phi_{\jmath,P}\rangle \langle \phi_{\jmath,P}, \phi_{\jmath,P'}\rangle \langle \phi_{\jmath,P'}, f\rangle| \ .$$

Here we have used that $\langle \phi_{\jmath,P}, \phi_{\jmath,P'}\rangle$ is nonzero only if $\omega_{\jmath,P}$and $\omega_{\jmath,P'}$ are not disjoint. Using that the $\langle f, \phi_{\jmath,P}\rangle\,|I_P|^{-\frac{1}{2}}$ are essentially constant we can estimate the previously displayed expression by:

$$C\,\|f\|_2^2 \sum_{P\in\mathbf{P}^\kappa} |\langle f, \phi_{\jmath,P}\rangle|^2 \sum_{P,P'\in\mathbf{P}^\kappa:\omega_{\jmath,P}\subset\omega_{\jmath,P'}} |\,\langle \phi_{\jmath,P}, \phi_{\jmath,P'}\rangle\,||I_{P'}|^{\frac{1}{2}}|I_P|^{-\frac{1}{2}} \ .$$

Now we have for fixed P:

$$\sum_{P'\in\mathbf{P}_I:\omega_{\jmath,P}\subset\omega_{\jmath,P'}} |\,\langle \phi_{\jmath,P}, \phi_{\jmath,P'}\rangle\,||I_{P'}|^{\frac{1}{2}}|I_P|^{-\frac{1}{2}}$$

$$\le \sum_{P':\omega_{\jmath,P}\subset\omega_{\jmath,P'}} |I_{P'}| \left(1 + \frac{\mathrm{dist}(I_P, I_{P'})}{|I_P|}\right)^{-m}$$

$$\le \left\|\sum_{P':\omega_{\jmath,P}\subset\omega_{\jmath,P'}} \left(1 + \frac{\mathrm{dist}(I_P, I_{P'})}{|I_P|}\right)^{-m} 1_{I_P}\right\|_1 \ .$$

Since the $I_{P'}$ appearing here are pairwise disjoint, smaller in size than I_P, and - with the exception of at most two intervals that are adjacent to I_P - have distance at least $|I_{P'}|$ from I_P, this is bounded by

$$C \left\|\left(1 + \frac{\mathrm{dist}(I_P, .)}{|I_P|}\right)^{-m} 1_{I_P}(.)\right\|_1 + C \le C \quad .$$

This proves (1.22).

We prove (1.21). Define

$$\mathbf{P}' = \bigcup_{T\in\mathrm{T}^1_\lambda} \{P \in T : |\,\langle f, \phi_{\jmath,P}\rangle\,| \le \lambda |I_P|^{\frac{1}{2}}\} \quad .$$

We have similarly to before

$$\left(\sum_{P\in\mathbf{P}'} |\langle f, \phi_{\jmath,P}\rangle|^2\right)^2$$

$$\le \|f\|_2^2 \sum_{P,P'\in\mathbf{P}':\omega_{\jmath,P}=\omega_{\jmath,P'}} |\langle f, \phi_{\jmath,P}\rangle \langle \phi_{\jmath,P}, \phi_{\jmath,P'}\rangle \langle \phi_{\jmath,P'}, f\rangle|$$

$$+2\|f\|_2^2 \sum_{P,P'\in \mathbf{P}':\omega_{\jmath,P}\subset\neq\omega_{\jmath,P'}} |\langle f,\phi_{\jmath,P}\rangle \langle \phi_{\jmath,P},\phi_{\jmath,P'}\rangle \langle \phi_{\jmath,P'},f\rangle| \ .$$

For the first summand we observe

$$\sum_{P,P'\in \mathbf{P}':\omega_{\jmath,P}=\omega_{\jmath,P'}} |\langle f,\phi_{\jmath,P}\rangle \langle \phi_{\jmath,P},\phi_{\jmath,P'}\rangle \langle \phi_{\jmath,P'},f\rangle|$$

$$\leq 2\sum_{P\in\mathbf{P}'} |\langle f,\phi_{\jmath,P}\rangle|^2 \sum_{\phi_{\jmath,P'}:\omega_{\jmath,P}=\omega_{\jmath,P'}} |\langle\phi,\phi'\rangle| \leq C\sum_{\phi} |\langle f,\phi_{\jmath,P}\rangle|^2 \ .$$

For the second summand we observe

$$\sum_{P,P'\in \mathbf{P}'':\omega_{\jmath,P}\subset\neq\omega_{\jmath,P'}} |\langle f,\phi_{\jmath,P}\rangle \langle \phi_{\jmath,P},\phi_{\jmath,P'}\rangle \langle \phi_{\jmath,P'},f\rangle|$$

$$= 2\sum_{T\in \mathrm{T}_\lambda^1} \left(\sum_{P\in T} |\langle f,\phi_{\jmath,P}\rangle|^2\right)^{\frac{1}{2}} \left(\sum_{P\in T}\left(\sum_{P':\xi_T\in\omega_{\jmath,P'}} |\langle \phi_{\jmath,P},\phi_{\jmath,P'}\rangle \langle \phi_{\jmath,P'},f\rangle|\right)^2\right)^{\frac{1}{2}}$$

$$\leq 2\sum_{T\in \mathrm{T}_\lambda^1} \left(\sum_{P\in T} |\langle f,\phi_{\jmath,P}\rangle|^2\right) \left(\sum_{P\in T}\left(\sum_{P':\xi_T\in\omega_{\jmath,P'}} |\langle \phi_{\jmath,P},\phi_{\jmath,P'}\rangle|\, |I_{P'}|^{\frac{1}{2}}|I_T|^{-\frac{1}{2}}\right)^2\right)^{\frac{1}{2}} .$$

Observe that if $\xi_T \in \omega_{\jmath,P'}$, then $I_{P'}\cap I_T$ is empty, since otherwise the rectangles of P' and the top element of T would intersect. Hence we have similarly to before

$$\sum_{P':\xi_T\in\omega_{\jmath,P'}} |\langle \phi_{\jmath,P},\phi_{\jmath,P'}\rangle|\, |I_{P'}|^{\frac{1}{2}}|I_T|^{-\frac{1}{2}}$$

$$\leq |I_P|^{\frac{1}{2}}|I_T|^{-\frac{1}{2}} \left\| \left(1+\frac{\mathrm{dist}(I_P,.)}{|I_P|}\right)^{-m} 1_{I_P}(.)\right\|_{L^1(I_T^c)}$$

$$\leq C|I_P|^{\frac{1}{2}}|I_T|^{-\frac{1}{2}} \left(1+\frac{\mathrm{dist}(I_P,I_T^c)}{|I_P|}\right)^{-m}$$

Squaring this and summing over all $P\in T$ and using that for all $P\in T$ with $|I_P|$ fixed the I_P are pairwise disjoint:

$$C\sum_{\kappa>0}\sum_{P\in T:|I_T|=2^\kappa|I_P|} |I_P||I_T|^{-1}\left(1+\frac{\mathrm{dist}(I_P,I_T^c)}{|I_P|}\right)^{-m}$$

$$\leq C\sum_{\kappa>0}\left\|\left(1+\frac{\mathrm{dist}(.,I_T^c)}{|I_P|}\right)^{-m}\right\|_{L^1(I_T)} \leq C\sum_{\kappa>0}2^{-\kappa}\leq C$$

This proves (1.21) and thereby finishes the proof of Lemma 1.8.2 and Theorem 1.1.1.

Chapter 2

Uniform estimates for the bilinear Hilbert transform

2.1 The Hilbert forms

The *Hilbert form* with parameter

$$\beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{R}^3 \setminus \{(x, x, x) : x \in \mathbb{R}\}$$

is the continuous trilinear form defined on $S(\mathbb{R}) \times S(\mathbb{R}) \times S(\mathbb{R})$ by:

$$\Lambda_\beta(f_1, f_2, f_3) := \int \left(p.v. \int \left(\prod_{j=1}^{3} f_j(x - \beta_j t) \right) \frac{dt}{t} \right) dx \quad .$$

We assume without restriction that β is a unit vector perpendicular to $\gamma := (1, 1, 1)$.

Let $1 \leq p_1, p_2, p_3 \leq \infty$. We say that the trilinear form Λ_β is of type (p_1, p_2, p_3) if there is a constant C such that

$$|\Lambda_\beta(f_1, f_2, f_3)| \leq C \|f_1\|_{p_1} \|f_2\|_{p_2} \|f_3\|_{p_3} \tag{2.1}$$

for all Schwartz functions f_1, f_2, f_3.

It has been shown in [10] and [12] that in the case that β satisfies $\beta_j \neq \beta_{j'}$ for $j \neq j'$, which we will refer to as the non-degenerate case, the Hilbert form Λ_β is of type (p_1, p_2, p_3) if and only if

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1, \quad 0 \leq \frac{1}{p_1}, \frac{1}{p_2}, \frac{1}{p_3} < 1 \quad . \tag{2.2}$$

If $\beta_2 = \beta_3$, then Λ_β is trivially of type (p_1, p_2, p_3) if and only if

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1, \quad 0 < \frac{1}{p_1} < 1, \quad 0 \leq \frac{1}{p_2}, \frac{1}{p_3} < 1 \quad . \tag{2.3}$$

A natural conjecture is that, if we fix (p_1, p_2, p_3) in the range (2.3) and let β_2 approach β_3, the constant C in (2.1) can be chosen uniformly in β. The purpose of this paper is to prove this conjecture for some choices of (p_1, p_2, p_3). We prove:

Theorem 2.1.1 *There is a constant C such that for all unit vectors $\beta \in \mathbb{R}^3$ perpendicular to γ with $|\beta_2 - \beta_3| < \frac{1}{2}$ and all Schwartz functions f_1, f_2, f_3 we have:*

$$|\Lambda_\beta(f_1, f_2, f_3)| \le C \|f_1\|_2 \|f_2\|_2 \|f_3\|_\infty \quad . \tag{2.4}$$

Remark 1: It is no restriction to assume $\beta_1 < 0$.

Remark 2: The known proof for the non-degenerate case yields uniform bounds as long as β remains in a compact set away from the degenerate case. Hence it suffices to prove the theorem for $|\beta_2 - \beta_3| < \epsilon$ for some small $\epsilon > 0$.

Remark 3: By symmetry, the theorem also yields the uniform estimate

$$|\Lambda_\beta(f_1, f_2, f_3)| \le C \|f_1\|_2 \|f_2\|_\infty \|f_3\|_2 \quad ,$$

as well as all estimates that follow from interpolation between this estimate and (2.4).

The main issue in proving this theorem is to combine the Fourier methods, which are needed in the non-degenerate case, with the simple argument of Hölder's inequality which works in the degenerate case. We follow largely the proof in the non-degenerate case, with the decomposition of the Hilbert form adapted to the parameter α, and then extract large pieces of the Hilbert form which can be estimated by Hölders inequality.

This paper is essentially self contained, however a knowledge of [10] and [12] may help. In this paper, C and m denote universal constants, whose values may change from line to line. All other constants are fixed throughout the paper.

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2.2 The decomposition of the Hilbert forms

Define the following isometries of $L^p(\mathbb{R})$:

$$L_y f(x) := f(x - y) \quad ,$$

$$U_\eta f(x) := f(x) e^{i\eta x} \quad ,$$

$$D_\lambda^p f(x) := \lambda^{-\frac{1}{p}} f(\lambda^{-1} x) \quad .$$

Define the Fourier transform $\widehat{f}$ of a function f on $\mathbb{R}^d$ by

$$\widehat{f}(\xi) = \int f(x) e^{-i\xi \cdot x}\, dx \quad .$$

If $F \in S(\mathbb{R}^2)$, $G \in S'(\mathbb{R}^2)$, and A is a regular linear transformation of $\mathbb{R}^2$, then an easy application of Plancherel's theorem shows:

$$\int_{\mathbb{R}^2} F(Ax) G(x)\, dx = C \int_{\mathbb{R}^2} \widehat{F}(\xi) \widehat{G}(-A^T \xi)\, d\xi \quad .$$

Applying this with $F(x_1,x_2)=f_1(x_1)f_2(x_2)$ and $G(x_1,x_2)=f_3(x_1)(p.v.\frac{1}{x_2})$ gives:

$$\Lambda_\beta(f_1,f_2,f_3)=\int\int f_1(x-(\beta_1-\beta_3)t)f_2(x-(\beta_2-\beta_3)t)f_3(x)\left(p.v.\frac{1}{t}\right)\,dt\,dx$$

$$=C\int\int \widehat{f_1}(\eta_1)\widehat{f_2}(\eta_2)\widehat{f_3}(-\eta_1-\eta_2)\mathrm{sign}((\beta_1-\beta_3)\eta_1+(\beta_2-\beta_3)\eta_2)\,d\eta_1\,d\eta_2$$

$$=C\int_{\eta\cdot\gamma=1}\widehat{f_1}(\eta_1)\widehat{f_2}(\eta_2)\widehat{f_3}(\eta_3)\mathrm{sign}(\beta\cdot\eta)d\sigma\quad . \tag{2.5}$$

Here $d\sigma$ denotes the Lebesgue measure on the hyperplane defined by $\eta\cdot\gamma=0$. By a trivial decomposition of $\mathrm{sign}(\beta\cdot\eta)$ and by symmetry it suffices to prove the estimate of the theorem for the trilinear form

$$\int_{\eta\cdot\gamma=1}\widehat{f_1}(\eta_1)\widehat{f_2}(\eta_2)\widehat{f_3}(\eta_3)1_{\mathbb{R}^+}(\beta\cdot\eta)\,d\sigma\quad .$$

Define $\epsilon:=2^{-100}$ and $N:=\epsilon^{-100}$. We can assume $0<|\beta_2-\beta_3|<N^{-100}$ and $\beta_1<0$. Pick $\alpha\in\mathbb{R}^3$ perpendicular to β and γ with $\alpha_1=1$. Observe that α_2 and α_3 have large modulus.

Let ϕ be a real Schwartz function such that $\widehat{\phi}$ maps into $[0,1]$, is supported in $[-2,2]$, and is equal to 1 in $[-1,1]$.

Consider the function Φ defined on the hyperplane $\eta\cdot\gamma=0$ by

$$\Phi(\eta_1,\eta_2,\eta_3):=\prod_{j=1}^{3}L_{\epsilon\beta_j}D^\infty_{\epsilon^2\alpha_jN^{-1}}\widehat{\phi}(\eta_j)\quad .$$

The support of Φ is contained in the rectangle of all $\eta=\mu\beta+\nu\alpha$ with

$$|\nu|\le 2\epsilon^2\ ,\quad |\mu-\epsilon|\le 16\epsilon^2\quad .$$

To see this, observe that a necessary condition for η to be in the support of Φ is:

$$|\eta_j-\epsilon\beta_j|\le 2|\alpha_j|\epsilon^2 \tag{2.6}$$

for each $j\in\{1,2,3\}$. Hence

$$\|\eta-\epsilon\beta\|_2\le 2\epsilon^2\|\alpha\|_2\quad ,$$

from which the estimate on ν follows. Writing $\eta_j=\mu\beta_j+\nu\alpha_j$ in (2.6) and using the estimate on ν gives:

$$|(\mu-\epsilon)\beta_j|\le 4|\alpha_j|\epsilon^2\quad .$$

This inequality for $j=1$ implies the condition on μ, since $4|\beta_1|>1$ and $\alpha_1=1$.

This discussion yields in particular that Φ vanishes outside the half plane $\eta\cdot\beta>0$. Hence averaging gives for all η with $\eta\cdot\gamma=0$:

$$c_\beta 1_{\mathbb{R}_+}(\eta\cdot\beta)=\int_{\mathbb{R}}\int_{\mathbb{R}}\prod_{j=1}^{3}D^\infty_{N^{-k}}L_{\alpha_j\xi}L_{\epsilon\beta_j}D^\infty_{\epsilon^2\alpha_jN^{-1}}\widehat{\phi}(\eta_j)\,dk\,d\xi\quad .$$

The estimate on the support of Φ and the fact that Φ maps into $[0,1]$ implies that c_β is bounded above uniformly in β. To see that c_β is also bounded below uniformly in β we discuss the preimage of 1 under Φ. A sufficient condition on η for $\Phi(\eta)=1$ is

$$|\eta_j - \epsilon\beta_j| \le |\alpha_j|\epsilon^2 N^{-1}$$

for each j. This is in particular satisfied if $\eta = \mu\beta + \nu\alpha$ with

$$|\mu - \epsilon| < \frac{\epsilon^2}{2}N^{-1}, \quad |\nu| < \frac{\epsilon^2}{2}N^{-1} \quad .$$

Therefore we clearly have $c_\beta > c > 0$ for some universal c.

Hence it suffices to prove uniform bounds on the trilinear form $\Lambda = \Lambda(\beta)$ defined by

$$\Lambda(f_1,f_2,f_3) := \int\limits_{\mathbb{R}^3} \prod_{j=1}^{3} (f_j * \phi_{j,k,\xi})(x)\, dk\, d\xi\, dx$$

$$= C \int\limits_{\eta\cdot\gamma=0} \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \prod_{j=1}^{3} \left(D^{\infty}_{\epsilon^2\alpha_j N^{-k-1}} L_{\epsilon^{-2}N\xi+\epsilon^{-1}\alpha_j^{-1}N\beta_j}\widehat{\phi}(\eta_j)\right) \widehat{f}_j(\eta_j)\, dk\, d\xi\, d\sigma \ ,$$

Here we have set:

$$\phi_{j,k,\xi} := D^1_{\epsilon^{-2}\alpha_j^{-1}N^{k+1}} U_{\epsilon^{-2}N\xi+\epsilon^{-1}\alpha_j^{-1}N\beta_j}\phi \quad .$$

2.3 The truncated trilinear form

Let κ be an integer such that

$$N^{\kappa-\frac{2}{3}} < |\alpha_2|, |\alpha_3| < N^{\kappa+\frac{2}{3}} \quad . \tag{2.7}$$

Such an integer exists and is greater than 5, because $|\alpha_2|$ and $|\alpha_3|$ are large and their distance is 1.

Let ψ be a positive Schwartz function such that $\widehat{\psi}$ maps into $[0,1]$, is supported in $[-1,1]$, and $\widehat{\psi}(0)=1$. For $E \subset \mathbb{R}$ and $k \in \mathbb{R}$ define

$$E_k := \{x \in E : \mathrm{dist}(x, E^c) \ge N^k\} \quad ,$$

$$\psi_{1,k} := 1_{(E_k)^c} * D^1_{N^{k+2}}\psi \ , \quad \psi_{2,k} := \psi_{3,k} := \psi_{1,k-\kappa} \quad ,$$

$$\Lambda_E(f_1,f_2,f_3) := \int\limits_{\mathbb{R}^3} \prod_{j=1}^{3} \psi_{j,k}(x)\,(f_j * \phi_{j,k,\xi})(x)\, dk\, d\xi\, dx \quad .$$

We will prove:

Lemma 2.3.1 *For $\jmath \in \{1,2,3\}$ let $f_\jmath$ be a Schwartz function with $\|f_\jmath\|_2 = 1$ and define:*

$$E := \bigcup_{\jmath=1}^{3} \{x : M_2 f_\jmath(x) > 1\} \quad .$$

Then:

$$|\Lambda_E(f_1, f_2, f_3)| \leq C$$

for a universal constant C.

