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Arrival times in a zero-range process with injection and decay

B Hertz Shargel[1], **M R D'Orsogna**[2] **and T Chou**[1,3]

[1] Department of Mathematics, UCLA, Los Angeles, CA 90095-1555, USA
[2] Department of Mathematics, CSUN, Los Angeles, CA 91330-8313, USA
[3] Department of Biomathematics, UCLA, Los Angeles, CA 90095-1766, USA

E-mail: dorsogna@csun.edu (Maria D'Orsogna) and tomchou@ucla.edu (Tom Chou)

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Abstract

Explicit expressions for arrival times of particles moving in a one-dimensional zero-range process (ZRP) are computed. Particles are fed into the ZRP from an injection site and can evaporate from any site of the one-dimensional lattice. Two dynamics are considered: bulk dynamics, where particle hopping and decay are proportional to the number of particles at each site, and surface dynamics, where only the top particle at each site can hop or evaporate. We find exact solutions in the bulk dynamics case and for a single-site ZRP obeying surface dynamics. For a multisite ZRP obeying surface dynamics, we compare simulations with approximations obtained from the steady-state limit, where mean interarrival times for both models are equivalent. Our results highlight the competition between injection and evaporation in the arrival times of particles to an absorbing site.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The zero-range process (ZRP) is a stochastic model to describe the dynamics of far from equilibrium, interacting particles hopping between lattice sites [1–3]. The ZRP has been used in many applications as a paradigm for transport processes, including traffic flows [4], shaken granular gases [5], network dynamics [6], phase separation [7, 8], and particle condensation and clustering [9]. Mathematical interest also arises from the fact that a simple connection can be made between the ZRP and the totally asymmetric exclusion process (TASEP) [1] and that in certain cases—particularly for conserved systems—exact factorizable steady-state solutions can be derived [10].

In this paper, we compute the multiple passage times of particles obeying ZRP dynamics to reach a final absorbing site. We treat a nonconserved system in which particles are injected at

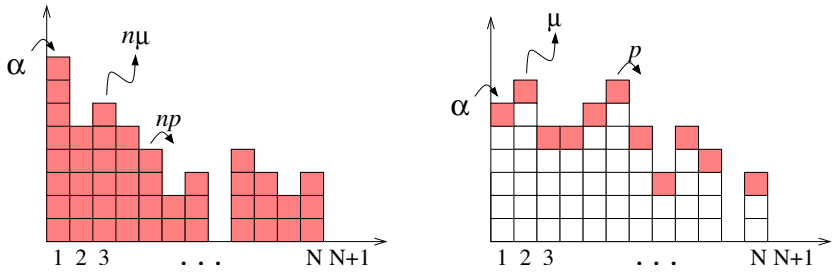

Figure 1. Two realisations of a zero-range process. In (*a*), any one of the red particles in the bulk can hop to the right or decay, while in (*b*), only the surface particle (in red) in each pile can hop or decay, the underlying particles being protected by the top particle.

the origin and evaporate as they hop toward the end site of the lattice, as shown in figure 1. This type of dynamics may be applied to many specific microbiological systems. For example, molecular motors may attach at one end of a microtubule, but desorb while traversing it. The distribution of arrival times of the motors will depend on their speed, and injection and desorption rates. Other examples include virus entry and transport to the nucleus, where the viral cargo is transported by molecular motors while being subject to degradation [11], and sperm entry into egg cells, where the first sperm to penetrate all layers of the cell triggers a block for subsequent ones [12]. In all applications there is a flux of particles, or 'immigration', into the first site as well as particle annihilation at every location along the microtubule or layer within the egg cell.

Related theoretical analyses of the dynamic properties of the ZRP have been presented; however, these studies have been restricted to the steady-state [6, 13], hydrodynamic/asymptotic [14] or mean-field [6] limits. Moreover, most previous approaches only consider particle conservation [14]. Unlike the first passage time (FPT) problem of a single, conserved particle undergoing a simple random walk [15, 16], the first passage time of a nonconserved, multiparticle problem cannot be solved by analyzing steady states. In problems with injection and decay, the dynamics of particles reaching a specific 'absorbing' site are complicated by subsequent particle arrivals, as well as arrival times conditioned on particles reaching the absorbing site. Since we will be concerned with a specific initial configuration, and wish to understand how the ZRP first reaches another configuration, we must find time-dependent solutions for the dynamics of the ZRP. Nonetheless, despite the nonconserved nature of the problem, steady-state solutions can still sometimes provide useful approximations for first passage times of the ZRP in certain limits.

We present exact solutions for arrival times in a finite-sized ZRP obeying two specific dynamical rules. In the first case, which we denote as 'bulk dynamics' and which is illustrated in figure 1(*a*), all particles at a site are independent and equally likely to hop to the next one. In the second 'surface dynamics' case, depicted in figure 1(*b*), only one particle can hop to its neighboring site. This surface dynamics case has also been described as a 'chipping' process [13]. These two cases are limits of the ZRP and may serve as a model system for many physical systems.

We begin our analysis in the following section with bulk dynamics. In this case, particles are independent and we find exact analytic expressions for the distributions of the passage times for the kth particle to arrive at the absorbing site $N+1$. For mathematical completeness, we present two derivations of the solution. The first involves explicit enumeration of the number of particles injected, evaporated and having reached the absorbing site by time t. The second involves writing down a Master equation, which is solved using generating functions

and the method of characteristics. All results for the bulk case are exact. In the third section, we define a ZRP with surface dynamics, where only one particle at each site, if it exists, is allowed to hop or desorb. In this case, we can only find exact solutions for a single-site ZRP. In the limit where desorption is neglected, the position of the rightmost occupied site can be found by representing it by a single forward-hopping particle and tracking its dynamics. However, in the presence of detachments, a lead particle can desorb requiring us to keep track of mobile particles behind the lead site. Even though explicit solutions are not available in the surface dynamics case with desorption, we approximate particle arrival times in certain limits using a steady-state assumption, and compare our estimates with results derived from Monte-Carlo simulations.

2. Zero-range model with bulk dynamics

In our bulk-dynamic ZRP, beginning at time $t = 0$, particles are injected into the first site of an empty lattice whose positions are denoted by $\{1, 2, \ldots, N+1\}$. The injection occurs as a Poisson process with rate α. Each particle can then hop one site to the right with rate p or evaporate with rate μ, independently of others. The forward hopping and evaporation processes continue until the particle reaches site $N+1$ or is desorbed from the lattice. We wish to calculate the distribution of times for the kth particle to reach site $N+1$.

