Math 184 Week 2

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TA information:
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1. **Derangements and inclusion/exclusion** Recall that a permutation is a *derangement* if it has no *fixed points*; that is, $\sigma(i) \neq i$ for all i. We will count the number of derangements in S_n. For every $i \in [n]$, let $A_i = \{\sigma \in S_n \ : \ \sigma(i) = i\}$. Then the set of derangements is $S_n - \bigcup A_i$.

   (a) What is $|A_i|$? And if $i \neq j$, what is $|A_i \cap A_j|$?

   (b) Show that the number of derangements in S_n is

   $$n! \cdot \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

   (Hint: inclusion/exclusion)

   (c) Show that as $n \to \infty$ the probability that a random permutation is a derangement approaches $1/e$. Can you give a bound for the error in this approximation?

   (d) Show that as $n \to \infty$ the probability that a random permutation has exactly m fixed points approaches $1/(em!)$.

2. **Counting** Suppose we pick a random permutation $\sigma \in S_n$ for n some fixed constant.

   (a) What is the probability that the first three entries in the permutation $\sigma(1), \sigma(2), \sigma(3)$ are in increasing order?

   (b) What's the expected number of i such that $\sigma(i), \sigma(i+1), \sigma(i+2)$ are in increasing order? (Hint: Write this value as a sum of things you can take the expectation of)