Here M_p denotes the maximal operator

$$M_p f(x) := \left(\sup_{x \in I} \frac{1}{|I|} \int_I |f(x)|^p \, dx\right)^{\frac{1}{p}} \quad .$$

We postpone the proof of Lemma 2.3.1 and prove that Lemma 2.3.1 implies Theorem 2.1.1. Define the operator H mapping $S(\mathbb{R}) \times S(\mathbb{R}) \to S'(\mathbb{R})$ by

$$H(f_1, f_2)(f_3) := \Lambda(f_1, f_2, f_3) \quad . \tag{2.8}$$

The estimate of Theorem 2.1.1 is equivalent to the estimate

$$\|H(f_1, f_2)\|_p \leq C \|f_1\|_{p_1} \|f_2\|_{p_2} \tag{2.9}$$

with $p = 1$ and $p_1, p_2 = 2$. Estimate (2.9) has been proved in [12], with C replaced by a constant C_β depending on β, for all (p, p_1, p_2) which are contained in a small neighborhood of $(1,2,2)$ and satisfy $1/p = 1/p_1 + 1/p_2$. We will prove

$$|\{x : |H(f_1, f_2)(x)| > \lambda\}| \leq C' \lambda^{-1} \|f_1\|_2 \|f_2\|_2 \tag{2.10}$$

with a universal constant C'. Then we can use interpolation as in [8] to conclude estimate (2.9) with constant $2C'$ in a small neighborhood of $(1,2,2)$, and then use interpolation again to conclude the estimate for $(1,2,2)$ itself with constant $2C'$. Hence it suffices to prove the weak type estimate (2.10)

By homogeneity and linearity it suffices to prove

$$|\{x : \Re(H(f_1, f_2)(x)) \geq C\}| \leq C \tag{2.11}$$

for all f_1, f_2 with $\|f_1\|_2 = \|f_2\|_2 = 1$. Fix such f_1 and f_2 and define:

$$E := \bigcup_{\jmath=1}^{2} \{x : M_2 f_\jmath(x) > 1\} \quad .$$

It suffices to prove the weak type estimate (2.11) for H_E and $H - H_E$ separately, where H_E is the dual of Λ_E as in (2.8). Consider the set

$$F := \{x : \Re(H_E(f_1, f_2)(x)) \geq 1\} \quad .$$

Let f_3 be a Schwartz function with $\|f_3\|_\infty \le 1$ and

$$\|f_3 - 1_F\|_2 \le \min\{1, \|H_E(f_2,f_3)\|_2^{-1}\} \quad .$$

We can assume that $\|f_3\|_2 > 1$, because we have an immediate bound on F otherwise. Hence $M_2(f_3/\|f_3\|_2)$ is bounded by 1 and we can apply Lemma 2.3.1 to conclude

$$|F| \le \Re\Lambda_E(f_1,f_2,1_F) \le \|f_3\|_2\, |\Lambda_E(f_1,f_2,f_3/\|f_3\|_2)| + C \le C|F|^{\frac{1}{2}} + C \quad .$$

This proves a uniform bound on $|F|$.

To complete the proof of Theorem 2.1.1 it now suffices to prove

$$\|(H-H_E)(f_1,f_2)\|_{L^1(E^c)} \le C \quad ,$$

which in turn follows from the a priori estimate

$$|(\Lambda - \Lambda_E)(f_1,f_2,f_3)| \le C \quad .$$

for all Schwartz functions f_3 supported in E^c with $\|f_3\|_\infty \le 1$. First observe that since $\psi_{j,k}$ takes values between 0 and 1, we have

$$\begin{aligned} |1-\psi_{1,k}\psi_{2,k}\psi_{3,k}| &\le |1-\psi_{1,k}| + |1-\psi_{2,k}| + |1-\psi_{3,k}| \\ &= 1_{E_k} * D^1_{N^{k+2}}\phi + 2\left(1_{E_{k-\kappa}} * D^1_{N^{k+2-\kappa}}\phi\right) \quad . \end{aligned} \tag{2.12}$$

Accordingly, $|\Lambda - \Lambda_E|(f_1,f_2,f_3)$ can be estimated by two terms, the first one being

$$\int_{\mathbb{R}}\int_{E_k}\int_{\mathbb{R}} L_{x_0} D^1_{N^{k+2}}\phi(x) \int_{\mathbb{R}} \left|\prod_{j=1}^3 \left(f_j * \phi_{j,k,\xi}\right)(x)\right| d\xi\, dx\, dx_0\, dk \quad . \tag{2.13}$$

By Hölder's inequality the inner integral is bounded by

$$\prod_{j=1}^2 \left(\int_{\mathbb{R}} |f_j * \phi_{j,k,\xi}(x)|^2\, d\xi\right)^{\frac{1}{2}} \sup_\xi |f_3 * \phi_{3,k,\xi}(x)| \quad .$$

A calculation using Plancherel's theorem and the symmetry of ϕ shows:

$$\begin{aligned} \|f_j * \phi_{j,k,\xi}(x)\|_{L^2(\xi)} &= \left\|\left((D^\infty_{\epsilon^2\alpha_j N^{-k-1}} L_{-x} f_j) * (U_{\epsilon^{-2}N\xi + \epsilon^{-1}\alpha_j^{-1}N\beta_j}\phi)\right)(0)\right\|_{L^2(\xi)} \\ &= C\left\|\left((D^\infty_{\epsilon^2\alpha_j N^{-k-1}} L_{-x} f_j)\phi\right)\hat{}\,\right\|_2 = C\left\|f_j L_x D^2_{\epsilon^{-2}\alpha_j^{-1}N^{k+1}}\phi\right\|_2 \quad . \end{aligned}$$

For $j \in \{1,2\}$ we estimate this by

$$C\left(1 + \epsilon^2\alpha_j N^{-k-1}\mathrm{dist}(x,E^c)\right) \sup_{y\in E^c} M_2 f_j(y) \le C\left(1 + N^{\kappa-k}\mathrm{dist}(x,E^c)\right) \quad .$$

For $j=3$ we calculate similarly

$$\|f_3 * \phi_{3,k,\xi}(x)\|_{L^\infty(\xi)} \le \left\|f_3 L_x D^1_{\epsilon^{-2}\alpha_3^{-1}N^{k+1}}\phi\right\|_1$$

$$\leq C\left(1+N^{\kappa-k}\operatorname{dist}(x,E^c)\right)^{-3} \quad .$$

In the last inequality we have used that f_3 is supported in E^c. Hence we can estimate (2.13) by

$$C\int\limits_{\mathbb{R}}\int\limits_{E_k}\int\limits_{\mathbb{R}}\left|L_{x_0}D^1_{N^{k+2}}\phi(x)\right|\left(1+N^{\kappa-k}\operatorname{dist}(x,E^c)\right)^{-1}\,dx\,dx_0\,dk \quad .$$

If we do the integration over x separately on the region $2|x-x_0|<\operatorname{dist}(x_0,E^c)$ and on the complement of this region, we see easily that this is bounded by

$$C\int\limits_{\mathbb{R}}\int\limits_{E_k}\left(1+N^{-k}\operatorname{dist}(x_0,E^c)\right)^{-1}\,dx_0\,dk$$

$$\leq C\int\limits_{E}\int\limits_{N^k<\operatorname{dist}(x_0,E^c)}\left(1+N^{-k}\operatorname{dist}(x_0,E^c)\right)^{-1}\,dk\,dx_0\leq C\int\limits_{E}dx_0\leq C \quad .$$

The modifications to the last three lines which are needed to estimate the term corresponding to the second summand in (2.12) are left to the reader. This completes the reduction of Theorem 2.1.1 to Lemma 2.3.1.

2.4 Discretization of the trilinear forms and the geometric picture

We write

$$\Lambda_E(f_1,f_2,f_3)=\int\limits_{[0,1]^2}\Lambda_{k_0,\xi_0}(f_1,f_2,f_3)\,dk_0\,d\xi_0$$

with

$$\Lambda_{k_0,\xi_0}(f_1,f_2,f_3):=\sum_{k,l\in\mathbb{Z}}\int\limits_{\mathbb{R}}\prod_{j=1}^{3}\psi_{j,k+k_0}(x)\left(f_j*\phi_{j,k+k_0,l+\xi_0}\right)(x)\,dx \quad .$$

Obviously it suffices to prove uniform bounds on Λ_{k_0,ξ_0}. For simplicity of notation we will only consider the case $k_0,\xi_0=0$, however it will be clear that the estimates we obtain hold for arbitrary $k_0,\xi_0\in[0,1]$.

Let φ be a nonnegative Schwartz function such that $\widehat{\varphi}$ is supported in $[-1,1]$ and $\widehat{\varphi}(0)=1$. Define for any $k,n\in\mathbb{Z}$:

$$I_{k,n}:=[N^k n,N^k(n+1)) \quad .$$

We call such an interval $I_{k,n}$ an N *-adic* interval. Define:

$$\varphi_{1,k,n}:=1_{I_{k,n}}*D^1_{N^{k+2}}\varphi$$

and

$$\varphi_{j,k,n}:=1_{I_{k,n}}*D^1_{N^{k+2-\kappa}}\varphi$$

for $\jmath \in \{2,3\}$. Then we can write

$$\Lambda_{0,0}(f_1,f_2,f_3) = \sum_{k,l\in\mathbb{Z}} \int_{\mathbb{R}} \prod_{\jmath=1}^{3} \left(\sum_{n\in\mathbb{Z}} \varphi_{\jmath,k,n}(x)\psi_{\jmath,k}(x)\,(f_\jmath * \phi_{\jmath,k,l})\,(x) \right) dx \quad . \tag{2.14}$$

We introduce intervals $\omega_{\jmath,k,l}$ such that the support of $D^{\infty}_{\alpha_\jmath^{-1}}\widehat{\phi}_{\jmath,k,l}$ is contained in $\omega_{\jmath,k,l}$. More precisely, we choose for each $k,l \in \mathbb{Z}$, $\jmath \in \{1,2,3\}$ an interval $\omega_{\jmath,k,l}$, such that the following five properties are satisfied:

1. The center $c(\omega_{\jmath,k,l})$ satisfies

$$\left|c(\omega_{\jmath,k,l}) - N^{-k}(l + \epsilon\alpha_\jmath^{-1}\beta_\jmath)\right| \le \epsilon^2 N^{-k} \quad . \tag{2.15}$$

2. The length $|\omega_{\jmath,k,l}|$ satisfies

$$N^{-k}\epsilon^{1.4} < |\omega_{\jmath,k,l}| < N^{-k}\epsilon^{1.6} \quad . \tag{2.16}$$

3. We have $\omega_{2,k,l} = \omega_{3,k,l}$ for all k,l,

4. If two intervals $\omega_{\jmath,k,l}$ and $\omega_{\jmath',k',l'}$, are not disjoint, then one is contained in the other.

5. If $\omega_{\jmath,k,l}$ strictly contains $\omega_{\jmath',k',l'}$, then $\omega_{\jmath,k,l}$ contains $\epsilon^{-0.1}\omega_{\jmath'',k',l'}$ for all $\jmath'' \in 1,2,3$.

Since $|\alpha_2|$ and $|\alpha_3|$ are very large, the third property obviously does not conflict with the first two properties. Hence it is obvious that one can choose intervals which satisfy the first three properties. The method to choose the intervals such that in addition properties 4 and 5 are satisfied is described in [10]. One first chooses intervals satisfying the first three properties, then one enlarges these intervals a little bit so that they in addition satisfy properties 4 and 5. We do not work out the details here.

Observe that the distance between $\omega_{1,k,l}$ and $\omega_{2,k,l}$ is approximately $\epsilon\beta_1 N^{-k}$, which is large compared to the size of the intervals. In particular the two intervals are disjoint. Since we have $\beta_1 < 0$, we have $\omega_{1,k,l} < \omega_{2,k,l}$ in the sense that $\xi_1 < \xi_2$ for all $\xi_1 \in \omega_{1,k,l}$ and $\xi_2 \in \omega_{2,k,l}$.

We will consider truncations of $\Lambda_{0,0}$ of the following type: Let $\mathbf{P}$ be a subset of $\mathbb{Z}^3$, and for each $k,l \in \mathbb{Z}$ let $\mathbf{P}_{k,l}$ be the set of integers n such that $(k,n,l) \in \mathbf{P}$. Define

$$\Lambda_{\mathbf{P}}(f_1,f_2,f_3) := \sum_{k,l\in\mathbb{Z}} \int_{\mathbb{R}} \prod_{\jmath=1}^{3} \left(\sum_{n\in\mathbf{P}_{k,l}} \varphi_{\jmath,k,n}(x)\psi_{\jmath,k}(x)\,(f_\jmath * \phi_{\jmath,k,l})\,(x) \right) dx \quad . \tag{2.17}$$

We will write P, P', P'' for the triples (k,n,l), (k',n',l'), and (k'',n'',l''). Moreover we will write I_P for $I_{k,n}$ and $\omega_{\jmath,P}$ for $\omega_{\jmath,k,l}$, and likewise for P' and P''. We call a set $\mathbf{P} \subset \mathbb{Z}^3$ *convex*, if for all $P \in \mathbf{P}$, $P' \in \mathbb{Z}^3$, $P'' \in \mathbf{P}$, $\imath \in \{1,2\}$ with $I_P \subset I_{P'} \subset I_{P''}$ and $\omega_{\imath,P''} \subset \omega_{\imath,P'} \subset \omega_{\imath,P}$ we have $P' \in \mathbf{P}$.

It clearly suffices to prove bounds on $\Lambda_{\mathbf{P}}$ for all convex sets $\mathbf{P}$, provided the bound does not depend on the set $\mathbf{P}$.

2.5 The selection of the trees

Let $\mathbf{P}$ be any convex subset of $\mathbb{Z}^3$. We will partition the set $\mathbf{P}$ into subsets for which we do have good estimates by the standard theory of singular integrals and paraproducts. A *tree* consists of the data $(T, \imath_T, I_T, \xi_T)$, where $T \subset \mathbb{Z}^3$, $\imath_T \in \{1,2\}$, I_T is an N-adic interval, and $\xi_T \in \mathbb{R}$ such that $I_P \subset I_T$ and $\xi_T \in \omega_{\imath_T,P}$ for all $P \in T$. We will also call the set T itself a tree. The number $\imath_T$ is called the type of the tree, I_T is called the top interval, and ξ_T is called the top frequency of the tree. Define k_T and n_T by $I_{k_T,n_T} = I_T$. If there is a $P \in T$ with $I_P = I_T$, then this P is unique and we write P_T for it. As before we define $T_{k,l}$ to be the set of all integers n such that $(k,n,l) \in T$.

Let $\mathbf{P}' \subset \mathbb{Z}^3$ be a convex set, $\imath \in \{1,2\}$, I an N-adic interval and $\xi \in$ T. The *maximal tree* in $\mathbf{P}'$ associated to the data $(\imath, I, \xi)$ consists of all elements $P \in \mathbf{P}'$ such that $I_P \subset I$ and $\xi \in \omega_{\imath,P}$.

We prove that this maximal tree T is convex. Let $P \in T$, $P' \in \mathbb{Z}^3$, $P'' \in T$, $\imath' \in \{1,2\}$, with $I_P \subset I_{P'} \subset I_{P''}$ and $\omega_{\imath' P''} \subset \omega_{\imath',P'} \subset \omega_{\imath',P}$. Then we have $P \in \mathbf{P}'$ by convexity of $\mathbf{P}'$, and we have $I_{P'} \subset I$. We can assume $P \neq P''$ and therefore $\omega_{\imath,P} \neq \omega_{\imath,P''}$. Then we have $\imath = \imath'$, because $\omega_{\imath,P} \cap \omega_{\imath,P''} \neq \emptyset$ and $\omega_{\imath',P} \cap \omega_{\imath',P''} \neq \emptyset$. Hence $\xi \in \omega_{\imath,P'}$ and therefore $P' \in T$.

Fix I and ξ and let $T^\imath$ be the maximal trees in $\mathbf{P}'$ which are associated to the data $(\imath, I, \xi)$ for $\imath = 1,2$. We prove that the set $\mathbf{P}'' = \mathbf{P}' \setminus (T^1 \cup T^2)$ is convex. Assume to get a contradiction that $\mathbf{P}''$ is not convex. Then there exist $P \in \mathbf{P}''$, $P' \in \mathbf{P}' \setminus \mathbf{P}''$, $\imath' \in \{1,2\}$, with $I_P \subset I_{P'}$ and $\omega_{\imath',P'} \subset \omega_{\imath',P}$. Then P' is in the tree $T^\imath$ for some $\imath$. Since we have $\omega_{\imath',P} \neq \omega_{\imath',P'}$, we conclude $\omega_{\imath,P'} \subset \omega_{\imath',P}$, and therefore $\xi \in \omega_{\imath',P}$. Moreover, $I_P \subset I_{P'} \subset I$, hence P qualifies to be in the tree $T^{\imath'}$. This is a contradiction to $P \in \mathbf{P}''$.

If T is a tree and $k \in \mathbb{Z}$, then there is at most one $l \in \mathbb{Z}$ such that $T_{k,l} \neq \emptyset$. If such an l exists, we define $T_k := T_{k,l}$ and $\phi_{\jmath,k,T} := \phi_{\jmath,k,l}$, if no such l exists, we define $T_k := \emptyset$ and $\phi_{\jmath,k,T} := 0$. Moreover, we write for brevity $(k,n) \in T$ iof there exists an $l \in \mathbb{Z}$ with $(k,n,l) \in T$. Hence we can write

$$\Lambda_T(f_1, f_2, f_3)$$
$$= \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \prod_{\jmath=1}^{3} \left(\sum_{n \in T_k} \varphi_{\jmath,k,n}(x) \psi_{\jmath,k}(x) \left(f_\jmath * \phi_{\jmath,k,T} \right)(x) \right) dx \tag{2.18}$$

Let T be a tree of type 2. Besides (2.18), we will study the following similar expression:

$$\sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \prod_{\jmath=1}^{3} \left(\sum_{n \in \tilde{T}_k} \varphi_{\jmath,k,n}(x) \psi_{\jmath,k}(x) \left(f_\jmath * \tilde{\phi}_{\jmath,k,T} \right)(x) \right) dx \quad , \tag{2.19}$$

where the objects $\tilde{T}_k$ and $\tilde{\phi}_{\jmath,k,T}$ are defined in the following paragraph.