2.1. Explicit enumeration of particle fates

Denoting by T_k the time at which the final absorbing site $N+1$ is reached for the kth time, we consider the accumulated number of hits $H(t)$ by time t defined by $P(H(t) = k) = P(T_k \leqslant t < T_{k+1})$. The primary result of this section is that $H(t)$ is Poisson distributed, with rate parameter

$$\lambda(t) = \alpha \left(\frac{p}{\mu+p}\right)^N \left(t - \frac{N}{\mu+p}\right) - \frac{\alpha}{\mu+p} \mathrm{e}^{-(\mu+p)t} \sum_{\ell=0}^{N-1} \left(\frac{p}{\mu+p}\right)^\ell \sum_{m=0}^{\ell} \left(\frac{\mu}{p+\mu} \sum_{i=0}^{m} \frac{((\mu+p)t)^i}{i!} - \frac{((\mu+p)t)^m}{m!}\right). \quad (1)$$

From this expression we can find the survival probability $S_k(t)$ that the final site has been hit by $k-1$ or fewer particles:

$$S_k(t) = \sum_{j=0}^{k-1} P(H(t) = j) = \mathrm{e}^{-\lambda(t)} \sum_{j=0}^{k-1} \frac{\lambda(t)^j}{j!}. \quad (2)$$

Similarly, the distribution of T_i can be represented as the sum

$$P(T_i \leqslant t) = \sum_{j=i}^{\infty} P(H(t) = j) = \mathrm{e}^{-\lambda(t)} \sum_{j=i}^{\infty} \frac{\lambda(t)^j}{j!}. \quad (3)$$

Apart from the evaluation of $\lambda(t)$ in equation (1), the other main results of this section are the full expression for $S_1(t)$ given in equation (15) and the evaluation of the first passage time in equation (16). To derive the distribution of $H(t)$, we begin by noting that we may break up the event $\{H(t) = k\}$ according to how many particles n were injected before time t, each with a common probability $q(t)$ of reaching site $N+1$ by time t, before we consider the times at which it is injected (see figure 2). The probability that exactly k of those n particles reach site

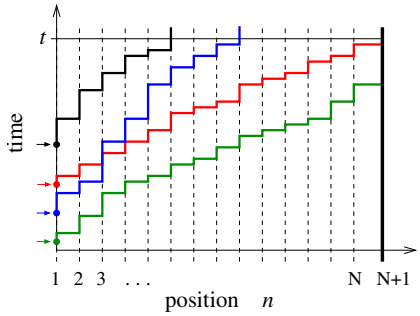

Figure 2. Spacetime plot of ZRP bulk dynamics. In this realization, four particles are injected by time t, and are the only ones capable of contributing to $H(t)$. Trajectories that intersect the vertical line have arrived at site $N+1$ within time t. Trajectories that intersect with the horizontal line at time t are those that failed to reach the absorbing site $N+1$ by time t.

$N+1$ is binomially distributed with parameter $q(t)$, so that

$$\begin{aligned} P(H(t)=k) &= \sum_{n=k}^{\infty} \frac{(\alpha t)^n \, \mathrm{e}^{-\alpha t}}{n!} \frac{n!}{k!(n-k)!} q(t)^k (1-q(t))^{n-k} \\ &= \frac{q(t)^k (\alpha t)^k}{k!} \mathrm{e}^{-\alpha t} \sum_{n=k}^{\infty} \frac{(\alpha(1-q(t))t)^{n-k}}{(n-k)!} \\ &= \frac{(\alpha q(t) t)^k}{k!} \mathrm{e}^{-\alpha q(t) t}. \end{aligned} \tag{4}$$

This implies that $H(t)$ is Poisson-distributed with parameter

$$\lambda(t) = \alpha q(t) t. \tag{5}$$

Deriving the probability $q(t)$ is therefore our task. The particle injected at time τ is characterized by its decay time ζ_τ, which is exponentially distributed with the mean $1/\mu$, and by its arrival time X_τ at the target site $N+1$, excluding the possibility of decay. As this latter random variable is the sum of exponentials, it is Gamma(N,k)-distributed [17]. The probability $q(t)$ of reaching site $N+1$ is then given by the probability that the arrival time precedes both the chosen time limit t or the time of decay, averaged over the possible injection times τ. Symbolically,

$$q(t) = \frac{1}{t} \int_0^t [P(X_\tau \leqslant t-\tau \leqslant \zeta_\tau) + P(X_\tau \leqslant \zeta_\tau \leqslant t-\tau)] \, \mathrm{d}\tau. \tag{6}$$

Since X_τ and ζ_τ are independent, the first probability $P(X_\tau \leqslant t-\tau \leqslant \zeta_\tau) = P(X_\tau \leqslant t-\tau) P(\zeta_\tau \geqslant t-\tau)$. And since ζ_τ is exponentially distributed with the mean μ, and X_τ is Gamma-distributed, the first probability in the integrand of equation (6) is simply

$$\begin{aligned} P(X_\tau \leqslant t-\tau \leqslant \zeta_\tau) &= F_\Gamma(t-\tau; N, p)[1 - F_{\zeta_\tau}(t-\tau)] \\ &= F_\Gamma(t-\tau; N, p) \, \mathrm{e}^{-\mu(t-\tau)}, \end{aligned} \tag{7}$$

where $F_\Gamma(s; N, \beta)$ is the Gamma distribution function. The second probability and many of the computations to follow rely on the following equivalent representations of this function:

$$F_\Gamma(s; N, \beta) = \frac{\gamma(N, \beta s)}{\Gamma(N)} = 1 - \mathrm{e}^{-\beta s} \sum_{\ell=0}^{N-1} \frac{(\beta s)^\ell}{\ell!}. \tag{8}$$

Here $\gamma(z, w) = \int_0^w t^{z-1}\,\mathrm{e}^{-t}\,\mathrm{d}t$ is the lower incomplete Gamma function, and the right-hand equality with the Erlang distribution holds because N is an integer [17]. Expression (8) leads to the following useful identities for F_Γ:

$$\int_0^s F_\Gamma(u; N, \beta)\,\mathrm{d}u = \int_0^s \left(1 - \mathrm{e}^{-\beta u}\sum_{\ell=0}^{N-1}\frac{(\beta u)^\ell}{\ell!}\right)\mathrm{d}u = s - \sum_{\ell=0}^{N-1}\frac{1}{\ell!}\int_0^s (\beta u)^\ell\,\mathrm{e}^{-\beta u}\,\mathrm{d}u$$
$$= s - \frac{1}{\beta}\sum_{\ell=0}^{N-1}\frac{\gamma(\ell+1, \beta s)}{\Gamma(\ell+1)} \equiv s - \frac{1}{\beta}\sum_{\ell=0}^{N-1} F_\Gamma(s; \ell+1, \beta) \tag{9}$$

and

$$\int_0^s \mathrm{e}^{-\eta u}\,F_\Gamma(u; N, \beta)\,\mathrm{d}u = \frac{1}{\eta}(1 - \mathrm{e}^{-\eta s}) - \frac{1}{\eta+\beta}\sum_{\ell=0}^{N-1}\left(\frac{\beta}{\eta+\beta}\right)^\ell F_\Gamma(s; \ell+1, \eta+\beta). \tag{10}$$

