Counting and Expectation

1. **Counting cards.** Suppose we draw 13 cards out of a standard deck of 52. How many possible hands are there with

   (a) All four aces?
   (b) Exactly two aces?
   (c) No clubs?
   (d) At least eight spades?
   (e) More black cards than red cards?
   (f) The five of diamonds and the ten of hearts?
   (g) The five of diamonds or the ten of hearts (or both)?
   (h) The five of diamonds or the ten of hearts, but not both?

   (Please don't simplify any of your answers). Suppose we draw cards uniformly at random. Pick one of these examples and find the probability that we get a hand of that type.

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2. **The St. Petersburg paradox.** Consider the following game: We start with \$1 in a pot. A fair coin is flipped. If the coin comes up *heads*, the amount of money in the pot is doubled and the coin is flipped again. If the coin comes up *tails*, you win the money in the pot. Let X be the random variable representing your winnings.

   (a) How much would you be willing to pay to play this game? (What is your gut feeling as to the value of X?).
   (b) Find $\mathbb{P}(X = 2^k)$.
   (c) Compute $\mathbb{E}X$. How does this compare to your answer from part (a)?

3. **What expectation is not**. Give examples of random variables X such that

   (a) $\mathbb{P}(X = \mathbb{E}X) = 0$
   (b) $\mathbb{P}(|X - \mathbb{E}X| > 10^9) = 1$
   (c) $\mathbb{P}(X > \mathbb{E}X) > 0.999$

4. **Challenge**. (As usual, this is mostly in case you are done with the other problems and want something to think about.). Let X be a random variable that only takes nonnegative integer values. Show that

$$\mathbb{E}X = \sum_{k=0}^{\infty} \mathbb{P}(X > k)$$

   This is sometimes called the *tail sum* formula for expectation, and can be handy. Note: you may need to make some assumptions about the behavior of infinite sums.