Define for $k \in \mathbb{Z}$ and $\jmath \in \{2,3\}$:

$$\tilde{\phi}_{\jmath,k,T} := U_{\alpha_\jmath \xi_T} D^1_{\epsilon^{-2}\alpha_\jmath^{-1} N^{k+1}} \phi \quad ,$$

$$\tilde{\phi}_{1,k,T} := \lambda_{2,k,T} \lambda_{3,k,T} \phi_{1,k,T} \quad ,$$

where $\lambda_{j,k,T}$ is defined by

$$\left(\phi_{j,k,T} - \lambda_{j,k,T}\tilde{\phi}_{j,k,T}\right)\hat{}(\alpha_j \xi_T) = 0 \quad . \tag{2.20}$$

Observe that $|\lambda_{j,k,T}|$ is bounded by 1. Define

$$\omega_{k,T} := [\xi_T - 8\epsilon^2 N^{-k}, \xi_T + 8\epsilon^2 N^{-k}] \quad ,$$

and observe that $\mathrm{supp}(D^{\infty}_{\alpha_j^{-1}}\hat{\tilde{\phi}}_{j,k}) \subset \omega_{k,T}$ and $\mathrm{supp}(D^{\infty}_{\alpha_j^{-1}}\hat{\tilde{\phi}}_{j,k-1}) \subset \omega_{k,T}$. Define $\tilde{T}$ to be the set of pairs $(k,n) \in \mathbb{Z}^2$ such that $I_{k,n} \subset I_T$ and there is a $P' \in T$ such that $I_{P'} \subset I_{k,n}$ and $8\omega_{k,T} \subset \omega_{2,P'}$. Here AJ for any $A > 0$ and any interval J denotes the interval which has length $A|J|$ and the same center as J. Define $\tilde{T}_k$ to be the set of integers n such that $(k,n) \in \tilde{T}$. Observe that if T is convex and $\tilde{\phi}_{1,k} \neq 0$, then $\tilde{T}_k = T_k$.

We are aiming to do Littlewood Paley theory near the frequencies $\alpha_j \xi_T$, hence we define $\theta^+_{j,k,T}$ and $\theta^-_{j,k,T}$ by:

$$\widehat{\theta^+_{j,k,T}} := (\tilde{\phi}_{j,k-1,T} - \tilde{\phi}_{j,k,T})\hat{}\ 1_{\{\alpha_j\xi:\ \xi \geq \xi_T\}} \quad ,$$

$$\widehat{\theta^-_{j,k,T}} := (\tilde{\phi}_{j,k-1,T} - \tilde{\phi}_{j,k,T})\hat{}\ 1_{\{\alpha_j\xi:\ \xi \leq \xi_T\}} \quad .$$

Define

$$\varphi^*(x) := \psi^*(x) := (1 + x^2)^{-m} \quad ,$$

where m is a sufficiently large number. By replacing the generating functions ψ and φ by ψ^* and φ^* we define the functions $\psi^*_{j,k}$ and $\varphi^*_{j,k,n}$. Observe that

$$|\psi_{j,k}| \leq C \left(\psi^*_{j,k+\tilde{k}}\right)^m$$

for all $\tilde{k} \in \{-1, 0, 1\}$.

Define Δ_k to be the set of all intervals which are either of the form $\frac{1}{2}J$ where J is a connected component of $E_k \setminus E_{k+1}$ of length at least N^{k-2}, or of the form $[c(J) - N^{k-2}, c(J) + N^{k-2}]$ where J is a connected component of $E_k \setminus E_{k+1}$ of length less than N^{k-2}. Observe that the union of all Δ_k for $k \in \mathbb{Z}$ is a set of pairwise disjoint intervals.

Define $\Delta_{k,T}$ to be the set of all elments J of Δ_k such that there exists a $(k+\kappa+1, n) \in \tilde{T}$ such that $J \subset I_{k+\kappa+1,n}$. For $J \in \Delta_{k,T}$ define

$$\rho_{k,J} := 1_J * D^1_{N^{-k}}\phi^* \quad .$$

We begin to describe the algorithm for splitting $\mathbf{P}$ into a disjoint union of trees and a remainder set $\mathbf{P}^r$ which satisfies

$$\Lambda_{\mathbf{P}^r}(f_1, f_2, f_3) = 0 \quad .$$

We select the trees by iterating a selection rule described below, which selects two trees at each iteration step. Let $\nu \in \mathbb{N}_0$ be even and assume that we have already selected the trees $T^{\nu'}$ for all $\nu' < \nu$. Define

$$\mathbf{P}^\nu := \mathbf{P} \setminus \bigcup_{\nu' < \nu} T^{\nu'} \quad .$$

If $\mathbf{P}^\nu = \emptyset$, then the algorithm stops. Assume $\mathbf{P}^\nu \neq \emptyset$. Let μ be the maximal integer for which there exists an N-adic interval I and a number ξ such that at least one of the following ten conditions is satisfied:

1. There is a $(k,n,l) \in \mathbf{P}^\nu$ with $I_{k,n} = I$ and $\xi \in \omega_{1,k,l}$ such that at least one of the following two inequalities holds:

$$\left\| \varphi^*_{1,k,n} \psi^*_{1,k} (f_1 * \phi_{1,k,l}) \right\|_2 \geq 2^\mu |I|^{\frac{1}{2}} \quad , \tag{2.21}$$

$$\left\| \varphi^*_{1,k,n} \psi^*_{1,k} \left(f_1 * U_{\alpha_1 \xi} (U_{-\alpha_1 \xi} \phi_{1,k,l})' \right) \right\|_2 \geq 2^\mu |I|^{-\frac{1}{2}} \quad . \tag{2.22}$$

2. There is a $\jmath \in \{2,3\}$ such that for the maximal tree T in $\mathbf{P}^\nu$ associated to $(1, I, \xi)$ the following inequality holds:

$$\sum_{(k,n,l)\in T} \left\| \varphi^*_{\jmath,k,n} \psi^*_{\jmath,k} (f_\jmath * \phi_{\jmath,k,l}) \right\|_2^2 \geq 5 \cdot 2^{2\mu} |I| \quad . \tag{2.23}$$

3. The maximal tree T in $\mathbf{P}^\nu$ associated to the data $(2, I, \xi)$ satisfies at least one of the following two inequalities:

$$\sum_{(k,n,l)\in T} \left\| \varphi^*_{1,k,n} \psi^*_{1,k} (f_1 * \phi_{1,k,l}) \right\|_2^2 \geq 5 \cdot 2^{2\mu} |I| \quad , \tag{2.24}$$

$$\sum_{(k,n,l)\in T} |I_{k,n}|^2 \left\| (\varphi^*_{1,k,n} \psi^*_{1,k} (f_1 * U_{\alpha_1 \xi_T} (U_{-\alpha_1 \xi_T} \phi_{1,k,l})') \right\|_2^2 \geq 5 \cdot 2^{2\mu} |I| \quad . \tag{2.25}$$

4. There is a $\jmath \in \{2,3\}$ and a $(k,n,l) \in \mathbf{P}^\nu$ with $I_{k,n} = I$ and $\xi \in \omega_{2,k,l}$ such that the following inequality holds:

$$\left\| \varphi^*_{\jmath,k,n} \psi^*_{\jmath,k} (f_\jmath * \phi_{\jmath,k,l}) \right\|_2 \geq 2^\mu |I|^{\frac{1}{2}} \quad . \tag{2.26}$$

5. There is a $\jmath \in \{2,3\}$ and a $z \in \mathbb{Z}$, such that the maximal tree T in $\mathbf{P}^\nu$ associated to $(2, I, \xi)$ satisfies the following inequality:

$$\sum_{(k,n,l)\in T} \left\| \varphi^*_{\jmath,k,n} \psi^*_{\jmath,k} \left(f_\jmath * (\phi_{\jmath,k,l} - \lambda_{\jmath,k} \tilde{\phi}_{\jmath,k,T}) * \theta^+_{\jmath,k+z,T} \right) \right\|_2^2 \geq 2^{-\frac{z}{2}} 2^{2\mu} |I| \quad . \tag{2.27}$$

6. There is a $\jmath \in \{2,3\}$ and a $z \in \mathbb{Z}$, such that the maximal tree T in $\mathbf{P}^\nu$ associated to $(2, I, \xi)$ satisfies the following inequality:

$$\sum_{(k,n,l)\in T} \left\| \varphi^*_{\jmath,k,n} \psi^*_{\jmath,k} \left(f_\jmath * (\phi_{\jmath,k,l} - \lambda_{\jmath,k} \tilde{\phi}_{\jmath,k,T}) * \theta^-_{\jmath,k+z,T} \right) \right\|_2^2 \geq 2^{-\frac{z}{2}} 2^{2\mu} |I| \quad . \tag{2.28}$$

7. There is a $\jmath \in \{2,3\}$ and a $\tilde{k} \in \{-1,0,1,2,3,4\}$ such that, if T is the maximal tree in $\mathbf{P}^\nu$ associated to $(2,I,\xi)$, then $(k_T,n_T) \in \tilde{T}$ and at least one of the following three inequalities holds:

$$\left\|\varphi^*_{\jmath,k_T,n_T}\psi^*_{\jmath,k_T+\tilde{k}}(f_\jmath * \tilde{\phi}_{\jmath,k_T+\tilde{k},T})\right\|_2 \geq 2^\mu |I|^{\frac{1}{2}} \quad , \tag{2.29}$$

$$\left\|\varphi^*_{1,k_T,n_T}\psi^*_{\jmath,k_T+\kappa+\tilde{k}}(f_\jmath * \tilde{\phi}_{\jmath,k_T+\kappa+\tilde{k},T})\right\|_2 \geq 2^\mu |I|^{\frac{1}{2}} \quad . \tag{2.30}$$

$$\left\|\varphi^*_{1,k_T,n_T}\psi^*_{\jmath,k_T+\kappa+\tilde{k}}(f_\jmath * U_{\alpha_2\xi_T}(U_{-\alpha_2\xi_T}\tilde{\phi}_{\jmath,k_T+\kappa+\tilde{k},T})')\right\|_2 \geq 2^\mu |I|^{-\frac{1}{2}} \quad . \tag{2.31}$$

8. There is a $\jmath \in \{2,3\}$ such that the maximal tree T in $\mathbf{P}^\nu$ associated to $(2,I,\xi)$ satisfies the following inequality:

$$\sum_{(k,n)\in\tilde{T}} \left\|\varphi^*_{\jmath,k,n}\psi^*_{\jmath,k}\left(f_\jmath * \theta^+_{\jmath,k,T}\right)\right\|_2^2 \geq 2^{2\mu}|I_T| \quad . \tag{2.32}$$

9. There is a $\jmath \in \{2,3\}$ such that the maximal tree T in $\mathbf{P}^\nu$ associated to $(2,I,\xi)$ satisfies the following inequality:

$$\sum_{(k,n)\in\tilde{T}} \left\|\varphi^*_{\jmath,k,n}\psi^*_{\jmath,k}\left(f_\jmath * \theta^-_{\jmath,k,T}\right)\right\|_2^2 \geq 2^{2\mu}|I_T| \quad . \tag{2.33}$$

10. There is a $\jmath \in \{2,3\}$ and a $\tilde{k} \in \{-1,0,1,2,3,4\}$ such that for the maximal tree T in $\mathbf{P}^\nu$ associated to $(2,I,\xi)$ the following inequality is satisfied:

$$\sum_k \sum_{J\in\Delta_{k-\kappa,T}} \left\|\rho_{k-\kappa,J}(f_\jmath * \tilde{\phi}_{\jmath,k+\tilde{k},T})\right\|_2^2 \geq 2^{2\mu}|I| \quad . \tag{2.34}$$

If no such number μ exists, then we have necessarily

$$\Lambda_{\mathbf{P}^\nu}(f_1,f_2,f_3) = 0 \quad .$$

We set $\mathbf{P}^r := \mathbf{P}^\nu$ and stop the selection algoritm. If such a μ exists, we pick a number $\sigma \in \{1,\ldots,10\}$ such that condition σ is satisfied for this μ and some pair I, ξ. If $\sigma \in \{1,4,7,10\}$, pick a pair (I,ξ) satisfying condition σ such that $|I|$ is maximal. If $\sigma \in \{2,5,8\}$, pick a pair (I,ξ) satisfying condition σ such that ξ_T is maximal. If $\sigma \in \{3,6,9\}$, pick a pair (I,ξ) satisfying condition σ such that ξ_T is minimal.

Let $T^{\nu+\imath-1}$ be the maximal tree in $\mathbf{P}^\nu$ which is associated to $(\imath,I,\xi)$ for $\imath = 1,2$. Observe that at least one of the two trees T^ν and $T^{\nu+1}$ is nonempty.

If $\mathbf{P}^\nu$ is convex, then also T^ν, $T^{\nu+1}$, and $\mathbf{P}^\nu \setminus (T^\nu \cup T^{\nu+1})$ are convex. It follows by induction that all these sets are indeed convex.

This completes the description of the decomposition of $\mathbf{P}$ into trees. For each selected tree the selection rule provides a parameter μ, which is called the size of the selected tree. Let T_μ be the set of selected trees with size μ. Denote the size of a selected tree T^ν by μ_ν, and denote the top interval of T^ν by I_ν.

We outline the end of the proof of Lemma 2.3.1. This calculation uses various estimates which are proved in the subsequent sections. Recall that we have to prove a universal bound on

$$|\Lambda_{\mathbf{P}}(f_1, f_2, f_3)| \quad ,$$

where $\mathbf{P}$ is the given convex subset of $\mathbb{Z}^3$. Using the previously described partition of $\mathbf{P}$ we can estimate this by

$$\sum_{\nu=0}^{r-1} \left| \Lambda_{\mathbf{P}}^{\nu}(f_1, f_2, f_3) - \Lambda_{\mathbf{P}}^{\nu+1}(f_1, f_2, f_3) \right| \quad .$$

By Lemma 2.8.1 this is bounded by

$$C \sum_{\nu=0}^{r-1} |I_\nu| \, 2^{\frac{7}{3}\mu_\nu} \quad .$$

By Lemmata 2.6.1 and 2.9.1 this is bounded by

$$C \sum_{\mu \leq \mu_0} 2^{\frac{1}{4}\mu} \leq C \quad .$$

Hence it remains to prove Lemmata 2.8.1, 2.6.1, and 2.9.1.

2.6 An upper bound for μ

In this section we prove estimates which make use the cutoff functions $\psi_{j,k}$. Most notably the effect of these functions is that the size μ of the selected trees is bounded by a universal constant μ_0.

Lemma 2.6.1 *The size of a selected tree is bounded by μ_0.*

We proof this Lemma. This requires to prove a priori bounds on the various quantities which appear in the selection algorithm for the trees.

We start with the quantity on the left hand side of (2.21). Pick any $(k, n, l) \in \mathbf{P}$. If there exists an $x \in I_{k,n} \setminus E$, then we simply have by Jensen's inequality, using the symmetry of $|\phi_{1,k,l}|$,

$$\left\| \varphi_{1,k,n}^* \psi_{1,k}^* (f_1 * \phi_{1,k,l}) \right\|_2^2 \leq C \left\| \left({\varphi_{1,k,n}^*}^2 * |\phi_{1,k,l}| \right) |f_1|^2 \right\|_1$$

$$\leq C \inf_{x \in I_{k,n}} \left\| (L_x D_{N^k}^\infty \varphi^*) |f_1|^2 \right\|_1 \leq C N^k \inf_{x \in I_{k,n}} (M_2 f_1(x))^2 \leq C N^k \quad .$$

This is the desired estimate in the case $I_{k,n} \not\subset E$. If $I_{k,n}$ is contained in E, pick a number $A \geq 1$ such that $A I_{k,n} \subset E$ and $2A I_{k,n} \not\subset E$. We have

$$\left\| \varphi_{1,k,n}^* \psi_{1,k}^* (f_1 * \phi_{1,k,l}) \right\|_2^2 \leq \left\| \varphi_{1,k,n}^* \psi_{1,k}^* \right\|_\infty \left\| \varphi_{1,k,n}^* |f_1 * \phi_{1,k,l}|^2 \right\|_1 \quad .$$

This is bounded by CN^k, since

$$\left\|\varphi^*_{1,k,n}\psi^*_{1,k}\right\|_\infty \le CA^{-m} \quad ,$$

$$\left\|\varphi^*_{1,k,n}|f_1 * \phi_{1,k,l}|^2\right\|_1 \le C \inf_{x\in 2AI_{k,n}} \left\|(L_x D^\infty_{AN^k}\varphi^*)|f_1|^2\right\|_1 \le CAN^k \quad .$$

This completes the discussion of the quantity on the left hand side of (2.21).

Now consider the quantity on the left hand side of (2.24). Since we discuss a fixed tree T, we drop the indices T, for example we write $\phi_{\jmath,k}$ for $\phi_{\jmath,k,T}$.

Pick $A \ge 1$ such that $2AI_T \not\subset E$ and either $A = 1$ or $A > 1$ and $AI_T \subset E$. We have

$$\sum_{(k,n)\in T} \left\|\varphi^*_{1,k,n}\psi^*_{1,k}(f * \phi_{1,k})\right\|_2^2 \le \sup_{(k,n)\in T} \left\|\varphi^*_{1,k,n}\psi^*_{1,k}\right\|_\infty \left(\sum_{(k,n)\in T} \left\|\varphi^*_{1,k,n}|f * \phi_{1,k}|^2\right\|_1\right) \quad .$$

Again, the first factor on the right hand side is bounded by CA^{-m}. We split

$$f_1 = f_1 1_{2I_T} + f 1_{(2I_T)^c} \quad .$$

Then we have

$$\sum_{(k,n)\in T} \left\|\varphi^*_{1,k,n}|(f_1 1_{2I_T}) * \phi_{1,k}|^2\right\|_1 = \sum_k \left\|\left(\sum_{n\in T_k} \varphi^*_{1,k,n}\right) |(f_1 1_{2I_T}) * \phi_{1,k}|^2\right\|_1$$

$$\le \sum_k \|(f_1 1_{2I_T}) * \phi_{1,k}\|_2^2 \le \|f_1 1_{2I_T}\|_2^2 \le CA|I_T| \quad .$$

Here we have used that the tree T is of type 2 and hence the supports of the $\widehat{\phi_{1,k}}$ are pairwise disjoint as k varies. We have with Jensen's inequality for $(k,n) \in T$:

$$\left\|\varphi^*_{1,k,n}|(f_1 1_{(2I_T)^c}) * \phi_{1,k}|^2\right\|_1 \le \left\|\left(\varphi^*_{1,k,n} * |\phi_{1,k}|\right) |f_1|^2 1_{(2I_T)^c}\right\|_1$$

$$\le \left\|\left(\varphi^*_{1,k,n} * |\phi_{1,k}|\right)^{\frac{1}{2}} 1_{(2I_T)^c}\right\|_\infty \left\|\left(\varphi^*_{1,k,n} * |\phi_{1,k}|\right)^{\frac{1}{2}} |f_1|^2\right\|_1$$

$$\le CN^{m(k-k_T)}\ AN^{k_T-k}N^k \quad .$$

Summing this over $(k,n) \in T$ gives

$$\sum_{(k,n)\in T} \left\|\varphi^*_{1,k,n}|(f 1_{(2I_T)^c}) * \phi_{1,k}|^2\right\|_1 \le CA|I_T| \quad .$$

This ends the discussion of the quantity in (2.24).