Returning to the second probability in equation (6), equation (10) yields

$$P(X_\tau \leqslant \zeta_\tau \leqslant t-\tau) = \int_0^{t-\tau} \mu\,\mathrm{e}^{-\mu s} F_\Gamma(s; N, p)\,\mathrm{d}s$$
$$= 1 - \mathrm{e}^{-\mu(t-\tau)} - \frac{\mu}{\mu+p}\sum_{\ell=0}^{N-1}\left(\frac{p}{\mu+p}\right)^\ell F_\Gamma(t-\tau; \ell+1, \mu+p), \tag{11}$$

so that upon substituting equations (7) and (11) into equation (6), we obtain

$$q(t) = 1 + \frac{1}{t}\int_0^t \Bigg[F_\Gamma(t-\tau, N, p)\,\mathrm{e}^{-\mu(t-\tau)} - \mathrm{e}^{-\mu(t-\tau)}$$
$$- \frac{\mu}{\mu+p}\sum_{\ell=0}^{N-1}\left(\frac{p}{\mu+p}\right)^\ell F_\Gamma(t-\tau; \ell+1, p+\mu)\Bigg]\mathrm{d}\tau.$$

Evaluating the integral termwise using equations (9) and (10) we find

$$q(t) = 1 - \frac{1}{t(\mu+p)}\sum_{\ell=0}^{N-1}\left(\frac{p}{\mu+p}\right)^\ell$$
$$\times\left[\mu t + F_\Gamma(t; \ell+1, \mu+p) - \frac{\mu}{\mu+p}\sum_{m=0}^{\ell} F_\Gamma(t; m+1, \mu+p)\right].$$

We can now expand the distribution functions in terms of a finite sum. Performing some algebraic simplifications yields the closed-form expression

$$q(t) = 1 - \frac{1}{t(\mu+p)}\sum_{\ell=0}^{N-1}\left(\frac{p}{\mu+p}\right)^\ell\Bigg[\frac{p-\ell\mu}{\mu+p} + \mu t$$
$$+ \mathrm{e}^{-(\mu+p)t}\sum_{m=0}^{\ell}\left(\frac{\mu}{\mu+p}\sum_{i=0}^{m}\frac{((\mu+p)t)^i}{i!} - \frac{((\mu+p)t)^m}{m!}\right)\Bigg]. \tag{12}$$

Upon further algebraic simplification, we finally end up with

$$q(t) = \left(\frac{p}{\mu+p}\right)^N\left(1 - \frac{N}{t(\mu+p)}\right) - \frac{1}{t(\mu+p)}$$
$$\times\,\mathrm{e}^{-(\mu+p)t}\sum_{\ell=0}^{N-1}\left(\frac{p}{\mu+p}\right)^\ell\sum_{m=0}^{\ell}\left(\frac{\mu}{p+\mu}\sum_{i=0}^{m}\frac{((\mu+p)t)^i}{i!} - \frac{((\mu+p)t)^m}{m!}\right), \tag{13}$$

which yields equation (1).

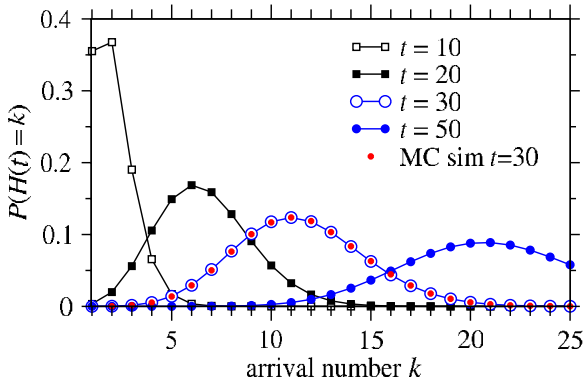

Figure 3. Plots of the distribution $P(H(t) = k)$ of the number of hits $H(t)$ that have occurred up to time t. Parameters used were $p \equiv 1$, $N = 10$, $\mu = 0.2$ and $\alpha = 3$. The distribution is plotted for times $t = 5, 10, 20, 30, 50$. Monte-Carlo simulation (filled red circles) (300 000 runs) was used to verify the results for $t = 30$.

In all of our following analyses, we nondimensionalize all rates in terms of p and times in terms of p^{-1}. The distribution $P(H(t) = k)$ plotted in figure 3 shows that the probability of a finite number k of arrivals first increases with time after the start of injection, and then decreases for long times when an excess of particles have arrived. Using 300 000 independent Monte-Carlo simulation runs implemented with the Bortz–Kalos–Lebowitz algorithm [18], we verified our results at time $t = 30$, with parameters $N = 10$, $p = 1$, $\mu = 0.2$ and $\alpha = 3$.

From the survival probability $S_k(t)$ found from equation (2), all moments σ of the kth passage times can be computed:

$$\left\langle T_k^{\sigma} \right\rangle = -\int_0^{\infty} t^{\sigma} \frac{\mathrm{d}S_k}{\mathrm{d}t}\,\mathrm{d}t. \tag{14}$$

The mean ($\sigma = 1$) *first* ($k = 1$) passage time to the absorbing site can be found from

$$S_1(t) = P(H(t) = 0) = \exp\left[\alpha\left(\frac{p}{\mu+p}\right)^N\left(\frac{N}{\mu+p} - t\right) + \frac{\alpha}{\mu+p}\,\mathrm{e}^{-(\mu+p)t} \times \right.$$
$$\left. \sum_{\ell=0}^{N-1}\left(\frac{p}{\mu+p}\right)^{\ell}\sum_{m=0}^{\ell}\left(\frac{\mu}{\mu+p}\sum_{i=0}^{m}\frac{((\mu+p)t)^i}{i!} - \frac{((\mu+p)t)^m}{m!}\right)\right] \tag{15}$$

and equation (14). An explicit expression can be found for a single-site ZRP ($N = 1$):

$$\langle T_1 \rangle = \frac{\mathrm{e}^x x^{-x}}{\mu+p}\gamma(x,x), \qquad N = 1, \tag{16}$$

where

$$x \equiv \frac{\alpha p}{(\mu+p)^2}. \tag{17}$$

Upon approximating the lower incomplete gamma function $\gamma(x,x) \equiv \Gamma(x) - \Gamma(x,x)$ in the $x \to 0$ limit, we find

$$\langle T_1 \rangle = \frac{1}{\mu+p}\left[\frac{1}{x} + 1 + \mathcal{O}(x)\right], \qquad N = 1. \tag{18}$$

