Expectation and Variance

1. **Uncorrelated does not imply independent.** Suppose we have random variables X and Y with the following joint distribution:

$$\mathbb{P}(X = x, Y = y) = \begin{cases} 1/3 & x = 1, y = \pm 1 \text{ or } x = 0, y = 0 \\ 0 & \text{otherwise} \end{cases}$$

   Show that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ and that $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$, but that the random variables X and Y are not independent. Pairs of random variables with the former two properties (which are equivalent) are called *uncorrelated*.

2. **A fact about variances**. Show that

$$\mathrm{Var}(\alpha X) = \alpha^2 \,\mathrm{Var}(X)$$

   for any real α and any X with finite variance.

3. **Using the law of total expectation.** Suppose we flip two fair coins, Coin A and Coin B, until both coins come up tails. Find the expected total number of heads we flip. (For example, we could get $\{T, H\}$, $\{H, H\}$, $\{H, T\}$, $\{T, H\}$, $\{T, T\}$ for a total of 5 heads.) For a much harder problem, you can try to find the expected number of heads before each coin has come up tails at least once.

4. **Computing a variance**. Show that the variance of a binomial random variable with parameters n and p is $np(1 - p)$. *Hint*: Direct computation from the definition of variance is probably not the right way to do this.

5. **Joint distributions**. For any $\alpha \in [-1/12, 1/6]$, consider a pair of random variables X, Y with the following joint distribution:

$$\mathbb{P}(X = x, Y = y) = \begin{cases} 1/4 - \alpha & x = 0, y = 0 \\ 1/12 + \alpha & x = 0, y = 1 \\ 1/2 + \alpha & x = 1, y = 0 \\ 1/6 - \alpha & x = 1, y = 1 \\ 0 & \text{otherwise} \end{cases}$$

   Show that this gives an (uncountable) family of joint distributions with the same marginal distributions. Which value of α corresponds to independence?