Now we consider the quantity on the left hand side of (2.27) for some $\jmath \in \{2,3\}$. Pick again $A \ge 1$ such that $2AI_T \not\subset E$ and either $A = 1$ or $A > 1$ and $AI_T \subset I$. Observe that the left hand side of (2.27) vanishes if $z < 0$. Therefore we assume $z \ge 0$. We have

$$\sum_{(k,n)\in T} \left\|\varphi^*_{\jmath,k,n}\psi^*_{\jmath,k}\left((f 1_{2^z I_T}) * (\phi_{\jmath,k} - \lambda_{\jmath,k}\tilde{\phi}_{\jmath,k}) * \theta^+_{\jmath,k+z}\right)\right\|_2^2$$

$$\leq CA^{-m}\sum_k \left\|(f1_{2^zI_T}) * (\phi_{\jmath,k}-\lambda_{\jmath,k}\tilde{\phi}_{\jmath,k}) * \theta^+_{\jmath,k+z}\right\|_2^2$$

$$\leq CA^{-m}2^{-2z}\left\|f1_{2^zI_T}\right\|_2^2 \leq CA^{1-m}2^{-z}|I_T| \quad .$$

Here we have used that, as k varies, the Fourier transforms of the functions

$$(\phi_{\jmath,k}-\lambda_{\jmath,k}\tilde{\phi}_{\jmath,k}) * \theta^+_{\jmath,k+z}$$

are bounded by $C2^{-z}$ and their supports have bounded overlap. Observe that

$$\left|(\phi_{\jmath,k}-\lambda_{\jmath,k}\tilde{\phi}_{\jmath,k}) * \theta^+_{\jmath,k+z}\right| \leq C2^{-z}D^1_{N^{k-\kappa+z}}\varphi^* \quad . \tag{2.35}$$

Hence

$$\left\|\varphi^*_{\jmath,k,n}\left|(f_1 1_{(2^zI_T)^c}) * (\phi_{\jmath,k}-\lambda_{\jmath,k}\tilde{\phi}_{\jmath,k}) * \theta^+_{\jmath,k+z}\right|^2\right\|_1 \leq$$

$$\leq C2^{-2z}\left\|\left(\varphi^*_{\jmath,k,n} * D^1_{N^{k-\kappa+z}}\varphi^*\right)|f_1|^2 1_{(2^zI_T)^c}\right\|_1$$

$$\leq C2^{-2z}N^k\left\|\left(L_{c(I_{k,n})}D^1_{N^{k-\kappa+z}}\varphi^*\right)|f_1|^2 1_{(2^zI_T)^c}\right\|_1$$

$$\leq C2^{-2z}N^kN^{m(k-k_T)}A \quad .$$

Hence we have as before:

$$\sum_{(k,n)\in T}\left\|\varphi^*_{k,n}\psi^*_{\jmath,k}\left((f1_{(2^zI_T)^c}) * (\phi_{\jmath,k}-\lambda_{\jmath,k}\tilde{\phi}_{\jmath,k}) * \theta^+_{\jmath,k+z}\right)\right\|_2^2 \leq C2^{-z}|I_T| \quad .$$

This completes the discussion of the quantity in (2.27).

Now we consider the quantity on the left hand side of (2.34) for some $\jmath \in \{2,3\}$. Fix some k and let $J \in \Delta_{k-\kappa,T}$. Then we have

$$\left\|\rho_{k-\kappa,J}(f_\jmath * \tilde{\phi}_{\jmath,k})\right\|_2^2 \leq \left\|(\rho^2_{k-\kappa,J} * |\tilde{\phi}_{\jmath,k}|)|f_\jmath|^2\right\|_1$$

$$\leq C|J|\inf_{x\in N^3J}M_2f_\jmath(x)^2 \leq C|J| \quad .$$

The last inequality is true because construction of $\Delta_{k-\kappa}$ we have $N^3J\cap E\neq\emptyset$. Since $J\in I_T$ and since the union of the $\Delta_{k-\kappa}$ is a set of pairwise disjoint intervals, we obtain

$$\sum_k\sum_{J\in\Delta_{k-\kappa,T}}\left\|\rho_{k-\kappa,J}(f_\jmath * \tilde{\phi}_{\jmath,k})\right\|_2^2 \leq C|I_T| \quad .$$

This ends the discussion of (2.34). The discussion of the remaining quantities does not require any new ideas, we therefore take Lemma 2.6.1 for granted.

We will also need the following lemma:

Lemma 2.6.2 *Let $(T,2,I_T,\xi_T)$ be a tree and $\jmath\in\{2,3\}$. We have*

$$\left\|\psi^*_{\jmath,k}(f_\jmath * \tilde{\phi}_{\jmath,k})\right\|_\infty \leq C$$

for each $k \in \mathbb{Z}$, and

$$\left\| \left(\sum_k \left| \sum_{n \in \tilde{T}_k} \varphi_{J,k,n} \psi_{J,k} (f_J * (\tilde{\phi}_{J,k-1} - \tilde{\phi}_{J,k}) \right|^2 \right)^{\frac{1}{2}} \right\|_{BMO} \leq C \quad .$$

Here BMO is understood as N-adic BMO.

We prove the first estimate of this lemma. For each $x \in \mathbb{R}$ we have

$$\psi^*_{J,k}(x) \leq C(1 + N^{\kappa - k} \mathrm{dist}(x, E^c))^{-m} \quad ,$$

$$f_J * \phi_{J,k}(x) \leq C(1 + N^{\kappa - k} \mathrm{dist}(x, E^c)) \quad .$$

This proves the desired estimate. We prove the second estimate of Lemma 2.6.2. Pick an N-adic interval $J = I_{k_J, n_J}$. Then we have

$$\inf_c \left\| \left(\sum\nolimits_k \left| \sum\nolimits_{n \in \tilde{T}_k} \varphi_{J,k,n} \psi_{J,k} (f_J * (\tilde{\phi}_{J,k-1} - \tilde{\phi}_{J,k}) \right|^2 \right)^{\frac{1}{2}} - c \right\|^2_{L^2(J)}$$

$$\leq \inf_c \left\| \sum\nolimits_{k-\kappa \geq k_J} \left| \sum\nolimits_{n \in \tilde{T}_k} \varphi_{J,k,n} \psi_{J,k} (f_J * (\tilde{\phi}_{J,k-1} - \tilde{\phi}_{J,k}) \right|^2 - c \right\|_{L^1(J)} \tag{2.36}$$

$$+ \left\| \sum\nolimits_{k-\kappa < k_J} \left| \sum\nolimits_{n \in \tilde{T}_k} \varphi_{J,k,n} \psi_{J,k} (f_J * (\tilde{\phi}_{J,k-1} - \tilde{\phi}_{J,k}) \right|^2 \right\|_{L^1(J)} \quad . \tag{2.37}$$

The first summand on the right hand side is bounded by

$$|J|^2 \sum\nolimits_{k-\kappa \geq k_J} \left\| \left(\left(\sum\nolimits_{n \in \tilde{T}_k} \varphi_{J,k,n} \psi_{J,k} (f_J * (\tilde{\phi}_{J,k-1} - \tilde{\phi}_{J,k}) \right)^2 \right)' \right\|_\infty$$

Each term in the sum can be estimated in the spirit of the first part of this Lemma by $C|J|N^{\kappa-k}$. This shows that the term (2.36) is bounded by $C|J|$.

In order to discuss the term (2.37) we split the function f_J into $f_J 1_{2J} + f_J 1_{(2J)^2}$ We have for $k - \kappa < k_J$:

$$\left\| (f_J 1_{(2J)^c}) * \tilde{\phi}_{J,k} \right\|_{L^\infty(J)}$$

$$\leq C N^{m(k-\kappa-k_J)} (1 + |J|^{-1} \mathrm{dist}(J, E^c)) \quad .$$

This estimate together with the estimate

$$\|\psi_{J,k}\|_{L^\infty(J)} \leq (1 + N^{\kappa-k} \mathrm{dist}(J, E^c))^{-m}$$

proves that the term (2.37) with f_J replaced by $f_J 1_{(2J)^c}$ is bounded by $C|J|$. We estimate (2.37) with f_J replaced by $f_J 1_{2J}$ by

$$\left\| \sum_{k-\kappa < k_J} \left| \sum_{n \in \tilde{T}_k} \varphi_{J,k,n} \psi_{J,k} ((f_J 1_{2J}) * (\tilde{\phi}_{J,k-1} - \tilde{\phi}_{J,k})) \right|^2 \right\|_{L^1(J)} \quad .$$

Since the Fourier transforms of the functions $\tilde{\phi}_{J,k-1} - \tilde{\phi}_{J,k}$ are supported on sets with bounded overlap as k varies, we can estimate the last expression by

$$\left(\sup_{k-\kappa < k_J} \|\psi_{J,k}\|_{L^\infty(J)} \right) \|f_J 1_{2J}\|_2^2 \leq C|J| \quad .$$

This completes the proof of Lemma 2.6.2.

2.7 The tree estimate

In this section we prove Lemmata 2.7.1 and 2.7.2, which state estimates on the selected trees.

Lemma 2.7.1 *Let $T \in \mathrm{T}_\mu$ be of type 1. Then*

$$|\Lambda_T(f_1, f_2, f_3)| \leq C|I_T|2^{3\mu} \quad .$$

We prove this lemma. We use the notation of (2.18). Since we only consider one fixed tree T, we write $\phi_{\jmath,k}$ for $\phi_{\jmath,k,T}$. We have

$$\left|\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\prod_{\jmath=1}^{3}\sum_{n\in T_k}\varphi_{\jmath,k,n}(x)\psi_{\jmath,k}(x)\left(f_\jmath * \phi_{\jmath,k}\right)(x)\,dx\right|$$

$$\leq \left(\sup_{k\in\mathbb{Z}}\left\|\sum_{n\in T_k}\varphi_{1,k,n}\psi_{1,k}\left(f_1*\phi_{1,k}\right)\right\|_\infty\right)\prod_{\jmath=2}^{3}\left(\sum_{k\in\mathbb{Z}}\left\|\sum_{n\in T_k}\varphi_{\jmath,k,n}\psi_{\jmath,k}\left(f_\jmath*\phi_{\jmath,k}\right)\right\|_2^2\right)^{\frac{1}{2}}$$

$$\leq C\left(\sup_{(n,k)\in T}\left\|\varphi^*_{1,k,n}\psi^*_{1,k}\left(f_1*\phi_{1,k}\right)\right\|_\infty\right)\prod_{\jmath=2}^{3}\left(\sum_{(k,n)\in T}\left\|\varphi^*_{\jmath,k,n}\psi^*_{\jmath,k}\left(f_\jmath*\phi_{\jmath,k}\right)\right\|_2^2\right)^{\frac{1}{2}} \quad .$$

Each of the last two factors in the last expression is bounded by $C|I_T|^{\frac{1}{2}}2^\mu$ because $T \in \mathrm{T}_\mu$. In order to estimate the first factor we observe that for $(k,n) \in T$ we have

$$\left\|\varphi^*_{1,k,n}\psi^*_{1,k}(f_1*\phi_{1,k})\right\|_\infty \leq \left\|\varphi^*_{1,k,n}\psi^*_{1,k}(f_1*\phi_{1,k})\right\|_2^{\frac{1}{2}}\left\|\left(\varphi^*_{1,k,n}\psi^*_{1,k}U_{-\alpha_1\xi_T}(f_1*\phi_{1,k})\right)'\right\|_2^{\frac{1}{2}} \quad .$$

The first factor on the right hand side is bounded by $C|I_{k,n}|^{\frac{1}{4}}2^{\frac{\mu}{2}}$ because $T \in \mathrm{T}_\mu$. For the second factor on the right hand side we observe

$$\left\|\left(\varphi^*_{1,k,n}\psi^*_{1,k}U_{-\alpha_1\xi_T}\left(f*\phi_{1,k}\right)\right)'\right\|_2$$

$$\leq C\left(N^{-k}\left\|\varphi^*_{1,k,n}\psi^*_{1,k}\left(f*\phi_{1,k}\right)\right\|_2 + \left\|\varphi^*_{1,k,n}\psi^*_{1,k}\left(f*\left(U_{\alpha_1\xi_T}(U_{-\alpha_1\xi_T}(\phi_{1,k}))'\right)\right)\right\|_2\right) \quad .$$

This is bounded by $C|I_{k,n}|^{-\frac{1}{2}}2^\mu$ because $T \in T_\mu$. This completes the proof of Lemma 2.7.1.

Lemma 2.7.2 *Let $T \in \mathrm{T}_\mu$ be of type 2. Then*

$$|\Lambda_T(f_1, f_2, f_3)| \leq C|I_T|2^{\frac{7}{3}\mu} \quad .$$

We prove this lemma. Recall that $\mu < \mu_0$, hence the main point is the decay of $2^{\frac{7}{3}\mu}$ as μ tends to $-\infty$. Again we use the notation of (2.18) and write $\phi_{\jmath,k}$, $\tilde{\phi}_{\jmath,k}$ and $\lambda_{\jmath,k}$ for $\phi_{\jmath,k,T}$, $\tilde{\phi}_{\jmath,k,T}$ and $\lambda_{\jmath,k,T}$. We write for the integrand of (2.18):

$$\sum_{k\in\mathbb{Z}} \prod_{\jmath=1}^{3} \sum_{n\in T_k} \varphi_{\jmath,k,n}\psi_{\jmath,k}\left(f_\jmath * \phi_{\jmath,k}\right)$$

$$= \sum_{k\in\mathbb{Z}} \left(f_1 * \phi_{1,k}\right)\left(f_2 * \phi_{2,k}\right)\left(f_3 * \left(\phi_{3,k} - \lambda_{3,k}\tilde{\phi}_{3,k}\right)\right) \prod_{\jmath=1}^{3} \sum_{n\in T_k} \varphi_{\jmath,k,n}\psi_{\jmath,k} \tag{2.38}$$

$$+ \sum_{k\in\mathbb{Z}} \lambda_{3,k}\left(f_1 * \phi_{1,k}\right)\left(f_2 * \left(\phi_{2,k} - \lambda_{2,k}\tilde{\phi}_{2,k}\right)\right)\left(f_3 * \tilde{\phi}_{3,k}\right) \prod_{\jmath=1}^{3} \sum_{n\in \tilde{T}_k} \varphi_{\jmath,k,n}\psi_{\jmath,k} \tag{2.39}$$

$$+ \sum_{k\in\mathbb{Z}} \left(f_1 * \tilde{\phi}_{1,k}\right)\left(f_2 * \tilde{\phi}_{2,k}\right)\left(f_3 * \tilde{\phi}_{3,k}\right) \prod_{\jmath=1}^{3} \sum_{n\in \tilde{T}_k} \varphi_{\jmath,k,n}\psi_{\jmath,k} \quad . \tag{2.40}$$

Here we have used $\tilde{\phi}_{1,k} = \lambda_{2,k}\lambda_{3,k}\phi_{1,k}$. In (2.39) and (2.40) we were able to replace T_k by $\tilde{T}_k$ because $T_k \neq \tilde{T}_k$ implies $\lambda_{3,k}\phi_{1,k} = 0$. We estimate the integral over (2.38) by

$$\sum_{k,n'\in\mathbb{Z}} \left(\left\| \sum_{n\in T_k} \varphi_{1,k,n}\psi_{1,k}\left(f_1 * \phi_{1,k}\right) \right\|_{L^\infty(I_{k,n'})} \left\| \sum_{n\in T_k} \varphi_{2,k,n}\psi_{2,k}\left(f_2 * \phi_{2,k}\right) \right\|_{L^2(I_{k,n'})} \right.$$

$$\left. \times \left\| \sum_{n\in T_k} \varphi_{3,k,n}\psi_{3,k}\left(f_3 * \left(\phi_{3,k} - \lambda_{3,k}\tilde{\phi}_{3,k}\right)\right) \right\|_{L^2(I_{k,n'})} \right)$$

$$\leq \left(\sum_{k,n'\in\mathbb{Z}} \left\| \sum_{n\in T_k} \varphi_{1,k,n}\psi_{1,k}\left(f_1 * \phi_{1,k}\right) \right\|^2_{L^\infty(I_{k,n'})} N^k \right)^{\frac{1}{2}}$$

$$\times \left(\sup_{k,n'\in\mathbb{Z}} \left\| \sum_{n\in T_k} \varphi_{2,k,n}\psi_{2,k}\left(f_2 * \phi_{2,k}\right) \right\|_{L^2(I_{k,n'})} N^{-\frac{k}{2}} \right)$$

$$\times \left(\sum_{k,n'\in\mathbb{Z}} \left\| \sum_{n\in T_k} \varphi_{3,k,n}\psi_{3,k}\left(f_3 * \left(\phi_{3,k} - \lambda_{3,k}\tilde{\phi}_{3,k}\right)\right) \right\|^2_{L^2(I_{k,n'})} \right)^{\frac{1}{2}}$$

$$\leq C \left(\sum_{(k,n)\in T} \left\| \varphi^*_{1,k,n}\psi^*_{1,k}\left(f_1 * \phi_{1,k}\right) \right\|^2_\infty N^k \right)^{\frac{1}{2}} \left(\sup_{(k,n)\in T} \left\| \varphi^*_{2,k,n}\psi^*_{2,k}\left(f_2 * \phi_{2,k}\right) \right\|_2 N^{-\frac{k}{2}} \right) \tag{2.41}$$

$$\times \left(\sum_{(k,n)\in T} \left\| \varphi^*_{3,k,n}\psi^*_{3,k}\left(f_3 * \left(\phi_{3,k} - \lambda_{3,k}\tilde{\phi}_{3,k}\right)\right) \right\|^2_2 \right)^{\frac{1}{2}}$$

In the last step we have used that

$$\left\| \sum_{n\in T_k} \varphi_{\jmath,k,n}\psi_{\jmath,k}\left(f_\jmath * \phi_{\jmath,k}\right)\right\|_{L^p(I_{k,n'})} \leq C(n'-n'')^{-m}\left\|\varphi^*_{\jmath,k,n''}\psi^*_{\jmath,k}\left(f_\jmath * \phi_{\jmath,k}\right)\right\|_p \quad , \qquad (2.42)$$

where n'' is a number in T_k which minimizes the distance to n'.