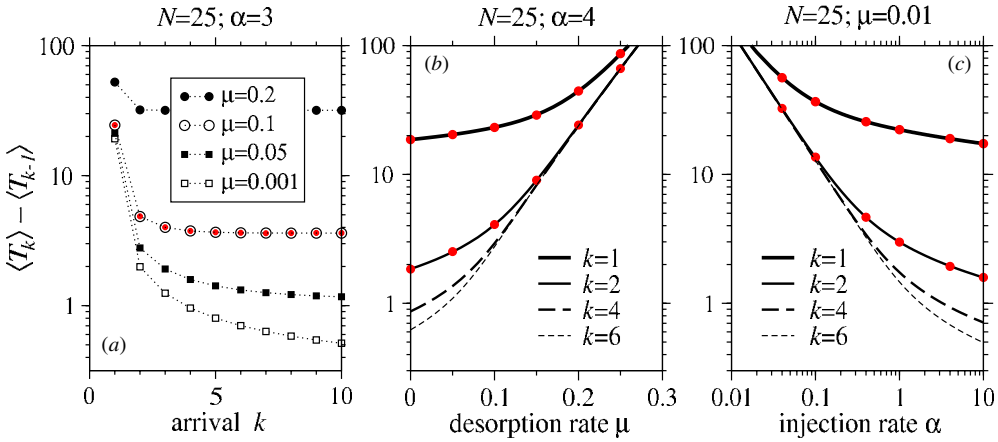

Figure 4. Interarrival times $\langle T_k\rangle - \langle T_{k-1}\rangle$ for the ZRP obeying bulk dynamics (with $p \equiv 1$). Here $T_0 = 0$, so that for $k = 1$ we are simply plotting T_1. (a) Interarrival times as a function of arrival k various values of the desorption rate μ for fixed $\alpha = 3$. (b) Interarrival times of the first, second, fourth and sixth particles as a function of μ for fixed $\alpha = 4$. (c) First, second, fourth and sixth arrival times as a function of the injection rate α for $\mu = 0.01$. All plots were evaluated using $N = 25$. The curves asymptote to the interarrival time $(p+\mu)^N/\alpha p^N$. Results were verified using Monte-Carlo simulations as depicted by the filled red circles. The plot label to the far left applies to all three panels.

In the $x \to \infty$ limit, we apply the method of steepest descents [19] to the integral definition of the Γ-function to find

$$\langle T_1\rangle = \frac{1}{\mu + p}\sqrt{\frac{\pi}{2x}}\left[1 + \frac{12}{x} + \mathcal{O}(x^{-2})\right], \qquad N = 1. \tag{19}$$

For $N > 1$, the integral in equation (14) does not have a simple representation in the non-integral form, nor can the mean kth passage times be calculated explicitly in the $N = 1$ case for $k > 1$. However, we can find some asymptotic results for large k in the 'fast dynamics' limit. If we denote by τ the characteristic time $\tau = N/(\mu + p)$ for a particle just injected to reach the final site, and consider times t such that $t \gg \tau$, equation (1) may be written as

$$\lambda(t) = \alpha_{\text{eff}}(t - \tau) + \mathcal{O}(\mathrm{e}^{-t(\mu+p)}), \tag{20}$$

where $\alpha_{\text{eff}} = \alpha p^N(\mu + p)^{-N}$ is an effective injection rate from the perspective of the final site that takes decay into account. Because $t \gg \tau$ holds for all but a negligible part of the range of the integral in equation (14), we have

$$\langle T_k\rangle = \frac{k}{\alpha_{\text{eff}}} + \tau + \mathcal{O}(\tau^2). \tag{21}$$

The assumption on t translates onto a condition on k, so that equation (21) remains valid as long as $\langle T_k\rangle \gg \tau$, or $k \gg \alpha_{\text{eff}}\tau$.

In figure 4, we plot the interarrival times $\langle T_k\rangle - \langle T_{k-1}\rangle$ as a function of k, μ and α.[1] Figure 4(*a*) shows that larger desorption rates μ, keeping α fixed, provide more time for more particles to be injected before the the first one reaches the end site. Therefore, the interarrival times approach their steady-state limit $\langle T_k\rangle - \langle T_{k-1}\rangle \approx 1/\alpha_{\text{eff}}$ at smaller k. Large desorption rates μ decrease the number of lattice particles, increasing arrival times to the end

[4] Equations (2) and (14) define $T_0 \equiv 0$.

site. As a consequence, the initial empty-lattice condition is 'forgotten' after a fewer number of particles have reached site $N+1$. In figure 4(*b*), we see that the mean interarrival times, including the mean first passage time, increase exponentially for large μ. This behavior is in contrast to the case where only a single particle is injected, in which the *conditional* mean first passage time of that single particle *decreases* with increasing desorption μ. Because the mean arrival time for a single particle needs to be conditioned on exiting through site N, only very fast trajectories can survive the desorption process, leading to a mean arrival time that decreases rapidly with increasing μ. Finally, figure 4(*c*) plots the mean interarrival times as a function of injection rate α. Note that interarrival times grow only geometrically as α decreases, indicating that the injection rate has less of an impact on them than the particle decay rate.

In the next section we rederive the same results above by solving the corresponding master equation using generating functions. Using this approach, we not only recover the mean passage times to site $N+1$ but also the full particle occupation distribution function $P(n_1, \ldots, n_{N+1}, t)$.

2.2. Solving the bulk dynamics via generating functions

In this section, for mathematical completeness, we rederive the survival probability using generating function methods applied to the Master equation for describing the probability $P(\{n_\ell\}, t)$ of having $\{n_\ell\}$ particles on each of the $1 \leqslant \ell \leqslant N+1$ sites:

$$\begin{aligned}\dot{P}(\{n_\ell\}, t) = &-\alpha\big[P(\{n_\ell\}, t) - P(n_1 - 1, \ldots, n_{N+1}, t)\big(1 - \delta_{n_1,0}\big)\big] \\ &- (\mu + p)\sum_{j=1}^{N} n_j\, P(\{n_\ell\}, t) + \mu \sum_{j=1}^{N}(n_j + 1)P(n_1, \ldots, n_j + 1, \ldots, t) \\ &+ p\sum_{j=1}^{N}(n_j + 1)P(n_1, \ldots, n_j + 1, n_{j+1} - 1, \ldots, t)\big(1 - \delta_{n_{j+1},0}\big). \end{aligned} \tag{22}$$

The survival probability $S_1(t)$ is defined by the probability of having no particles in the absorbing site

$$S_1(t) = \sum_{n_1, n_2 \cdots n_N} P(n_1, \ldots, n_{N+1} = 0, t).$$

After setting $n_{N+1} = 0$ in equation (22), multiplying by $z_1^{n_1} \cdots z_N^{n_N}$, and summing over all possible values of n_j, $1 \leqslant j \leqslant N$, we find a first-order partial differential equation for the constrained generating function

$$G_0(z_1, \ldots, z_N, t) = \sum_{n_1, \ldots, n_N = 0}^{\infty} z_1^{n_1} \cdots z_N^{n_N} P(n_1, \ldots, n_N, n_{N+1} = 0, t).$$

Upon solving this partial differential equation using the method of characteristics [20], we find that G_0 obeys

$$\frac{\mathrm{d}G_0}{\mathrm{d}t} = \alpha(z_1 - 1)G_0 \tag{23}$$

along the trajectories defined by

$$\begin{aligned}\frac{\mathrm{d}z_1(t)}{\mathrm{d}t} &= (\mu + p)z_1(t) - \mu - pz_2(t), \\ \frac{\mathrm{d}z_j(t)}{\mathrm{d}t} &= (\mu + p)z_j(t) - \mu - pz_{j+1}(t), \\ \frac{\mathrm{d}z_N(t)}{\mathrm{d}t} &= (\mu + p)z_N(t) - \mu. \end{aligned} \tag{24}$$

The initial condition $P(n_1, \ldots, n_N, n_{N+1} = 0, t = 0) = \delta_{n_1,0} \cdots \delta_{n_N,0}$ gives $G_0(z_0, \ldots, z_N, t = 0) = 1$. Equations (24) can be written in the form

$$\frac{\mathrm{d}\mathbf{Z}(t)}{\mathrm{d}t} = \mathbf{M}\mathbf{Z}(t) - \mu\mathbf{I}, \tag{25}$$

where $\mathbf{Z}(t) = (z_1(t), \ldots, z_N(t))^T$ is the vector of trajectories, $\mathbf{I}$ is the $N \times N$ identity matrix and $\mathbf{M}$ is a tridiagonal matrix with elements $m_{j,j} = \mu + p$, $m_{j,j+1} = -p$ and $m_{i,j} = 0$ otherwise. Upon defining the Laplace transform $\tilde{\mathbf{Z}}(s) = \int_0^\infty \mathbf{Z}(t)\,\mathrm{e}^{-st}\,\mathrm{d}t$ and the initial values $\mathbf{Z}(t = 0) = (z_1(t = 0), \ldots, z_N(t = 0))^T$, equation (25) can be written in the form

$$s\tilde{\mathbf{Z}}(s) = \mathbf{M}\tilde{\mathbf{Z}}(s) - \frac{\mu}{s}\mathbf{I} + \mathbf{Z}(t = 0), \tag{26}$$

and solved explicitly by first inverting $s\mathbf{I}-\mathbf{M}$ and then calculating the inverse Laplace transform of $\tilde{Z}(s)$. After performing the algebra, the solution to equation (25) can be expressed as

$$(z_j - R_{N-j}) = \sum_{k=0}^{N-j} (z_{j+k}(t = 0) - R_{N-j-k}) \frac{(-pt)^k}{k!}\,\mathrm{e}^{(\mu+p)t}, \tag{27}$$

where

$$R_k \equiv 1 - \left(\frac{p}{\mu + p}\right)^{k+1}. \tag{28}$$

Upon using $z_1(t)$ from equation (27) in equation (23), we find G_0 as a function of the z_j values implicitly expressed through the starting positions $z_j(t = 0)$ of the trajectories:

$$G_0(z_1, \ldots, z_N, t) = \exp\left[-\alpha t \left(\frac{p}{\mu + p}\right)^N - \alpha \sum_{k=0}^{N-1} \frac{(z_{k+1}(t = 0) - R_{N-k-1})p^k}{(\mu + p)^{k+1}k!} \gamma[k + 1, -(\mu + p)t] \right]. \tag{29}$$

We are thus left with explicitly determining $z_j(t = 0)$ as a function of the independent variables z_j. We do this by inverting equation (27):

$$(z_j(t = 0) - R_{N-j}) = \sum_{k=0}^{N-j} (z_{j+k} - R_{N-j-k}) \frac{(pt)^k}{k!}\,\mathrm{e}^{-(\mu+p)t}. \tag{30}$$

Using this result in equation (29) we find

$$G_0(z_1, \ldots, z_N, t) = \exp\left[-\alpha t \left(\frac{p}{p + \mu}\right)^N - \alpha\,\mathrm{e}^{-(\mu+p)t} \sum_{k=0}^{N-1} \sum_{j=0}^{N-k-1} \frac{p^{j+k}t^j}{(\mu + p)^{k+1}k!j!} \times (z_{j+k+1} - R_{N-j-k-1})\gamma[k + 1, -(\mu + p)t] \right]. \tag{31}$$

Finally, since the survival probability is obtained by imposing $z_\ell = 1$ for all ℓ, we obtain

$$S_1(t) = \exp\left[-\alpha t \left(\frac{p}{p + \mu}\right)^N - \frac{\alpha p^N \mathrm{e}^{-(\mu+p)t}}{(\mu + p)^{N+1}} \sum_{j=0}^{N-1} \sum_{\ell=0}^{N-1-j} \frac{t^\ell(\mu + p)^\ell}{j!\ell!} \gamma[j + 1, -(\mu + p)t] \right],$$

which is equivalent to equation (15). We can now successively determine the dynamics of the probability distribution function conditioned on the absorbing site containing a finite number

of particles $n_{N+1} \geqslant 1$. For $n_{N+1} = 1$, the corresponding generating function $G_1(z_1, \ldots, z_N, t)$ can be obtained from the master equation for $P(n_1, \ldots, n_{N+1} = 1, t)$:

$$G_1(z_1, \ldots, z_N, t) = \sum_{n_1, \cdots n_N = 0}^{\infty} z_1^{n_1} \cdots z_N^{n_N} P(n_1, \ldots, n_N, n_{N+1} = 1, t).$$

Upon summing equation (22) over all values of n_j we find that the dynamics of G_1 is given by

$$\frac{\partial G_1}{\partial t} = \alpha G_1(z_1 - 1) + \frac{\partial G_0}{\partial z_1}, \tag{32}$$

where G_0 is the generating function associated with the adsorbing site having no particles, $n_{N+1} = 0$, and where the evolution of the trajectories $z_j(t)$ are unchanged from those described by equation (24). The solution to equation (32) can be expressed in the form

$$G_1(z_1, \ldots, z_N, t) = \lambda(t) G_0(z_1, \ldots, z_N, t), \tag{33}$$

where $\lambda(t)$ obeys

$$\frac{\mathrm{d}\lambda(t)}{\mathrm{d}t} = \frac{p}{G_0} \frac{\partial G_0}{\partial z_1}. \tag{34}$$

The solution for $\lambda(t)$ turns out to be precisely that given in equation (1). Similarly, it can be found that the generating function with the constraint $n_{N+1} = j$ is given by

$$\begin{aligned} G_j(z_1, \ldots, z_N, t) &= \sum_{n_1, \cdots n_N = 0}^{\infty} z_1^{n_1} \cdots z_N^{n_N} P(n_1, \ldots, n_{N+1} = j, t) \\ &= \frac{\lambda(t)^j}{j!} G_0(z_1, \ldots, z_N, t). \end{aligned} \tag{35}$$

The survival probability $S_k(t)$ is given by $S_k(t) = \sum_{j=0}^{k-1} G_j(1, \ldots, 1, t)$ and moments of the kth arrival times, found previously, can also be obtained using equation (14). In addition, an advantage of the generating function approach is that the particle occupations can also be determined. For example, the mean occupation at site ℓ when exactly j particles have entered site $N + 1$ is given by

$$\begin{aligned} \langle n_\ell(t | n_{N+1} = j) \rangle &= \sum_{n_1, \cdots, n_N = 0}^{\infty} n_\ell P(n_1, \ldots, n_{N+1} = j, t) \\ &= \frac{\lambda(t)^j}{j!} \left. \frac{\partial G_0(1, \ldots, z_\ell, \ldots, 1, t)}{\partial z_\ell} \right|_{z_\ell = 1} \\ &= -\frac{\lambda(t)^j}{j!} \mathrm{e}^{-\lambda(t)} \alpha p^{\ell - 1} \mathrm{e}^{-(\mu + p)t} \sum_{k=0}^{\ell - 1} \frac{\gamma[k + 1, -(\mu + p)t]\, t^{\ell - k - 1}}{(\mu + p)^{k+1} k! (\ell - k - 1)!}. \end{aligned} \tag{36}$$