The first factor of (2.41) is bounded by

$$\left(\sum_{(k,n)\in T}\left\|\varphi^*_{1,k,n}\psi^*_{1,k}\left(f_1 * \phi_{1,k}\right)\right\|_2^2\right)^{\frac14}\left(\sum_{(k,n)\in T}\left\|\left(\varphi^*_{1,k,n}\psi^*_{1,k}U_{-\alpha_1\xi_T}\left(f_1*\phi_{1,k}\right)\right)'\right\|_2^2 N^{2k}\right)^{\frac14} \quad .$$

This is bounded by $C|I_T|^{\frac12}2^\mu$ because $T\in \mathrm{T}_\mu$. The second factor in (2.41) is bounded by $C2^\mu$ because $T\in\mathrm{T}_\mu$. We estimate the third factor of (2.41) by:

$$\sum_{z\geq 0}\left(\sum_{(k,n)\in T}\left\|\varphi^*_{3,k,n}\psi^*_{3,k}\left(f_3*(\phi_{3,k}-\lambda_{3,k}\tilde\phi_{3,k})*\theta^+_{\jmath,k+z}\right)\right\|_2^2\right)^{\frac12}$$

$$+\sum_{z\geq 0}\left(\sum_{(k,n)\in T}\left\|\varphi^*_{3,k,n}\psi^*_{3,k}\left(f_3*(\phi_{3,k}-\lambda_{3,k}\tilde\phi_{3,k})*\theta^-_{\jmath,k+z}\right)\right\|_2^2\right)^{\frac12} \quad .$$

Since $T\in\mathrm{T}_\mu$, this is bounded by

$$\sum_{z\geq 0}2^{-\frac{z}{4}}C|I_T|^{\frac12}2^\mu \leq C|I_T|^{\frac12}2^\mu \quad .$$

The term (2.39) is estimated similarly as the term (2.38), hence it remains to estimate the term (2.40).

Define

$$f_{\jmath,k} := \sum_{n\in\tilde T_k}\varphi_{\jmath,k,n}\psi_{\jmath,k}(f_\jmath * \tilde\phi_{\jmath,k}) \quad .$$

Then we can write for (2.40):

$$\sum_{k\in\mathbb{Z}} f_{1,k}f_{2,k+\kappa+1}f_{3,k+\kappa+1} \qquad (2.43)$$

$$+\sum_{k\in\mathbb{Z}} f_{1,k}\sum_{\tilde k=0}^{\kappa}\left(f_{2,k+\tilde k}f_{3,k+\tilde k}-f_{2,k+\tilde k+1}f_{3,k+\tilde k+1}\right) \quad . \qquad (2.44)$$

First we consider the term (2.43). The Fourier transform of $f_{1,k}$ is supported in

$$[-\infty,\alpha_1\xi_T-\epsilon^{1.5}N^{-k}] \quad ,$$

whereas for $\jmath\in\{2,3\}$ the Fourier transform of $f_{\jmath,k+\kappa+1}$ is supported in

$$[\alpha_\jmath\xi_T-\epsilon^{1.9}N^{-k},\alpha_\jmath\xi_T+\epsilon^{1.9}N^{-k}] \quad .$$

Since $\alpha \cdot \gamma = 0$, the Fourier transform of the product of these three functions is supported away from 0. Hence the integral over (2.43) vanishes.

We consider (2.44) and write for it:

$$\sum_k \left(\sum_{\tilde{k}=0}^{\kappa} f_{1,k-\tilde{k}} \right) (f_{2,k} f_{3,k} - f_{2,k+1} f_{3,k+1})$$

$$= \sum_k \left(\sum_{\tilde{k}=0}^{\kappa} f_{1,k-\tilde{k}} \right) (f_{2,k} - f_{2,k+1}) (f_{3,k} - f_{3,k+1}) \tag{2.45}$$

$$+ \sum_k \left(\sum_{\tilde{k}=0}^{\kappa} f_{1,k-\tilde{k}} \right) (f_{2,k+1} - f_{2,k+2}) (f_{3,k} - f_{3,k+1}) \tag{2.46}$$

$$+ \sum_k \left(\sum_{\tilde{k}=0}^{\kappa} f_{1,k-\tilde{k}} \right) (f_{2,k} - f_{2,k+1}) (f_{3,k+1} - f_{3,k+2}) \tag{2.47}$$

$$+ \sum_k \left(\sum_{\tilde{k}=0}^{\kappa} f_{1,k-\tilde{k}} \right) (f_{2,k+2}) (f_{3,k} - f_{3,k+1}) \tag{2.48}$$

$$+ \sum_k \left(\sum_{\tilde{k}=0}^{\kappa} f_{1,k-\tilde{k}} \right) (f_{2,k} - f_{2,k+1}) (f_{3,k+2}) \quad . \tag{2.49}$$

Each of the three terms (2.45),(2.46), and (2.47) is estimated pointwise by

$$\sup_k \left| \sum_{\tilde{k}=0}^{\kappa} f_{1,k-\tilde{k}} \right| \prod_{j=2}^{3} \left(\sum_k |f_{j,k} - f_{j,k+1}|^2 \right)^{\frac{1}{2}} \quad .$$

The integral over this expression is estimated by

$$\left\| \sup_k \left| \sum_{\tilde{k}=0}^{\kappa} f_{1,k-\tilde{k}} \right| \right\|_4 \prod_{j=2}^{3} \left\| \left(\sum_k |f_{j,k} - f_{j,k+1}|^2 \right)^{\frac{1}{2}} \right\|_{\frac{8}{3}} \quad . \tag{2.50}$$

Consider the first factor in (2.50). The supports of the Fourier transforms of the functions $f_{1,k}$ are contained in intervals which form a lacunary family as k varies. Hence we have

$$\left\| \sup_k \left| \sum_{\tilde{k}=0}^{\kappa} f_{1,k-\tilde{k}} \right| \right\|_4 \leq C \left\| \sum_{k \in \mathbb{Z}} f_{1,k} \right\|_4 \quad . \tag{2.51}$$

We estimate the right hand side of this inequality by interpolation between the two estimates

$$\left\| \sum_{k \in \mathbb{Z}} f_{1,k} \right\|_2 \leq C |I_T|^{\frac{1}{2}} 2^{\mu} \quad ,$$

$$\left\| U_{-\alpha_1 \xi_T} \sum_{k \in \mathbb{Z}} f_{1,k} \right\|_{BMO} \leq C 2^{\mu} \quad .$$

The L^2 estimate is seen as follows:

$$\left\| \sum_{(k,n)\in\tilde{T}_k} \varphi_{1,k,n}\psi_{1,k}(f_1 * \tilde{\phi}_{1,k}) \right\|_2^2 = \left\| \sum_{(k,n)\in T_k} \varphi_{1,k,n}\psi_{1,k}(f_1 * \tilde{\phi}_{1,k}) \right\|_2^2 \tag{2.52}$$

$$= \sum_{k\in\mathbb{Z}} \left\| \sum_{n\in T_k} \varphi_{1,k,n}\psi_{1,k}(f_1 * \tilde{\phi}_{1,k}) \right\|_2^2 \leq C \sum_{(k,n)\in T} \left\| \varphi^*_{1,k,n}\psi^*_{1,k}(f_1 * \phi_{1,k}) \right\|_2^2 \leq C|I_T|^{\frac{1}{2}} 2^{\mu} \quad ,$$

where the last inequality is true because $T \in \mathrm{T}_\mu$.

We prove the (N-adic) BMO estimate. For simplicity of notation assume $\xi_T = 0$, the easy modifications needed for general ξ_T are left to the reader. Fix an $N - adic$ interval J of length N^{k_J} and let T_J be the set of $(k,n) \in T$ such that $I_{k,n} \subset J$. Observe that $(T_J, 2, J, \xi_T)$ is again a tree, and its size is less than μ because T would not have been selected otherwise. We have

$$|J|^{-1} \inf_c \int_J \left| \sum\nolimits_{(k,n)\in\tilde{T}} \varphi_{1,k,n}(x)\psi_{1,k}(f_1 * \tilde{\phi}_{1,k})(x) - c \right| dx$$

$$\leq |J|^{-\frac{1}{2}} \left\| \sum\nolimits_{(k,n)\in T_J} \varphi_{1,k,n}\psi_{1,k}(f_1 * \tilde{\phi}_{1,k}) \right\|_2 \tag{2.53}$$

$$+|J|^{-\frac{1}{2}} \left\| \sum\nolimits_{(k,n)\in T\setminus T_J : k\leq k_J} \varphi_{1,k,n}\psi_{1,k}(f_1 * \tilde{\phi}_{1,k}) \right\|_{L^2(J)} \tag{2.54}$$

$$+|J|^{-1} \inf_c \int_J \left| \sum\nolimits_{(k,n)\in T : k > k_J} \varphi_{1,k,n}(x)\psi_{1,k}(x)(f_1 * \tilde{\phi}_{1,k})(x) - c \right| dx \quad . \tag{2.55}$$

The first summand (2.53) is bounded by $C2^\mu$ by the same arguments as in (2.52). For $(k,n) \in T \setminus T_J$ with $k \leq k_J$ we calculate:

$$\left\| \varphi_{1,k,n}\psi_{1,k}\left(f * \tilde{\phi}_{1,k}\right) \right\|_{L^2(J)}$$

$$\leq C \left(1 + |I_{k,n}|^{-1}\mathrm{dist}(J, I_{k,n})\right)^{-m} \left\| \varphi^*_{1,k,n}\psi^*_{1,k}\left(f * \phi_{1,k}\right) \right\|_2$$

$$\leq C \left(1 + |I_{k,n}|^{-1}\mathrm{dist}(J, I_{k,n})\right)^{-m} N^{\frac{k}{2}} 2^\mu \quad .$$

Again the last inequality follows from $T \in \mathrm{T}_\mu$. Summing over $(k,n) \in T \setminus T_J$ with $k \leq k_J$ proves that (2.54) is bounded by $C2^\mu$. For $(k,n) \in T$ with $k > k_J$ we calculate:

$$|J|^{-1} \inf_c \int_J \left| \varphi_{1,k,n}\psi_{1,k}\left(f * \phi_{1,k}\right) - c \right| dx$$

$$\leq C|J|^{\frac{1}{2}} \left\| \left(\varphi_{1,k,n}\psi_{1,k}\left(f * \phi_{1,k}\right)\right)' \right\|_{L^2(J)}$$

$$\leq C \left(N^{-k} \left\| {\varphi^*_{1,k,n}}^2 \psi^*_{1,k}\left(f * \phi_{1,k}\right) \right\|_{L^2(J)} + \left\| {\varphi^*_{1,k,n}}^2 \psi^*_{1,k}\left(f * (\phi_{1,k})'\right) \right\|_{L^2(J)} \right)$$

$$\leq C \left(1 + |I_{k,n}|^{-1}\mathrm{dist}(J, I_{k,n})\right)^{-m} \left(N^{-k} \left\| \varphi^*_{1,k,n}\psi^*_{1,k}\left(f * \phi_{1,k}\right) \right\|_2 + \left\| \varphi^*_{1,k,n}\psi^*_{1,k}\left(f * (\phi_{1,k})'\right) \right\|_2 \right)$$

$$\leq C\left(1+|I_{k,n}|^{-1}\mathrm{dist}(J,I_{k,n})\right)^{-m}N^{-\frac{k}{2}}2^{\mu} \quad .$$

Summing over $(k,n)\in T$ with $k>k_J$ proves that (2.55) is bounded by $C2^{\mu}$. This completes the proof of the N-adic BMO estimate and therefore proves that the first factor in (2.50) is bounded by $C|I_T|^{\frac{1}{4}}2^{\mu}$.

We consider the last two factors in (2.50). Let $\jmath\in\{2,3\}$. We have

$$\left(\sum_k |f_{\jmath,k}-f_{\jmath,k+1}|^2\right)^{\frac{1}{2}}$$

$$\leq\left(\sum_k\left|\sum_{n\in\tilde{T}_{k+1}}\varphi_{\jmath,k+1,n}\psi_{\jmath,k+1}\left(f_\jmath*(\tilde{\phi}_{\jmath,k}-\tilde{\phi}_{\jmath,k+1})\right)\right|^2\right)^{\frac{1}{2}} \tag{2.56}$$

$$+\left(\sum_k\left|\sum_{n\in\tilde{T}_{k+1}}\varphi_{\jmath,k+1,n}\left(\psi_{\jmath,k}-\psi_{\jmath,k+1}\right)\left(f_\jmath*\tilde{\phi}_{\jmath,k}\right)\right|^2\right)^{\frac{1}{2}} \tag{2.57}$$

$$+\left(\sum_k\left|\left(\sum_{n\in\tilde{T}_k}\varphi_{\jmath,k,n}-\sum_{n\in\tilde{T}_{k+1}}\varphi_{\jmath,k+1}\right)\psi_{\jmath,k}(f_\jmath*\tilde{\phi}_{\jmath,k})\right|^2\right)^{\frac{1}{2}} \tag{2.58}$$

We estimate the L^2- norm of the term (2.56) by

$$\left\|\left(\sum_k\left|\sum_{n\in\tilde{T}_k}\varphi_{\jmath,k,n}\psi_{\jmath,k}\left(f_\jmath*(\tilde{\phi}_{\jmath,k-1}-\tilde{\phi}_{\jmath,k})\right)\right|^2\right)^{\frac{1}{2}}\right\|_2$$

$$\leq\left(\sum_{(k,n)\in\tilde{T}}\left\|\varphi^*_{\jmath,k,n}\psi^*_{\jmath,k}\left(f_\jmath*(\theta^+_{\jmath,k})\right)\right\|_2^2\right)^{\frac{1}{2}}$$

$$\left(\sum_{(k,n)\in\tilde{T}}\left\|\varphi^*_{\jmath,k,n}\psi^*_{\jmath,k}\left(f_\jmath*(\theta^-_{\jmath,k})\right)\right\|_2^2\right)^{\frac{1}{2}} \quad .$$

This is bounded by $C|I_T|^{\frac{1}{2}}2^{\mu}$ because $T\in\mathrm{T}_{\mu}$. The BMO- norm of (2.56) is bounded by C by Lemma (2.6.2). Interpolation shows that the $L^{\frac{8}{3}}$- norm of (2.56) is bounded by $C|I_T|^{\frac{3}{8}}2^{\frac{3}{4}\mu}$.

In order to estimate (2.57), observe that $|\psi_{\jmath,k}-\psi_{\jmath,k+1}|$ is bounded by

$$\left|1_{E_{k-\kappa}\setminus E_{k-\kappa+1}}*D^1_{N^{k+2-\kappa}}\psi\right|+\left|1_{(E_{k-\kappa+1})^c}*\left(D^1_{N^{k+2-\kappa}}\psi-D^1_{N^{k+3-\kappa}}\psi\right)\right|$$

$$C\sum_{J\in\Delta_{k-\kappa}}{\rho_{k-\kappa,J}}^2 \quad .$$

Let V_k^+ be the set of integers $n\in\tilde{T}_k$ such that $n+1\notin\tilde{T}_k$, and let V_k^- be the set of integers $n\in\tilde{T}_k$ such that $n-1\notin\tilde{T}_k$. Observe that

$$\left|\sum_{n\in\tilde{T}_{k+1}}\varphi_{\jmath,k+1,n}\left(\psi_{\jmath,k}-\psi_{\jmath,k+1}\right)\right|$$

$$\leq C \sum_{n\in\Delta_{k-\kappa,T}} \rho_{k-\kappa,J}^2 + C \sum_{n\in V_{k+1}^+\cup V_{k+1}^-} {\varphi_{J,k+1,n}^*}^2 \psi_{J,k}^* \quad . \tag{2.59}$$

This gives for the L^2 norm of (2.57) the upper bound:

$$\leq C\left(\sum_k \sum_{J\in\Delta_{k-\kappa,T}} \left\|\rho_{k-\kappa,J}(f_J * \tilde{\phi}_{J,k})\right\|_2^2\right)^{\frac{1}{2}} \tag{2.60}$$

$$+\left(\sum_k \sum_{n\in V_{k+1}^+\cup V_{k+1}^-} \left\|\varphi_{J,k+1,n}^*\psi_{J,k}^*(f_J * \tilde{\phi}_{J,k})\right\|_2^2\right)^{\frac{1}{2}} \tag{2.61}$$

The term (2.60) is bounded by $C2^\mu|I_T|^{\frac{1}{2}}$ because $T\in \mathrm{T}_\mu$. The term (2.61) is bounded by

$$C2^\mu\left(\sum_k \sum_{n\in V_{k+1}^+} N^k\right)^{\frac{1}{2}} + C2^\mu\left(\sum_k \sum_{n\in V_{k+1}^-} N^k\right)^{\frac{1}{2}} \quad ,$$

because $T\in \mathrm{T}_\mu$. By symmetry it suffices to consider the first term. Consider the set of intervals $\frac{1}{3}I_{k,n+1}$ where $k\in\mathbb{Z}$ and $n\in V_k$. Observe that these intervals are pairwise disjoint by convexity of the tree T and contained in $3I_T$. Hence we have

$$\sum_{(n,k):n\in V_k^+} \frac{1}{3}N^k \leq 3|I_T| \quad .$$

This proves that (2.61) is bounded by $C2^\mu|I_T|^{\frac{1}{2}}$. Hence the L^2 norm of (2.57) is bounded by $C2^\mu|I_T|^{\frac{1}{2}}$.

With Lemma 2.6.2 we estimate (2.57) pointwise by

$$\left(\sum_{k\in\mathbb{Z}}\left|\sum_{n\in\tilde{T}_{k+1}} \varphi_{J,k+1,n}\left|\psi_{J,k}-\psi_{J,k+1}\right|^{\frac{1}{2}}\psi_{J,k}^*(f_J * \tilde{\phi}_{J,k})\right|^2\right)^{\frac{1}{2}}$$

$$\leq C\left(\sum_{k\in\mathbb{Z}}\left|\sum_{n\in\tilde{T}_{k+1}} \varphi_{J,k+1,n}\left|\psi_{J,k}-\psi_{J,k+1}\right|^{\frac{1}{2}}\right|^2\right)^{\frac{1}{2}}$$

$$\leq C\sum_k \sum_{J\in\Delta_{k-\kappa,I_T}} \rho_{k-\kappa,J} + C\sum_k \sum_{n\in V_{k+1}^+\cup V_{k+1}^-} \varphi_{1,k+1,n}^*$$

The L^1 norm of the last expression is bounded by $C|I_T|$, whereas the BMO norm is bounded by C. Hence the L^8 norm is bounded by $C|I_T|^{\frac{1}{8}}$. Hence the $L^{\frac{8}{3}}$- norm of (2.57) is bounded by $C|I_T|^{\frac{3}{8}}2^{\frac{2}{3}\mu}$.