Upon summing equation (36) over all j, we find that the unconditioned mean occupation $\langle n_\ell(t) \rangle$ is given by

$$\begin{aligned} \langle n_\ell(t) \rangle &= \sum_{n_1, \cdots, n_{N+1} = 0}^{\infty} n_\ell P(n_1, \ldots, n_{N+1}, t) \\ &= -\alpha p^{\ell - 1} \mathrm{e}^{-(\mu + p)t} \sum_{k=0}^{\ell - 1} \frac{\gamma[k + 1, -(\mu + p)t]\, t^{\ell - k - 1}}{(\mu + p)^{k+1} k! (\ell - k - 1)!}. \end{aligned} \tag{37}$$

Two limits are of interest: the mean occupation conditioned on no particles hitting site $N + 1$, which is given by

$$\langle n_\ell(t | n_{N+1} = 0) \rangle = \mathrm{e}^{-\lambda(t)} \langle n_\ell(t) \rangle,$$

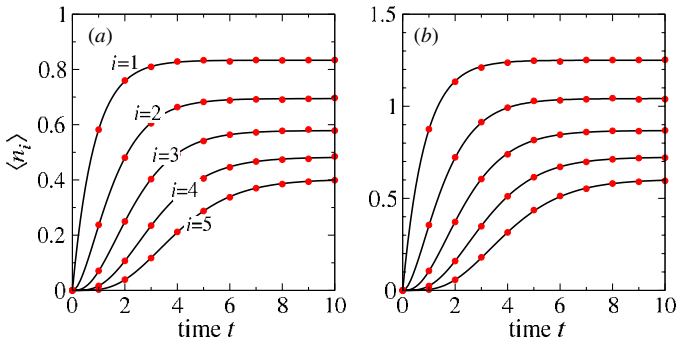

Figure 5. Time dependence of the mean site occupancies. Both panels display exact values (solid lines) from equation (37) and simulation (filled red circles) for parameter values $p \equiv 1$, $N = 5$ and $\mu = 0.2$. Each curve and approximating points correspond to mean occupations at different sites, with earlier site having higher occupations. In the left-hand panel $\alpha = 1 < p + \mu$ and particles are cleared out faster than they are injected, resulting in $\langle n_1 \rangle < 1$. In the right-hand panel $\alpha = 1.5 > p + \mu$, yielding the possibility of $\langle n_1 \rangle > 1$.

and the average occupation of site $N + 1$, regardless of the occupation state of all other sites, which is simply $\langle n_{N+1}(t) \rangle = \lambda(t)$. Thus, in the long time limit, the occupation of the final site $N + 1$ will scale as

$$\langle n_{N+1}(t) \rangle \sim \alpha t \left(\frac{p}{\mu + p} \right)^N,$$

indicating that at site $N + 1$ particles accumulate linearly at a rate that is proportional to the injection rate α attenuated by the evaporation probability for each of the N intervening sites.

In figure 5, we have plotted the mean occupations derived from equation (37) for $N = 5$ and $\mu = 0.2$. All mean occupancies are seen to reach a steady state by $t \approx 10$ and, for all times, the mean occupancy is a monotonically decreasing function of site index due to decay. In figure 5(*a*), $\alpha = 1 < p + \mu$, implying that particles are cleared out faster than they are injected, resulting in $\langle n_1 \rangle$ approaching a value less than 1. In (*b*), the $\alpha = 1.5 > p + \mu$, and $\langle n_1 \rangle$ (and $\langle n_2 \rangle$) asymptotes to values greater than 1. Our results are verified with Monte-Carlo simulations.

Finally, the full distribution for $P(n_1, \ldots, n_{N+1} = j, t)$ can be found by using the Cauchy integral [21] over equation (35):

$$P(n_1, \ldots, n_{N+1} = j, t) = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{G_j(z_1, \ldots, z_N, t)}{z_1^{n_1+1}, \ldots, z_N^{n_N+1}} \, \mathrm{d}z_1 \cdots \mathrm{d}z_N, \tag{38}$$

where the integral is closed along a path encircling the origin. Evaluating the residues, the above integral can be expressed as

$$P(n_1, \ldots, n_{N+1} = j, t) = \frac{\lambda(t)^j}{j!} \prod_{\ell=1}^{N} \left(\frac{\partial}{\partial z_\ell} \right)^{n_\ell} G_0(z_1, \ldots, z_N, t) \Big|_{z_\ell = 0}, \tag{39}$$

which can be calculated explicitly to yield

$$P(n_1, \ldots, n_{N+1} = j, t) = \frac{\lambda(t)^j G_0(0, \ldots, 0, t)}{j!} \times \prod_{\ell=1}^{N} \left[-\alpha \, \mathrm{e}^{-(\mu+p)t} \sum_{k=0}^{\ell-1} \frac{p^{\ell-1} t^{\ell-1-k}}{(p+\mu)^{k+1} k!} \gamma[k+1, -(\mu+p)t] \right]^{n_\ell}. \tag{40}$$

3. Zero-range model with surface dynamics

Surface dynamics, or 'chipping' processes, differ from bulk dynamics in that only the top particle at a given site is able to hop or decay, while the ones below remain inert. Such surface-dynamic rules generally preclude an analytic solution to the kth hitting time and the survival probability $S_k(t)$. In particular, we cannot use the strategy employed for bulk dynamics because it relied essentially on particles injected before time t having *independent* probabilities of reaching site $N+1$ by time t. In the case of surface dynamics, only the top particle in a pile attempts to move to a neighboring one. The difficulty arises in keeping track of which sites are empty and which ones contain at least one particle. The main results of this section are the first passage time in the case of one intervening site, equation (51), and the steady-state limit of the interarrival times, equation (54) below.

Beginning with the approach of the previous sub-section, we first consider the Master equation for the distribution of the site occupancies obeying surface dynamics

$$\begin{aligned}\dot{P}(\{n_\ell\},t) = &-\alpha\big[P(\{n_\ell\},t) - P(n_1-1,\ldots,n_{N+1},t)\big(1-\delta_{n_1,0}\big)\big]\\ &-\mu\sum_{j=1}^{N}\big[\big(1-\delta_{n_j,0}\big)P(\{n_\ell\},t) - P(n_1,\ldots,n_j+1,\ldots,t)\big]\\ &-p\sum_{j=1}^{N}\big(1-\delta_{n_j,0}\big)P(\{n_\ell\},t)\\ &+p\sum_{j=1}^{N}\big(1-\delta_{n_{j+1},0}\big)P(n_1,\ldots,n_j+1,n_{j+1}-1,\ldots,t).\end{aligned} \tag{41}$$

We now introduce the marginal probability

$$P_i(n_i,t) = \sum_{\{n_{j\neq i}\}=0}^{\infty} P(n_1,\ldots,n_{N+1},t), \tag{42}$$

where the sum is taken over all n_j for all sites $1\leqslant j\leqslant N+1$, except site $j=i$. Equation (42) represents the probability that site i has n_i particles regardless of the occupation of all other sites. Similarly, the joint probability for sites $i-1$ and i is defined as

$$P_{i-1,i}(n_{i-1},n_i,t) = \sum_{\{n_{j\neq i-1,i}\}=0}^{\infty} P(n_1,\ldots,n_{N+1},t).$$

Upon summing equation (41) over all values of $n_{j\neq i}$, we find the time evolution for the marginal probability $P_i(n_i,t)$ as a function of the two-site probabilities $P_{i-1,i}(n_{i-1},n_i,t)$ for $1<i\leqslant N$:

$$\begin{aligned}\dot{P}_i(n_i,t) = &\,p\sum_{n_{i-1}=1}^{\infty}\big(1-\delta_{n_i,0}\big)P(n_{i-1},n_i-1,t) - p\sum_{n_{i-1}=1}^{\infty}P(n_{i-1},n_i,t)\\ &+(\mu+p)P_i(n_i+1,t) - (\mu+p)P(n_i,t)\big(1-\delta_{n_i,0}\big).\end{aligned}$$

Continuing in this way, the equations for the marginal occupation probabilities form a hierarchy which is completed by the equation for the injection site $i=1$:

$$\begin{aligned}\dot{P}_1(n_1,t) = &-\alpha\big[P_1(n_1,t) - P_1(n_1-1,t)\big(1-\delta_{n_1,0}\big)\big]\\ &-(\mu+p)\big[P_1(n_1,t)\big(1-\delta_{n_1,0}\big) - P_1(n_1+1,t)\big].\end{aligned} \tag{43}$$

Note that the dynamics for site $i = 1$ is completely decoupled from that of the other sites so that the marginal occupation distribution of the first site can be solved directly. We now consider two cases where analytic results can be found.

3.1. Single-site ZRP densities and mean first passage times

Since equation (43) is decoupled from the hierarchy, we can solve it by taking its Laplace transform and using the initial condition $P(n_1, 0) = \delta_{n_1,0}$ to find

$$\begin{aligned}\tilde{P}_1(n_1 = 1, s) &= \frac{s+\alpha}{\mu+p}\tilde{P}_1(n_1 = 0, s) - \frac{1}{\mu+p}\\ \tilde{P}_1(n_1+1, s) &= \left(1 + \frac{s+\alpha}{\mu+p}\right)\tilde{P}_1(n_1, s) - \frac{\alpha}{\mu+p}\tilde{P}_1(n_1-1, s).\end{aligned} \tag{44}$$

The solution can be expressed in the form

$$\tilde{P}_1(n_1, s) = \left[\frac{1 - z_1(s)}{s}\right] z_1(s)^{n_1}, \tag{45}$$

where

$$z_1(s) = \frac{1}{2(\mu+p)}\Big((s+\alpha+\mu+p) - \sqrt{(s+\alpha+\mu+p)^2 - 4\alpha(\mu+p)}\Big). \tag{46}$$

Upon inverting the Laplace transform, we find

$$z_1(t) = \frac{\mathrm{e}^{-(\alpha+\mu+p)t}\sqrt{\alpha}}{t\sqrt{\mu+p}} I_1(2\sqrt{\alpha(\mu+p)}t), \tag{47}$$

where $I_1(t)$ is the first-order modified Bessel Function of the first kind. From equation (45) and using the fact that the inverse Laplace transform of a product is a convolution in time, we can iteratively construct $P_1(n_1, t)$ starting from $P_1(0, t)$:

$$P_1(n_1 = 0, t) = 1 - \int_0^t z_1(t')\,\mathrm{d}t', \tag{48}$$

where $z_1(t)$ is given by equation (47) and

$$P_1(n_1, t) = \int_0^t P_1(n_1 - 1, t')z_1(t')\,\mathrm{d}t'. \tag{49}$$

The integrals in equations (48) and (49) do not have simple closed forms. However, the functions $P_1(n_1, t)$ can also be obtained from differentiation using the relation

$$\begin{aligned}P_1(n_1+1, t) = &\left(1 - \delta_{n_1,0} + \frac{\alpha}{\mu+p}\right) P_1(n_1, t)\\ &+ \frac{\dot{P}_1(n_1, t) - \alpha P_1(n_1 - 1, t)\big(1 - \delta_{n_1,0}\big)}{\mu+p}.\end{aligned}$$

For instance, we may recursively write

$$P_1(n_1 = 1, t) = \frac{\alpha}{\mu+p}\left(1 - \int_0^t z_1(t')\,\mathrm{d}t'\right) - \frac{z_1(t)}{\mu+p}.$$

In the case of $N = 1$ we can also solve for the *first* ($k = 1$) passage times by observing that the equation for the two-site distribution function, conditioned on $n_2 = 0$, is also decoupled from the hierarchy:

$$\begin{aligned}\dot{P}(n_1, 0, t) = \alpha\big[P(n_1 - 1, 0, t)\big(1 - \delta_{n_1,0}\big) - P(n_1, 0, t)\big] + \mu P(n_1 + 1, 0, t)\\ - (\mu+p)P(n_1, 0, t))\big(1 - \delta_{n_1,0}\big),\end{aligned}$$

where we have dropped the subscripts on the two-site distribution function so that $P(n_1, n_2, t) \equiv P_{1,2}(n_1, n_2, t)$. Using Laplace transforms, we find

$$\tilde{P}(n_1, n_2 = 0, s) = \frac{y_1(s)^{n_1}}{s + \alpha - \mu y_1(s)}, \tag{50}$$

where

$$y_1(s) = \frac{\alpha + \mu + p + s - \sqrt{(\alpha + \mu + p + s)^2 - 4\alpha\mu}}{2\mu}.$$

From the above solution of $\tilde{P}(n_1, n_2 = 0, s)$, we can obtain the Laplace transformed probability that site $i = 2$ has not been hit by a particle $\tilde{S}_1(s) = \sum_{n_1=0}^{\infty} \tilde{P}(n_1, n_2 = 0, s)$. Thus, in the $N = 1$ case, the mean first passage time is

$$\langle T_1 \rangle = \sum_{n_1=0}^{\infty} \tilde{P}(n_1, 0, s = 0) = \frac{\alpha + \mu + p + \sqrt{(\alpha + \mu + p)^2 - 4\alpha\mu}}{2\alpha p}. \tag{51}$$

Note that in the case of $\mu = 0$, equation (51) simplifies to $\langle T_1 \rangle = \alpha^{-1} + p^{-1}$. In surface dynamics without desorption, the first passage time is determined by the dynamics of the lead particle. Therefore, the mean first arrival time in the case of an N site system for $\mu = 0$ is simply the total time it takes for the lead particle to reach site $N + 1$ and is given by $\langle T_1 \rangle = \alpha^{-1} + Np^{-1}$.