We turn to the term (2.58). Observe that $\tilde{T}_k\subset\tilde{T}_{k+1}$ unless $k=k_T$. We have $\tilde{T}_{k_T}=\{k_T,n_T\}$ and $\tilde{T}_{k_T+1}=\emptyset$. Hence we can estimate the square of the L^2 norm of (2.58) by

$$\sum_k \sum_{n'\in\mathbb{Z}} \left\|\left(\sum_{n\in\tilde{T}_k}\varphi_{J,k,n} - \sum_{n\in\tilde{T}_{k+1}}\varphi_{J,k+1}\right)\psi_{J,k}(f_J * \tilde{\phi}_{J,k})\right\|_{L^2(I_{k,n'})}^2 \tag{2.62}$$

$$\leq C \sum_{k\neq k_T} \sum_{n'\in\mathbb{Z}} \left\| \left| \sum_{n\in\tilde{T}_k} \varphi_{\jmath,k,n} - \sum_{n\in\tilde{T}_{k+1}} \varphi_{\jmath,k+1} \right|^{\frac{1}{2}} \right\|_{L^\infty(I_{k,n'})} \left\| \sum_{n\in\tilde{T}_{k+1}} {\varphi^*_{\jmath,k+1,n}}^2 \psi^*_{\jmath,k}(f_\jmath * \tilde{\phi}_{\jmath,k}) \right\|^2_{L^2(I_{k,n'})} \tag{2.63}$$

$$+ \left\| \varphi_{\jmath,k_T,n_T} \psi_{\jmath,k}(f_\jmath * \tilde{\phi}_{\jmath,k}) \right\|^2_{L^2} \quad . \tag{2.64}$$

The term (2.64) is bounded by $C|I_T|2^{2\mu}$ because $T \in \mathrm{T}_\mu$. The second factor in each summand of (2.63) is estimated by

$$\left\| \sum_{n\in\tilde{T}_{k+1}} {\varphi^*_{\jmath,k+1,n}}^2 \psi^*_{\jmath,k}(f_\jmath * \tilde{\phi}_{\jmath,k}) \right\|^2_{L^2(I_{k,n'})} \leq C \sup_{n\in\tilde{T}_{k+1}} \left\| \varphi^*_{\jmath,k+1,n} \psi^*_{\jmath,k}(f_\jmath * \tilde{\phi}_{\jmath,k}) \right\|^2_2 \quad , \tag{2.65}$$

which is bounded by $C2^{2\mu}N^k$ because $T \in \mathrm{T}_\mu$. Hence we can estimate (2.63) by

$$C2^{2\mu} \sum_{k\neq k_T} \sum_{n'\in\mathbb{Z}} N^k \left\| \left| \sum_{n\in\tilde{T}_k} \varphi_{\jmath,k,n} - \sum_{n\in\tilde{T}_{k+1}} \varphi_{\jmath,k+1} \right|^{\frac{1}{2}} \right\|_{L^\infty(I_{k,n'})} \quad .$$

Let W_k be the set of integers n such that $I_{k,n} \in \tilde{T}_k$ but $I_{k,n}$ is not covered by intervals of the type $I_{k-1,n'}$ with $n' \in \tilde{T}_{k-1}$. Then the previously displayed expression is bounded by

$$C2^{2\mu} \sum_{k\neq k_T, n'\in\mathbb{Z}} N^k \left(\left\| \sum_{n\in W_{k+1}} \varphi^*_{\jmath,k+1,n} \right\|_{L^\infty(I_{k,n'})} + \left\| \sum_{n\in {V^+}_{k+1}\cup {V^-}_{k+1}} \varphi^*_{\jmath,k+1,n} \right\|_{L^\infty(I_{k,n'})} \right) \quad . \tag{2.66}$$

The term involving the sets W_{k+1} is estimated by

$$C2^{2\mu} \sum_{k\neq k_T} \sum_{n\in W_{k+1}} \sum_{n'\in\mathbb{Z}} N^k \left\| \varphi^*_{\jmath,k+1,n} \right\|_{L^\infty(I_{k,n'})}$$

$$\leq C2^{2\mu} \sum_{k\neq k_T} \sum_{n\in W_{k+1}} N^k \quad .$$

To each $k \neq k_T$ and $n \in W_{k+1}$ pick an interval $I_{k,n'}$ with $I_{k,n'} \subset I_{k+1,n}$ and $I_{k,n'} \notin \tilde{T}_k$. By convexity of the tree these intervals are pairwise disjoint as k runs through $\mathbb{Z}$ and n through W_k. Hence the previously displayed expression is bounded by $C|I_T|2^{2\mu}$. The terms in (2.66) involving the sets V^+_{k+1} and V^-_{k+1} are estimated similarly.

We estimate (2.58) pointwise by

$$\left(\sum_k \left| \left(\sum_{n\in\tilde{T}_k} \varphi_{\jmath,k,n} - \sum_{n\in\tilde{T}_{k+1}} \varphi_{\jmath,k+1,n} \right) \left\| \psi^*_{\jmath,k}(f_\jmath * \tilde{\phi}_{\jmath,k}) \right\|_\infty \right|^2 \right)^{\frac{1}{2}}$$

$$\leq C \left| \varphi_{\jmath,k_T,n_T} \right| + C \sum_{k\neq k_T} \left| \sum_{n\in W_{k+1}\cup V^+_{k+1}\cup V^-_{k+1}} \varphi^*_{\jmath,k+1,n} \right| \quad .$$

The function on the right hand side of this inequality is easily seen to have L^1 norm bounded by $C|I_T|$ and BMO norm bounded by C. By the same argument as for the term

(2.57) we obtain that the $L^{\frac{8}{3}}$ norm of (2.58) is bounded by $C|I_T|^{\frac{3}{8}}2^{\frac{2}{3}\mu}$. This completes the discussion of (2.50).

We turn to the term (2.48), which is equal to

$$\sum_k \left(\sum_{\tilde{k}=0}^{\kappa-3} f_{1,k-\tilde{k}}\right)(f_{2,k+2}) \sum_{n\in\tilde{T}_{k+1}} \varphi_{3,k+1,n}\psi_{3,k+1}(f_3 * (\tilde{\phi}_{3,k}-\tilde{\phi}_{3,k+1})) \tag{2.67}$$

$$+\sum_k \left(\sum_{\tilde{k}=0}^{\kappa-3} f_{1,k-\tilde{k}}\right)(f_{2,k+2}) \sum_{n\in\tilde{T}_{k+1}} \varphi_{3,k+1,n}(\psi_{3,k}-\psi_{3,k+1})(f_3 * \tilde{\phi}_{3,k}) \tag{2.68}$$

$$+\sum_k \left(\sum_{\tilde{k}=0}^{\kappa-3} f_{1,k-\tilde{k}}\right)(f_{2,k+2}) \left(\sum_{n\in\tilde{T}_k} \varphi_{3,k,n} - \sum_{n\in\tilde{T}_{k+1}} \varphi_{3,k+1,n}\right)\psi_{3,k}(f_3 * \tilde{\phi}_{3,k}) \tag{2.69}$$

$$+\sum_k \left(\sum_{\tilde{k}=\kappa-2}^{\kappa} f_{1,k-\tilde{k}}\right)(f_{2,k+2})(f_{3,k}-f_{3,k+1}) \quad . \tag{2.70}$$

The Fourier transforms of the functions

$$\sum_{\tilde{k}=0}^{\kappa-3} f_{1,k-\tilde{k}}\ ,\quad f_{2,k+2}\ ,\quad \sum_{n\in\tilde{T}_{k+1}} \varphi_{3,k+1,n}\psi_{3,k+1}(f_3 * (\tilde{\phi}_{3,k}-\tilde{\phi}_{3,k+1}))$$

are supported in the sets

$$[\alpha_1\xi_T - \epsilon^{0.9}N^{-k+\kappa-2}, \alpha_1\xi_T + \epsilon^{0.9}N^{-k+\kappa-2}] \quad ,$$

$$\left[\alpha_2\xi_T - \epsilon^{1.9}|\alpha_2|N^{-k-3}, \alpha_2\xi_T + \epsilon^{1.9}|\alpha_2|N^{-k-3}\right] \quad ,$$

$$\mathbb{R}\setminus\left[\alpha_3\xi_T - \epsilon^{2.1}|\alpha_3|N^{-k-2}, \alpha_3\xi_T + \epsilon^{2.1}|\alpha_3|N^{-k-2}\right] \quad .$$

Therefore it is easy to see that the Fourier transform of (2.67) vanishes at the origin, hence the integral over (2.67) is zero.

The integral over (2.68) is estimated by

$$\left\|\sup_k\left|\sum_{\tilde{k}=0}^{\kappa-3} f_{1,k-\tilde{k}}\right|\right\|_4 \left\|\sum_k\left|f_{2,k+2}\sum_{n\in\tilde{T}_{k+1}}\varphi_{3,k+1,n}(\psi_{3,k}-\psi_{3,k+1})(f_3*\tilde{\phi}_{3,k})\right|\right\|_{\frac{4}{3}} \quad .$$

The first factor was seen before to be bounded by $C|I_T|^{\frac{1}{4}}2^{\mu}$. The second factor is estimated by

$$\left\|\sum_k\left|(f_2*\tilde{\phi}_{2,k+2})\left(\prod_{j=2}^{3}\left|\sum_{n\in\tilde{T}_{k+1}}\varphi_{j,k+1,n}(\psi_{j,k}-\psi_{j,k+1})\right|^{\frac{1}{2}}\right)(f_3*\tilde{\phi}_{3,k})\right|\right\|_{\frac{4}{3}}$$

$$\leq \left\|\left(\sum_k\left|\sum_{n\in\tilde{T}_{k+1}}\varphi_{2,k+1,n}(\psi_{2,k}-\psi_{2,k+1})\right|\left|f_2*\tilde{\phi}_{2,k+2}\right|^2\right)^{\frac{1}{2}}\right\|_{\frac{8}{3}}$$

$$\times \left\| \left(\sum\nolimits_k \left| \sum\nolimits_{n \in \tilde{T}_{k+1}} \varphi_{3,k+1,n} (\psi_{3,k} - \psi_{3,k+1}) \right| \left| f_3 * \tilde{\phi}_{3,k} \right|^2 \right)^{\frac{1}{2}} \right\|_{\frac{8}{3}}$$

Here each factor is estimated by $C|I_T|^{\frac{3}{8}} 2^{\frac{2}{3}\mu}$ with the arguments used in the discussion of (2.57). This completes the discussion of the term (2.68).

The integral over (2.69) is estimated by

$$\left\| \sup_k \left| \sum_{\tilde{k}=0}^{\kappa-3} f_{1,k-\tilde{k}} \right| \right\|_4 \left\| \sum_k \left| f_{2,k+2} \left(\sum_{n \in \tilde{T}_{k+1}} \varphi_{3,k,n} - \sum_{n \in \tilde{T}_{k+1}} \varphi_{3,k+1,n} \right) \psi_{3,k} (f_3 * \tilde{\phi}_{3,k}) \right| \right\|_{\frac{4}{3}} . \quad (2.71)$$

The first factor is estimated as before. For the second factor, we again consider only the sum over $k \neq k_T$, the term $k = k_T$ being easy and left to the reader. We calculate for the L^1- norm:

$$\sum_{k \neq k_T} \sum_{n' \in \mathbb{Z}} \left\| f_{2,k+2} \left(\sum_{n \in \tilde{T}_{k+1}} \varphi_{3,k,n} - \sum_{n \in \tilde{T}_{k+1}} \varphi_{3,k+1,n} \right) \psi_{3,k} (f_3 * \tilde{\phi}_{3,k}) \right\|_{L^1(I_{k,n'})}$$

$$\leq \sum_{k,n' \in \mathbb{Z}} \|f_{2,k+2}\|_{L^2(I_{k,n'})} \left\| \sum_{n \in \tilde{T}_{k+1}} {\varphi^*_{3,k+1,n}}^2 \psi^*_{3,k} (f_3 * \tilde{\phi}_{3,k}) \right\|_{L^2(I_{k,n'})}$$

$$\times \left\| \left| \sum_{n \in \tilde{T}_{k+1}} \varphi_{3,k,n} - \sum_{n \in \tilde{T}_{k+1}} \varphi_{3,k+1,n} \right|^{\frac{1}{2}} \right\|_{L^\infty(I_{k,n'})}$$

As in (2.65), the first two factors in each summand are bounded by $C|I_T|^{\frac{1}{2}} 2^\mu$. Hence the whole expression is bounded by $C 2^{2\mu} |I_T|$ by the same arguments as used for the term (2.58).

Moreover we can prove a pointwise estimate for the function in the second factor of (2.71) as in the argument used to estimate (2.58), and thereby concluding the appropriate bound for (2.71).

In the term (2.70), the index $\tilde{k}$ runs through three values. Each of the three summands can be estimated separately. We have for $\tilde{k} \in \{0, 1, 2\}$:

$$\sum_k \left| f_{1,k-\tilde{k}} f_{2,k+2} (f_{3,k} - f_{3,k+1}) \right|$$

$$\leq \sum_k \left| \left(\sum_{n \in \tilde{T}_{k-\tilde{k}}} {\varphi^*_{1,k-\tilde{k},n}}^2 \right)^2 \psi_{1,k-\tilde{k}} (f_1 * \phi_{1,k-\tilde{k}}) \psi_{2,k+2} (f_2 * \phi_{2,k+2}) (f_{3,k} - f_{3,k+1}) \right|$$

The integral over this expression is bounded by

$$\leq \left(\sum_k \left\| \sum_{n \in \tilde{T}_k} {\varphi^*_{1,k,n}}^2 \psi_{1,k} (f_1 * \phi_{1,k}) \right\|_2^2 \right)^{\frac{1}{2}}$$

$$\times \left(\sup_k \left\| \sum_{n \in \tilde{T}_k} {\varphi^*_{1,k,n}}^2 \psi_{2,k+\tilde{k}+2} (f_2 * \phi_{2,k+\tilde{k}+2}) \right\|_\infty \right) \left(\sum_k \|f_{3,k} - f_{3,k+1}\|_2^2 \right)^{\frac{1}{2}}$$

The first and the last factor have been discussed before, they are both bounded by $C|I_T|^{\frac{1}{2}} 2^\mu$. The second factor is estimated by

$$\sup_{(k,n) \in \tilde{T}_k} \left\| \varphi^*_{1,k,n} \psi_{2,k+\tilde{k}+2} (f_2 * \phi_{2,k+\tilde{k}+2}) \right\|_\infty$$

$$\leq \sup_{(k,n) \in \tilde{T}_k} \left\| \varphi^*_{1,k,n} \psi_{2,k+\tilde{k}+2} (f_2 * \phi_{2,k+\tilde{k}+2}) \right\|_2^{\frac{1}{2}} \left\| (\varphi^*_{1,k,n} \psi_{2,k+\tilde{k}+2} U_{-\alpha_2 \xi_T} (f_2 * \phi_{2,k+\tilde{k}+2}))' \right\|_2^{\frac{1}{2}} \quad .$$

This is bounded by $C 2^\mu$, which completes the discussion of the term (2.48). The term (2.49) is estimated in the same way as (2.48).

This completes the proof of Lemma 2.7.2.

2.8 The tree estimate again

Lemma 2.8.1 *We have for all $0 \leq \nu < r$:*

$$|\Lambda_{\mathbf{P}^\nu}(f_1, f_2, f_3) - \Lambda_{\mathbf{P}^{\nu+1}}(f_1, f_2, f_3)| \leq C |I_\nu| 2^{\frac{7}{3}\mu_\nu} \quad .$$

We prove this lemma. Fix ν. Recall that $\mathbf{P}^\nu \setminus \mathbf{P}^{\nu+1} = T^\nu$. Hence for each $k \in \mathbb{Z}$ there is at most on l such that $\mathbf{P}^\nu_{k,l} \setminus \mathbf{P}^{\nu+1}_{k,l} \neq \emptyset$. Define $\phi_{j,k} = \phi_{j,k,l}$ if $T^\nu_{k,l} \neq \emptyset$ and $\phi_{j,k} = 0$ otherwise. Define $\mathbf{P}^\nu_k := \mathbf{P}^\nu_{k,l}$ and $\mathbf{P}^{\nu+1}_k := \mathbf{P}^{\nu+1}_{k,l}$ if $T^\nu_{k,l} \neq \emptyset$, and define $\mathbf{P}^\nu_k := \mathbf{P}^{\nu+1}_k := \emptyset$ otherwise. Observe that we have for all $k \in \mathbb{Z}$:

$$\left| \prod_{j=1}^3 \sum_{n \in \mathbf{P}^\nu_k} \varphi_{j,k,n} - \prod_{j=1}^3 \sum_{n \in \mathbf{P}^{\nu+1}_k} \varphi_{j,k,n} \right| \leq$$

$$\left(\prod_{j=1}^3 \sum_{n \in T^\nu_k} \varphi_{j,k,n} \right) + \left(\left| 1_{I_\nu} * D^1_{N^k}(\varphi^{*\prime}) \right| \prod_{j=1}^3 \sum_{n \in \mathbf{P}^\nu_k} {\varphi^*_{j,k,n}}^2 \right) \quad .$$

Hence we can estimate the left hand side of the inequality in the lemma by

$$|\Lambda_{T^\nu}(f_1, f_2, f_3)| \tag{2.72}$$

$$+ C \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} \left| \left| 1_{I^\nu} * D^1_{N^k}(\varphi^{*\prime}) \right| \prod_{j=1}^3 \sum_{n \in \mathbf{P}^\nu_k} {\varphi^*_{j,k,n}}^2 \psi_{j,k}(x) \left(f_j * \phi_{j,k} \right)(x) \right| dx \tag{2.73}$$

The term (2.72) satisfies the appropriate bound by Lemmata 2.7.1 and 2.7.2. Fix k and consider the corresponding term in (2.73). We have

$$\int_{\mathbb{R}} \left| \left| 1_{I^\nu} * D^1_{N^k}(\varphi^{*\prime}) \right| \prod_{j=1}^3 \sum_{n \in \mathbf{P}^\nu_k} {\varphi^*_{j,k,n}}^2 \psi_{j,k}(x) \left(f_j * \phi_{j,k} \right)(x) \right| dx$$

$$\leq \sum_{n'\in\mathbb{Z}} \left\|1_{I^\nu} * D^1_{N^k}(\varphi^{*\prime})\right\|_{L^\infty(I_{n',k})} \left\|\prod_{\jmath=1}^{3} \sum_{n\in\mathbf{P}^\nu_k} {\varphi^*_{\jmath,k,n}}^2 \psi_{\jmath,k}(x)\,(f_\jmath * \phi_{\jmath,k})(x)\right\|_{L^1(I_{n',k})}$$

The first factor is bounded by

$$C\left(1+N^{-k}\mathrm{dist}(n',\partial I_\nu)\right)^{-m} \quad .$$

We estimate the second factor by Hölder. Observe that for $\jmath \in \{2,3\}$:

$$\left\|\sum_{n\in\mathbf{P}^\nu_k} {\varphi^*_{\jmath,k,n}}^2 \psi^*_{\jmath,k}(x)\,(f_\jmath * \phi_{\jmath,k})(x)\right\|_{L^2(I_{n',k})}$$

$$\leq C\left\|\varphi^*_{\jmath,k,n''}\psi^*_{\jmath,k}(x)\,(f_\jmath * \phi_{\jmath,k})(x)\right\|_2 \leq C2^{\mu_\nu}N^{\frac{k}{2}}$$

Here n'' is the integer in $\mathbf{P}^\nu_k$ which minimizes the distance to n', and the last inequality followed from $T \in \mathrm{T}_{\mu_\nu}$. Similarly we have

$$\left\|\sum_{n\in\mathbf{P}^\nu_k} {\varphi^*_{1,k,n}}^2 \psi_{1,k}(x)\,(f_1 * \phi_{1,k})(x)\right\|_{L^\infty(I_{n',k})}$$

$$\leq C\left\|\varphi^*_{1,k,n''}\psi_{1,k}(x)\,(f_1 * \phi_{1,k})(x)\right\|_\infty \leq C2^{\mu_\nu} \quad .$$

Hence we can estimate (2.73) by

$$\sum_{k:N^k\leq|I_\nu|}\sum_{n'\in\mathbb{Z}} C\left(1+N^{-k}\mathrm{dist}(n',\partial I_\nu)\right)^{-m} 2^{3\mu}N^k \leq C2^{3\mu}|I_\nu| \quad .$$

Since $\mu \leq \mu_0$, this ends the proof of Lemma 2.8.1

2.9 Counting the trees, part 1

In the remaining sections we prove an estimate on the amount of selected trees. Recall that T_μ is the set of selected trees with size μ. We have:

Lemma 2.9.1 *For all $\mu \in \mathbb{Z}$:*

$$\sum_{T\in\mathrm{T}_\mu} |I_T| \leq C2^{-\frac{25}{12}\mu} \quad . \tag{2.74}$$

At each iteration step we have selected two trees. By virtue of the selection rule, one of the trees satisfies one of the conditions 1 to 10 in the selection algorithm. We call this tree the good tree. Since both trees of a selected pair have the same top interval, it suffices to prove the estimate of the Lemma for the good trees.