3.2. Steady-state limit

We have not been able to find closed-form solutions of the surface dynamics ZRP for general N and nonzero desorption rate $\mu > 0$. However, equation (43) can be solved in the steady-state limit by using the *ansatz* $P_1(n_1) = (1 - z_1)\, z_1^{n_1}$. The equation supports a solution when $z_1 = \alpha(\mu + p)^{-1}$, implying

$$P_1(n_1) = \left(1 - \frac{\alpha}{\mu + p}\right)\left(\frac{\alpha}{\mu + p}\right)^{n_1}.$$

The above expression is correct only for $\alpha < \mu + p$, so that $0 \leqslant P_1(n_1) \leqslant 1$. Physically this condition is simply a statement that if injection is too fast, particles continue to accumulate without bound. Steady-state levels arise only if the injection rate α is small enough such that hopping p and evaporation μ can keep up.

In order to solve equation (43), we need a closure relation for the two-site probability distribution $P_{i-1,i}(n_{i-1}, n_i)$. As shown in [10], the two-site probability distribution can be factorized in the steady-state limit and expressed as $P_{i-1,i}(n_{i-1}, n_i) = P_{i-1}(n_{i-1})\, P_i(n_i)$. If we impose that each $P_j(n_j)$ has a power law dependence in n_j, similar to what was done for n_1, it is easy to verify that the steady-state marginal probabilities are solved by

$$P_j(n_j) = \left(1 - \frac{\alpha p^{j-1}}{(\mu + p)^j}\right)\left(\frac{\alpha p^{j-1}}{(\mu + p)^j}\right)^{n_j}, \qquad \alpha < \mu + p. \tag{52}$$

The resulting steady-state mean occupation at each site is

$$\langle n_j \rangle = \frac{\alpha p^{j-1}}{(\mu + p)^j - \alpha p^{-1}}, \qquad \alpha < \mu + p.$$

From our steady-state results for $N > 1$, we can find an approximation to the passage times by a mean-field argument in which the probability of site $N+1$ surviving up to $k-1$ particles hitting it obeys $\dot{S}_k(t) = -J(t) S_k(t)$. The particle current $J(t) = p \sum_{n_N=1}^{\infty} P(n_N | T_k > t)$ is conditioned on fewer than k particles having arrived at site $N + 1$ by time t. Since neither $P(n_N | T_k > t)$

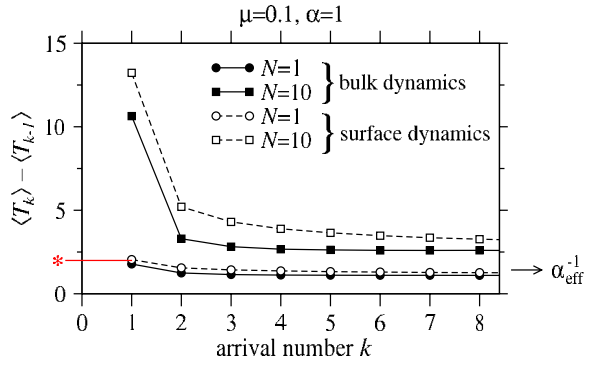

Figure 6. Interarrival times $\langle T_k\rangle - \langle T_{k-1}\rangle$ for the surface dynamics ZRP as a function of k. Results from Monte-Carlo simulations (open symbols) for both $N = 1$ and $N = 10$ are presented. Since fewer particle are mobile in surface dynamics, the arrival times are longer. The simulations match the analytic results found for the $N = k = 1$ (equation (51), indicated by the asterisk) and large k (equations (54) and (21)) limits. For comparison, numerical results for the bulk dynamics case (filled symbols) are also plotted. All plots were derived using $\mu = 0.1$ and $\alpha = 1$.

nor the unconditional distribution $P(n_N, t)$ are available, we must approximate $J(t)$ with its steady-state, 'mean-field' (single site marginal distribution) value through equation (52):

$$\begin{aligned} J &\approx p \sum_{n_N=1}^{\infty} P(n_N, t \to \infty) \\ &= p \sum_{n_N=1}^{\infty} \left(1 - \frac{\alpha p^{N-1}}{(\mu + p)^N}\right)\left(\frac{\alpha p^{N-1}}{(\mu + p)^N}\right)^{n_N} \\ &= \frac{\alpha p^N}{(\mu + p)^N}. \end{aligned} \tag{53}$$

In this approximation, J is independent of k and all interarrival times are approximately

$$\langle T_k\rangle - \langle T_{k-1}\rangle = \int_0^{\infty} S_k(t)\,\mathrm{d}t \approx \frac{(\mu + p)^N}{\alpha p^N}. \tag{54}$$

As expected, this estimate is precisely that given by α_{eff} in equation (20) for the bulk dynamics case and is accurate in the limit of $\alpha \ll (\mu + p)$ where the entry flux is slow compared to the internal dynamics and the first passage time is dominated by the contribution given by α^{-1}. Fast internal dynamics allows the system to quickly reach steady state, rendering the interarrival times equivalent for bulk and surface dynamics.

Upon taking the limit of slow injection rate $\alpha \to 0$ in equation (51) we find

$$\lim_{\alpha\to 0}\langle T_1\rangle = \frac{(\mu + p)}{\alpha p},$$

which is identical to the result in equation (54) for $N = k = 1$. Figure 6 plots simulated interarrival times and compares them with those from bulk dynamics. For large k, interarrival times for both bulk and surface dynamics approach the same value $\alpha_{\mathrm{eff}}^{-1}$ for each N. The only exact result for surface dynamics is that given by equation (51) for $N = k = 1$ and is indicated by the asterisk at $\langle T_1\rangle \approx 2.051\,25$.

4. Summary and conclusions

In this paper, we have provided detailed and explicit calculations of first passage times of an N-site, one-dimensional zero-range process. Both a Poissonian injection process at the first site and spontaneous desorption from all sites were included. We considered both bulk dynamic and surface dynamic ('chipping') rules as illustrated in figure 1.

For the ZRP obeying bulk dynamics, we computed the particle passage times using two methods. In the first method, we explicitly enumerated the random walks of each injected particle and evaluated their probability of reaching the final absorbing site within a time window. The probability that the absorbing site has not absorbed more than k particles by a certain time was constructed. The main results for the survival probabilities are given by equations (1) and (2), with explicit expressions for the mean first passage time given by equation (16) and its subsequent asymptotic limits.

We also derived the complete master equation for the probability distribution for a ZRP obeying bulk dynamics, and solved its corresponding generating function using the method of characteristics. In addition to the kth passage time distribution, this yielded the mean conditional occupancies of each site given by equation (36), and the full probability distribution given by equation (40).

Finally, for a single site ($N = 1$) ZRP obeying surface dynamics, we found exact results for the site density distribution (equations (48) and (49)) and the mean *first* passage times (equation (51)). Note that higher moments of the first passage time are readily obtained by evaluating higher derivatives of equation (50) at $s = 0$. For general N, only the steady-state particle currents and interarrival times could be found in the closed form (equation (54)).

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