Let T^σ_μ be the set of good trees in T_μ which are selected based on condition σ. It suffices to prove the estimate of the Lemma for T^σ_μ for each $\sigma \in \{1,\ldots,10\}$.

First we consider the set T^1_μ. Let T' be the set of those trees T in T^1_μ such that inequality (2.21) is satisfied for the element $P_T \in T$. Let $\mathbf{P}'$ be the set of elements P_T with $T \in \mathrm{T}'$.

For each $P \in \mathbf{P}'$ define

$$A_P f(x) := \varphi^*_{1,k,n} \psi^*_{1,k} \left(f * \phi_{1,k,l} \right) \quad .$$

Inequality (2.21) and the fact that μ was chosen to be maximal imply that

$$2^\mu |I_{k,n}|^{\frac{1}{2}} \leq \|A_P f_1\|_2 \leq 2^{\mu+1} |I_{k,n}|^{\frac{1}{2}} \quad . \tag{2.75}$$

Since $\mathrm{T}_\mu = \emptyset$ for $\mu > \mu_0$, it suffices to prove

$$\sum_{T \in \mathrm{T}'} |I_T| \leq C 2^{-2\mu} \quad . \tag{2.76}$$

By (2.75) this is equivalent to

$$\sum_{P \in \mathbf{P}'} \|A_P f_1\|_2^2 \leq C \quad .$$

We have

$$\left(\sum_{P \in \mathbf{P}'} \|A_P f_1\|_2^2 \right)^2 \leq \|f_1\|_2^2 \left\| \sum_{P \in \mathbf{P}'} A_P^* A_P f_1 \right\|_2^2$$

$$\leq 2 \|f_1\|_2^2 \sum_{P,P' \in \mathbf{P}' : |I_{P'}| \leq |I_P|} \|A_P f_1\|_2 \, \|A_P A_{P'}^*\| \, \|A_{P'} f_1\|_2$$

$$\leq C \|f_1\|_2^2 \sum_{P \in \mathbf{P}'} \|A_P f_1\|_2^2 \sum_{P' \in \mathbf{P}' : |I_{P'}| \leq |I_P|} \|A_P A_{P'}^*\| \, N^{\frac{k'-k}{2}} \quad .$$

In the last line we have used (2.75). We have to estimate the norm of $A_P A_{P'}^*$. The integral kernel of this operator is

$$K(x,y) = \varphi^*_{1,k,n}(x) \psi^*_{1,k}(x) \left(\phi_{1,k,l} * \check{\phi}_{1,k',l'} \right)(x-y) \varphi^*_{1,k',n'}(y) \psi^*_{1,k'}(y) \quad .$$

First we observe $\phi_{1,k,l} * \check{\phi}_{1,k',l'} = 0$ if $\omega_{1,P} \cap \omega_{1,P'} = \emptyset$, because then the supports of $\widehat{\phi}_{1,k,l}$ and $\widehat{\phi}_{1,k',l'}$ are disjoint. Since we have $k' \leq k$, the condition $\omega_{1,P} \cap \omega_{1,P'} \neq \emptyset$ implies $\omega_{1,P} \subset \omega_{1,P'}$. Let $\mathbf{P}_P$ be the set of all $P' \in \mathbf{P}'$ such that $|I_{P'}| \leq |I_P|$ and $\omega_{1,P} \subset \omega_{1,P'}$. For $P' \in \mathbf{P}_P$ we estimate the kernel K of $A_P A_{P'}^*$ pointwise by

$$|K(x,y)| \leq C \left((1_{I_{k,n}} * D^1_{N^k} \varphi^*)(x) \right) \left(D^1_{N^k} \varphi^*(y-x) \right) \left((1_{I_{k',n'}} * D^1_{N^{k'}} \varphi^*)(y) \right) \quad .$$

In preparation for Schur's lemma we estimate:

$$\int |K(x,y)| \, dx \leq C (1_{I_{k,n}} * D^1_{N^k} \varphi^*)(y) (1_{I_{k',n'}} * D^1_{N^{k'}} \varphi^*)(y)$$

$$\leq C (1 + N^{-k} \mathrm{dist}(I_{k,n}, I_{k',n'})^{-m} \quad ,$$

$$\int |K(x,y)|\,dy \le C(1_{I_{k',n'}} * D^1_{N^k}\varphi^*)(x)(1_{I_{k,n}} * D^1_{N^k}\varphi^*)(y)$$
$$\le CN^{k'-k}\left(1+N^{-k}\mathrm{dist}(I_{k,n},I_{k',n'})\right)^{-m} \quad .$$

Hence Schur's lemma implies:

$$\|A_P A^*_{P'}\| \le N^{\frac{k'-k}{2}}\left(1+N^{-k}\mathrm{dist}(I_{k,n},I_{k',n'})\right)^{-m} \quad .$$

These estimates on $A_P A^*_{P'}$ give for fixed $P \in \mathbf{P}'$:

$$\sum_{P'\in\mathbf{P}',|I_{P'}|\le|I_P|} \|A_P A^*_{P'}\| N^{\frac{k'-k}{2}}$$

$$\le \sum_{P'\in\mathbf{P}_P} N^{k'-k}(1+N^{-k}\mathrm{dist}(I_{k,n},I_{k',n'}))^{-m}$$

$$\le C + C\int_{\mathbb{R}} \sum_{P'\in\mathbf{P}_P} N^{-k}(1+N^{-k}\mathrm{dist}(I_{k,n},x))^{-m} 1_{I'_P}\,dx \quad .$$

Here we have approximated $\mathrm{dist}(I_{k,n},I_{k',n'})$ by $\mathrm{dist}(I_{k,n},x)$ for each $x \in I_{k',n'}$. This is inaccurate only if $\mathrm{dist}(I_{k,n},I_{k',n'}) < |I_{k',n'}|$. By the N-adic structure this happens only if $I_{k,n}$ and $I_{k',n'}$ are neighbouring intervals. Hence there are at most two intervals $I_{k',n'}$ for which this happens. The corresponding terms however can easily be estimated by a constant C. This justifies the previous calculation.

We claim that the intervals $I_{P'}$ whith $P' \in \mathbf{P}_P$ are pairwise disjoint. To see this assume to the contrary that $I_{P'} \subset I_{P''}$ for two such intervals. Then we can assume that the tree T'' corresponding to P'' has been selected first. Moreover we conclude $\omega_{1,P''} \subset \omega_{1,P'}$, because $\omega_{1,P} \subset \omega_{1,P'} \cap \omega_{1,P''}$ and $k' \le k''$. Hence $T'' \cup \{P'\}$ is a tree, a contradiction to the maximality of T''. Hence we can estimate the previously displayed expression by:

$$C + C\int_{\mathbb{R}} N^{-k}(1+N^{-k}\mathrm{dist}(I_{k,n},x))^{-m}\,dx \le C$$

This gives

$$\left(\sum_{P\in\mathbf{P}'} \|A_P f_1\|_2^2\right)^2 \le C\,\|f_1\|_2^2 \sum_{P\in\mathbf{P}'} \|A_P f_1\|_2^2 \quad ,$$

which proves the desired estimate since f_1 was normalized.

The estimate (2.76) for the set of those trees T in T^1_μ such that P_T satisfies inequality (2.22) is proved in the same way, using that the function $N^k U_{\alpha_1\xi}(U_{-\alpha_1\xi}\phi_{1,k,l})'$ satisfies the same estimates as the function $\phi_{1,k,l}$. This completes the discussion of the set T^1_μ.

We consider the set T^4_μ. Fix a $\jmath \in \{2,3\}$ and let T' be the set of those trees $T \in \mathrm{T}^4_\mu$ such that the element P_T satisfies inequality (2.26). We proceed as in the previous case. The only changes concern the kernel estimates, which become:

$$|K(x,y)| \le C\left((1_{I_{k,n}} * D^1_{N^{k-\kappa}}\varphi^*)(x)\right)\left(D^1_{N^{k-\kappa}}\varphi^*(y-x)\right)\left((1_{I_{k',n'}} * D^1_{N^{k'-\kappa}}\varphi^*)(y)\right) \quad .$$

$$\int |K(x,y)|\, dx \le C \int \left((1_{I_{k,n}} * D^1_{N^{k-\kappa}}\varphi^*)(y)\right) \left((1_{I_{k',n'}} * D^1_{N^{k'-\kappa}}\varphi^*)(y)\right)\, dy$$
$$\le C(1+N^{\kappa-k}\mathrm{dist}(I_{k,n},I_{k',n'}))^{-m} \quad ,$$
$$\int |K(x,y)|\, dy \le C \left((1_{I_{k',n'}} * D^1_{N^{k-\kappa}}\varphi^*)(x)\right) \left((1_{I_{k,n}} * D^1_{N^{k-\kappa}}\varphi^*)(y)\right)$$
$$\le C\min(1,N^{k'-k+\kappa})(1+N^{\kappa-k}\mathrm{dist}(I_{k,n},I_{k',n'}))^{-m} \quad .$$

Hence:
$$\|A_P A^*_{P'}\| \le \min(1,N^{\frac{k'-k+\kappa}{2}})(1+N^{\kappa-k}\mathrm{dist}(I_{k,n},I_{k',n'}))^{-m} \quad ,$$

and:
$$\sum_{P'\in\mathbf{P}',|I_{P'}|\le|I_P|} \|A_P A^*_{P'}\| N^{\frac{k'-k}{2}}$$
$$\le C + C\int_{\mathbb{R}} \sum_{P\in\mathbf{P}_P} N^{\frac{\kappa}{2}-k}(1+N^{\kappa-k}\mathrm{dist}(I_{k,n},x))^{-m} 1_{I_{k',n'}}(x)\, dx \le C \quad .$$

With these estimates one can argue as in the previous case. This completes the discussion of the set T^4_μ.

We consider the set T^7_μ. Fix a $\jmath \in \{2,3\}$ and a $\tilde{k}\in\{-1,0,1,2,3,4\}$ and let T' be the set of those trees $T\in\mathrm{T}^7_\mu$ such that inequality (2.29) is satisfied for this particular $\jmath$ and $\tilde{k}$. Define for $T\in\mathrm{T}'$:

$$A_T f(x) := \varphi^*_{1,k_T,n_T}\psi^*_{1,k_T}\,(f*\phi_{1,k_T,l}) \quad .$$

We proceed as in the previous cases. The only changes concern the argument that for fixed $T\in\mathrm{T}'$ the intervals $I_{T'}$ with $T'\in\mathrm{T}'$, $|I_{T'}|<|I_T|$, $A_TA^*_{T'}\neq 0$ are pairwise disjoint. Assume to the contrary that $I_{T'}\subset I_{T''}$ for two such intervals. We can assume that the tree T'' has been selected before the tree T'. We conclude from $A_TA^*_{T'}\neq 0$ that $\omega_{k_T,T}\cap\omega_{k_{T'},T'}\neq\emptyset$ because we have

$$\mathrm{supp} D^\infty_{\alpha_\jmath^{-1}}\hat{\hat{\phi}}_{\jmath,k_T+\tilde{k},T} \subset \omega_{k_T,T} \quad .$$

Likewise we have $\omega_{k_T,T}\cap\omega_{k_{T''},T''}\neq\emptyset$, and we conclude from $k_{T'},k_{T''}\ge k_T$ that $3\omega_{k_{T'},T'}\cap 3\omega_{k_{T''},T''}\neq\emptyset$. Since $k_{T''}\ge k_{T'}$, we have $\omega_{k_{T''},T''}\subset 8\omega_{k_{T'},T'}$. There exists a $P\in T'$ with $8\omega_{k_{T'},T'}\subset\omega_{\jmath,P}$ because $(k_{T'},n_{T'})\in\tilde{T}'$. Hence $\xi_{T''}\in\omega_{\jmath,P}$, which implies that $T''\cup\{P\}$ is a tree. This is a contradiction to the maximality of T''. This completes the discussion of the set T'. The cases involving inequalities (2.30) and (2.31) do not require any new ideas. This completes the discussion of the set T^7_μ.

2.10 Counting the trees, part 2

Let T' be the set of trees in T^3_μ which satisfy (2.24).

For $T \in \mathrm{T}'$ define $T^{\min}$ to be the set of all $P \in T$ such that I_P is minimal in $\{I_{P'} : P' \in T\}$ with respect to set inclusion. We have with (2.24) and the right hand side of (2.75):

$$\sum_{P \in T \setminus T^{\min}} \left\| \phi_{1,k,n}^* \psi_{1,k}^* (f_1 * \phi_{1,k}) \right\|_2^2$$
$$\geq 5|I_T| 2^{2\mu} - \sum_{P \in T^{\min}} |I_P| 2^{2(\mu+1)} \quad .$$

This is greater or equal $|I_T|2^{2\mu}$ because $I_P \cap I_{P'} = \emptyset$ for any $P \neq P' \in T^{\min}$.

Define $T^{\mathrm{red}} := T \setminus T^{\min}$. Let $\mathbf{P}'$ be the union of the sets T^{red} as T runs through T'. For each $P \in \mathbf{P}'$ define:

$$A_P f(x) := \varphi_{1,k,n}^* \psi_{1,k}^* \left(f_1 * \phi_{1,k,l}\right) \quad .$$

Again it suffices to prove:

$$\sum_{P \in \mathbf{P}'} \|A_P f_1\|_2^2 \leq C \quad .$$

We have

$$\left(\sum_{P \in \mathbf{P}'} \|A_P f_1\|_2^2 \right)^2 \leq \|f_1\|_2^2 \left\| \sum_{P \in \mathbf{P}'} A_P^* A_P f_1 \right\|_2^2$$

$$= \|f_1\|_2^2 \sum_{P,P' \in \mathbf{P}' : |I'_P| = |I_P|} \|A_P f_1\|_2 \, \|A_P A_{P'}^*\| \; \|A_{P'} f_1\|_2 \tag{2.77}$$

$$+2 \, \|f_1\|_2^2 \sum_{P,P' \in \mathbf{P}' : |I'_P| < |I_P|} \|A_P f_1\|_2 \, \|A_P A_{P'}^*\| \; \|A_{P'} f_1\|_2 \quad . \tag{2.78}$$

We consider the first summand (2.77). By symmetry it suffices to consider the sum over those pairs (P, P') which satisfy $\|A_{P'} f_1\|_2 \leq \|A_P f_1\|_2$, hence we have to estimate

$$\|f_1\|_2^2 \sum_{P \in \mathbf{P}'} \|A_P f_1\|_2^2 \sum_{P' \in \mathbf{P}' : |I'_P| = |I_P|} \|A_P A_{P'}^*\| \quad .$$

As before we have $A_P A_{P'}^* \neq 0$ only if $\omega_{1,P} \cap \omega_{1,P'}$ is nonempty. Since $|I_P| = |I_{P'}|$, this implies $\omega_{1,P} = \omega_{1,P'}$. This implies that the intervals $I_{P'}$ with if $P' \in \mathbf{P}'$, $|I'_P| = |I_P|$, and $A_P A_{P'}^*$ are pairwise disjoint. Hence we can estimate the previously displayed expression by the arguments of the previous section. We turn to the second summand (2.78). We have

$$\sum_{P,P' \in \mathbf{P}' : |I_{P'}| < |I_P|} \|A_P f_1\|_2 \, \|A_P A_{P'}^*\| \; \|A_{P'} f_1\|_2$$

$$\leq \sum_{T \in \mathrm{T}'} \left(\sum_{P \in T^{\mathrm{red}}} \|A_P f_1\|_2^2 \right)^{\frac{1}{2}} \left(\sum_{P \in T^{\mathrm{red}}} \left(\sum_{P' \in \mathbf{P}' : |I_{P'}| < |I_P|} \|A_P A_{P'}^*\| \; \|A_{P'} f_1\|_2 \right)^2 \right)^{\frac{1}{2}}$$

$$\leq C \sum_{T \in \mathrm{T}'} \left(\sum_{P \in T^{\mathrm{red}}} \|A_P f_1\|_2^2 \right) \left(\sum_{P \in T^{\mathrm{red}}} \left(\sum_{P' \in \mathbf{P}' : |I_{P'}| < |I_P|} \|A_P A_{P'}^*\| \; N^{\frac{k' - k_T}{2}} \right)^2 \right)^{\frac{1}{2}}$$

Fix a $T \in \mathrm{T}'$, $P \in T^{\mathrm{red}}$ and let $\mathbf{P}_P$ be the set of $P' \in \mathbf{P}'$ which satisfy $|I_{P'}| < |I_P|$ and $A_P A_P^* \neq 0$. We claim that the intervals $I_{P'}$ with $P' \in \mathbf{P}_P$ are pairwise disjoint. To see this, assume to the contrary that $I_{P'} \cap I_{P''}$ for two different elements $P', P'' \in \mathbf{P}_P$. Observe that $P' \neq P''$ and $\omega_{1,P'} \cap \omega_{1,P''} \neq \emptyset$ imply $|I_{P'}| < |I_{P''}|$. Moreover, $\omega_{1,P''}$ is a strict subset of $\omega_{1,P'}$, and therefore $\omega_{\jmath,P''} \subset \omega_{1,P'}$, which implies $\omega_{\jmath,P'} > \omega_{\jmath,P''}$. Hence $\xi_{T'} > \xi_{T''}$, where $T', T'' \in \mathrm{T}'$ are the trees which contain P' and P''. We conclude that T'' has been selected before T'. Pick $P^{\min} \in T'^{\min}$ such that $I_{P^{\min}} \subset I_{P'}$. We successively observe that $\omega_{\jmath,P^{\min}}$ contains $\omega_{\jmath,P'}$, $\omega_{1,P'}$, and $\omega_{\jmath,P''}$. Hence $P^{\min}$ qualifies to be in the tree T'', a contradiction to the maximality of T''.

Next we claim that any $P' \in \mathbf{P}_P$ satisfies $I_{P'} \cap I_T = \emptyset$. Assume to the contrary that $I_{P'} \subset I_T$ for some $P' \in \mathbf{P}_P$. Following the previous arguments we conclude that the tree T has been selected before T', where $T' \in \mathrm{T}'$ is the tree containing P'. Then any element $P^{\min} \in T'^{\min}$ which satisfies $I_{P^{\min}} \subset I_{P'}$ qualifies to be in the tree T, which gives a contradiction to the maximality of T.

Hence we have, using the estimates for $\|A_P A_{P'}^*\|$ from the previous section:

$$\sum_{P' \in \mathbf{P}', |I_{P'}| < |I_P|} \|A_P A_{P'}^*\| N^{\frac{k' - k_T}{2}}$$

$$\leq C \int_{\mathbb{R}} \sum_{P' \in \mathbf{P}_P} N^{\frac{-k_T - k}{2}} (1 + N^{-k} \mathrm{dist}(I_{k,n}, I_{k',n'}))^{-m} 1_{I'_P} \, dx$$

$$\leq C \int_{(I_T)^c} N^{\frac{-k_T - k}{2}} (1 + N^{-k} \mathrm{dist}(I_{k,n}, x))^{-m} \, dx$$

$$\leq C N^{\frac{k - k_T}{2}} \left(1 + N^{-k} \mathrm{dist}(I_{k,n}, (I_T)^c)\right)^{-m} \quad .$$

In this calculation we have neglected the difficulty coming from those $I_{k',n'}$ which are adjacent to I_T. However, the necessary modifications are easy and left to the reader. This gives:

$$\sum_{P \in T} \left(\sum_{P' \in \mathbf{P}' : |I'_P| < |I_P|} \|A_P A_{P'}^*\| \; N^{\frac{k' - k_T}{2}} \right)^2$$

$$\leq C \int_{\mathbb{R}} \sum_{P \in T} N^{-k_T} \left(1 + N^{-k} \mathrm{dist}(I_{k,n}, (I_T)^c)\right)^{-m} 1_{I_{k,n}}(x) \, dx$$

$$\leq C \sum_{k \leq k_T} N^{k - k_T} \int_{I_T} N^{-k} \left(1 + N^{-k} \mathrm{dist}(x, (I_T)^c)\right)^{-m} \, dx \leq C \quad .$$

As before this gives the desired estimate for the set T'. Similarly one estimates the trees satisfying inequality (2.25). This completes the discussion of the set T^3_μ.

The set T^2_μ is estimated similarly.

We turn to the set T^5_μ. For each tree T let T^{even} be the set of $(k,n,l) \in T$ with k even, and let T^{odd} be $T \setminus T^{\mathrm{even}}$. Fix $\jmath \in \{2,3\}$ and $z \in \mathbb{Z}$ and let T' be the set of all trees in T^5_μ which satisfy the inequality:

$$\sum_{(k,n,l) \in T^{\mathrm{even}}} \left\| \varphi^*_{\jmath,k,n} \psi^*_{\jmath,k} \left(f_\jmath * (\phi_{\jmath,k,l} - \lambda_{\jmath,k} \tilde{\phi}_{\jmath,k,T}) * \theta^+_{\jmath,k+z,T} \right) \right\|_2^2 \geq 2^{-1} 2^{-\frac{z}{2}} 2^{2\mu} |I| \quad . \tag{2.79}$$

We can assume $z \geq 0$ because the left hand side of this inequality would vanish otherwise. We will prove

$$\sum_{T\in \mathrm{T}'} |I_T| \leq C 2^{2\mu} 2^{-\frac{z}{4}} \quad . \tag{2.80}$$

The same proof will apply if we replace T^{even} by T^{odd} in (2.79). Therefore this proves the appropriate bound for T^5_μ.

We define $\mathbf{P}'$ to be the union of the sets T^{even} as T runs through T'. For $P \in \mathbf{P}'$ we define

$$A_P f := \varphi^*_{\jmath,k,n} \psi^*_{\jmath,k}(f_\jmath * (\phi_{\jmath,k,T} - \lambda_{\jmath,k,T}\tilde{\phi}_{\jmath,k,T}) * \theta^+_{\jmath,k+z,T}) \quad ,$$

where T is the tree in T' which satisfies $P \in T^{\mathrm{even}}$. Observe that for each $P \in \mathbf{P}'$ we have

$$\|A_P f_\jmath\|_2 \leq 2^{-\frac{z}{4}} 2^{2(\mu+1)} |I_P|^{\frac{1}{2}}$$

because otherwise the set $\{P\}$ itself would be a tree with size greater of equal $\mu + 1$, a contradiction.

As before we consider for fixed $T \in \mathrm{T}'$ and $P \in T^{\mathrm{even}}$ the sets

$$\tilde{\mathbf{P}}_P = \{P' \in \mathbf{P}' : |I_{P'}| = |I_P|, A_P A^*_{P'} \neq 0\} \quad ,$$

$$\mathbf{P}_P = \{P' \in \mathbf{P}' : |I_{P'}| < |I_P|, A_P A^*_{P'} \neq 0\} \quad .$$

Observe that $A_P A^*_{P'} \neq 0$ again implies $\omega_{\jmath,P} \cap \omega_{\jmath,P'} \neq \emptyset$. Hence we conclude as before that we have $I_{P'} \cap I_{P''} = \emptyset$ for any two different elements $P', P'' \in \tilde{\mathbf{P}}_P$.

Next, we claim that the intervals $I_{P'}$ with $P' \in \mathbf{P}_P$ are pairwise disjoint. Assume to the contrary that $I_{P'} \subset I_{P''}$ are not disjoint for two different elements $P', P'' \in \mathbf{P}_P$. Again we conclude from $\omega_{\jmath,P'} \cap \omega_{\jmath,P''} \neq \emptyset$ that $I_{P'}$ is strictly contained in $I_{P''}$. Observe that $A_P A^*_{P'} \neq 0$ implies that the supports of $\widehat{\theta}^+_{\jmath,k+z,T}$ and $\widehat{\theta}^+_{\jmath,k'+z,T'}$ have nonempty intersection. Observe that the size of the support of $\widehat{\theta}^+_{\jmath,k+z,T}$ is at most $4\epsilon^2 \alpha_\jmath N^{-k-z}$, the distance from the support of $\theta^+_{\jmath,k+z,T}$ to $\alpha_\jmath \xi_T$ is at least $\frac{1}{4}\epsilon^2 \alpha_\jmath N^{-k-z-1}$, and ξ_T lies below $\mathrm{supp}(D^\infty_{\alpha_\jmath^{-1}} \theta^+_{\jmath,k+z,T})$. Hence we conclude from $k' + 2 \leq k'' \leq k$ that $\xi_{T'} < \xi_{T''}$. Here T' and T'' are the trees in T' which contain P' and P''. Hence the tree T'' has been selected before the tree T'. Moreover we easily see that $\xi_{T''}$ is contained in $\omega_{\jmath,P'}$, hence P' qualifies to be in the tree T'', a contradiction to the maximality of T''.

Finally, one can show by similar arguments that $I_{P'} \cap I_T = \emptyset$ for any $P' \in \mathbf{P}_P$. Hence the geometric situation is as in the argument for T^3_μ.

In order to estimate $\|A_P A^*_{P'}\|$ we calculate with (2.35) for the integral kernel K of this operator:

$$|K(x,y)| \leq$$

$$C N^{-2z} \left((1_{I_{k,n}} * D^1_{N^{k-\kappa}}\varphi^*)(x)\right) \left(D^1_{N^{k-\kappa+z}}\varphi^*(y-x)\right) \left((1_{I_{k',n'}} * D^1_{N^{k'-\kappa}}\varphi^*)(y)\right) \quad .$$

$$\int |K(x,y)|\, dx \leq C N^{-2z} \left((1_{I_{k,n}} * D^1_{N^{k-\kappa+z}}\varphi^*)(y)\right) \left((1_{I_{k',n'}} * D^1_{N^{k'-\kappa}}\varphi^*)(y)\right)$$

$$\leq C N^{-2z} \min(1, N^{\kappa-z}) \left(1 + N^{\kappa-k-z} \mathrm{dist}(I_{k,n}, I_{k',n'})\right)^{-m} \quad ,$$

$$\int |K(x,y)|\, dy \leq C N^{-2z} \left((1_{I_{k',n'}} * D^1_{N^{k-\kappa+z}}\varphi^*)(x)\right) \left((1_{I_{k,n}} * D^1_{N^{k-\kappa}}\varphi^*)(y)\right)$$

$$\leq CN^{-2z}\min(1,N^{k'-k+\kappa-z})\left(1+N^{\kappa-k-z}\mathrm{dist}(I_{k,n},I_{k',n'})\right)^{-m} \quad .$$

This gives:

$$\sum_{P'\in\mathbf{P}',|I_{P'}|<|I_P|}\|A_PA_{P'}^*\|N^{\frac{k'-k_T}{2}}$$

$$\leq CN^{-2z}\sum_{P'\in\mathbf{P}_P}N^{k'}N^{\frac{-k-k_T}{2}}N^{\kappa-z}(1+N^{\kappa-k-z}\mathrm{dist}(I_{k,n},I_{k',n'}))^{-m}$$

$$\leq CN^{-2z}\int\limits_{(I_T)^c}N^{\frac{k-k_T}{2}}N^{\kappa-k-z}(1+N^{\kappa-k-z}\mathrm{dist}(I_{k,n},x))^{-m}\,dx$$

$$\leq CN^{-2z}N^{\frac{k-k_T}{2}}\left(1+N^{\kappa-k-z}\mathrm{dist}(I_{k,n},(I_T)^c)\right)^{-m} \quad ,$$

and:

$$\sum_{P\in T}\left(\sum_{P'\in\mathbf{P}':|I_P'|<|I_P|}\|A_PA_{P'}^*\|\;N^{\frac{k'-k_T}{2}}\right)^2$$

$$\leq CN^{-4z}\sum_{k\leq k_T}N^{k-k_T}\int_{I_T}N^{-k}\left(1+N^{\kappa-k-z}\mathrm{dist}(x,(I_T)^c)\right)^{-m}\,dx\leq CN^{-3z} \quad .$$

This proves the desired estimate for T' and completes the discussion of the set T_μ^5. The set T_μ^6 is estimated similarly.

Also the sets T_μ^8 and T_μ^9 are estimated with the above arguments, with modifications as in the case of T_μ^7.

2.11 Counting the trees, part 3

It remains to estimate the set T_μ^{10}. Fix a $\jmath\in\{2,3\}$ and a $\tilde{k}\in\{-1,0,1,2,3,4\}$ and let T' be the set of all trees in T_μ^{10} which satisfy inequality (2.34) for this particular pair $\jmath$, k'.

Let $\mathbf{Q}$ be the set of all triples (k,J,T) with $T\in\mathrm{T}'$, $k\in\mathbb{Z}$, and $J\in\Delta_{k-\kappa,T}$. We write Q for (k,J,T) and Q' for (k',J',T'). For each $Q\in\mathbf{Q}$ define

$$A_Qf(x):=\rho_{k-\kappa,J}\left(f*\tilde{\phi}_{\jmath,k+\tilde{k},T}\right) \quad .$$

By inequality (2.34) we have

$$\sum_{(k,J,T)\in\mathbf{Q}}\|A_Qf_\jmath\|_2^2\geq 2^{2\mu}\sum_{T\in\mathrm{T}'}|I_T| \quad .$$

For $\tilde{\mu}\in\mathbb{Z}$ let $\mathbf{Q}_{\tilde{\mu}}$ be the set of all triples $(k,J,T)\in\mathbf{Q}$ with

$$2^{\tilde{\mu}}|J|^{\frac{1}{2}}\leq\|A_Qf_\jmath\|_2<2^{\tilde{\mu}+1}|J|^{\frac{1}{2}} \quad .$$

Then we have

$$\sum_{\tilde{\mu}<\mu-5}\sum_{(k,J,T)\in\mathbf{Q}_{\tilde{\mu}}}\|A_Qf_\jmath\|_2^2$$

$$\leq \sum_{\tilde\mu<\mu-5} \sum_{(k,J,T)\in\mathbf{Q}_{\tilde\mu}} 2^{2\tilde\mu+2}|J|$$

$$\leq \sum_{\tilde\mu<\mu-5} \sum_{T\in \mathrm{T}'} 2^{2\tilde\mu+2}|I_T| \leq 2^{2\mu-1} \sum_{T\in\mathrm{T}'} |I_T| \quad .$$

Here we have used that the intervals $J \in \Delta_{k,T}$ are pairwise disjoint and contained in I_T.

Moreover by the arguments in Lemma 2.6.1 there is a universal $\tilde\mu_0$ such that $\mathbf{Q}_{\tilde\mu} = \emptyset$ for $\tilde\mu \geq \tilde\mu_0$. Hence we have

$$\sum_{\mu-4\leq\tilde\mu<\tilde\mu_0} \sum_{(k,J,T)\in\mathbf{Q}_{\tilde\mu}} \|A_Q f_\jmath\| \geq C 2^{2\mu} \sum_{T\in\mathrm{T}'} |I_T| \quad .$$

Hence it suffices to prove for fixed $\tilde\mu$ that

$$\sum_{(k,J,T)\in\mathbf{Q}_{\tilde\mu}} \|A_Q f_\jmath\|_2^2 \leq C \quad .$$

By the calculations of the last sections the square of the left hand side is bounded by

$$\leq C \|f_\jmath\|_2^2 \sum_{Q\in\mathbf{Q}_{\tilde\mu}} \|A_Q f_\jmath\|_2^2 \sum_{Q'\in\mathbf{Q}_{\tilde\mu}:|J'|\leq|J|} \left\| A_Q A_{Q'}^* \right\| N^{\frac{k'-k}{2}} \quad .$$

We have to estimate the norm of $A_Q A_{Q'}^*$. The integral kernel of this operator is

$$K(x,y) = \rho_{k-\kappa,J}(x) \left(\tilde\phi_{\jmath,k+\tilde k,T} * \check{\tilde\phi}_{\jmath,k'+\tilde k,T'} \right)(x-y) \rho_{k-\kappa,J}(y) \quad .$$

First we observe that $A_Q A_{Q'}^* = 0$ if $\omega_{k,T} \cap \omega_{k',T'} = \emptyset$, since then the supports of $\hat{\tilde\phi}_{\jmath,k+\tilde k,T}$ and $\hat{\tilde\phi}_{\jmath,k'+\tilde k,T'}$ are disjoint. Now we estimate the kernel K pointwise by

$$|K(x,y)| \leq C \left((1_J * D^1_{N^{k-\kappa}}\varphi^*)(x) \right) \left(D^1_{N^{k-\kappa}}\varphi^*(y-x) \right) \left((1_{J'} * D^1_{N^{k'-\kappa}}\varphi^*)(y) \right) \quad .$$

An application of Schur's lemma as before gives:

$$\|A_Q A_{Q'}^*\| \leq N^{\frac{k'-k}{2}} \left(1 + N^{\kappa-k}\mathrm{dist}(J,J')\right)^{-m} \quad .$$

These estimates on $A_P A_{P'}^*$ give for fixed $Q \in \mathbf{Q}_{\tilde\mu}$:

$$\sum_{Q'\in\mathbf{Q}',|J'|\leq|J|} \|A_Q A_{Q'}^*\| N^{\frac{k'-k}{2}}$$

$$\leq C \int_{\mathbb{R}} \sum_{Q'\in\mathbf{Q}_{\tilde\mu},|J'|\leq|J|,\omega_{k,T}\cap\omega_{k',T'}\neq\emptyset} N^{\kappa-k}(1+N^{\kappa-k}\mathrm{dist}(J,x))^{-m} 1_{J'}\, dx \quad .$$

Let $Q', Q'' \in \mathbf{Q}_{\tilde\mu}$ such that $|J'|, |J''| \leq |J|$, and $\omega_{k',T'}$ and $\omega_{k'',T''}$ have nonempty intersection with $\omega_{k,T}$. Since $k', k'' \leq k$, we have $3\omega_{k',T'} \cap 3\omega_{k'',T''} \neq \emptyset$. We want to prove that $J' \cap J'' = \emptyset$. Assume the contrary. Then we necessarily have $J' = J''$. Since $Q' \neq Q''$, we have $T' \neq T''$ and we can assume that T'' has been selected first. Pick n such that $J \in I_{k'+1,n}$, then $(k'+1,n) \in \tilde T'$ and $(k'+1,n) \in \tilde T''$. There exist $P \in T'$ such that $I_P \subset I_{k'+1,n}$ and $8\omega_{k'+1,T'} \subset \omega_{\jmath,P}$. Then also $\omega_{k''+1,T''} \subset \omega_{\jmath,P'}$, hence P qualifies to be in the tree T'', a contradiction to the maximality of T''. Hence $J' \neq J''$. Hence the previously displayed expression is bounded by C. This completes the discussion of the set T_μ^{10}.

This also completes the proof of Lemma 2.9.1, and hence the proof of Lemma 2.3.1 and of Theorem 2.1.1.

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